Reconfiguration Graphs for Dominating Sets

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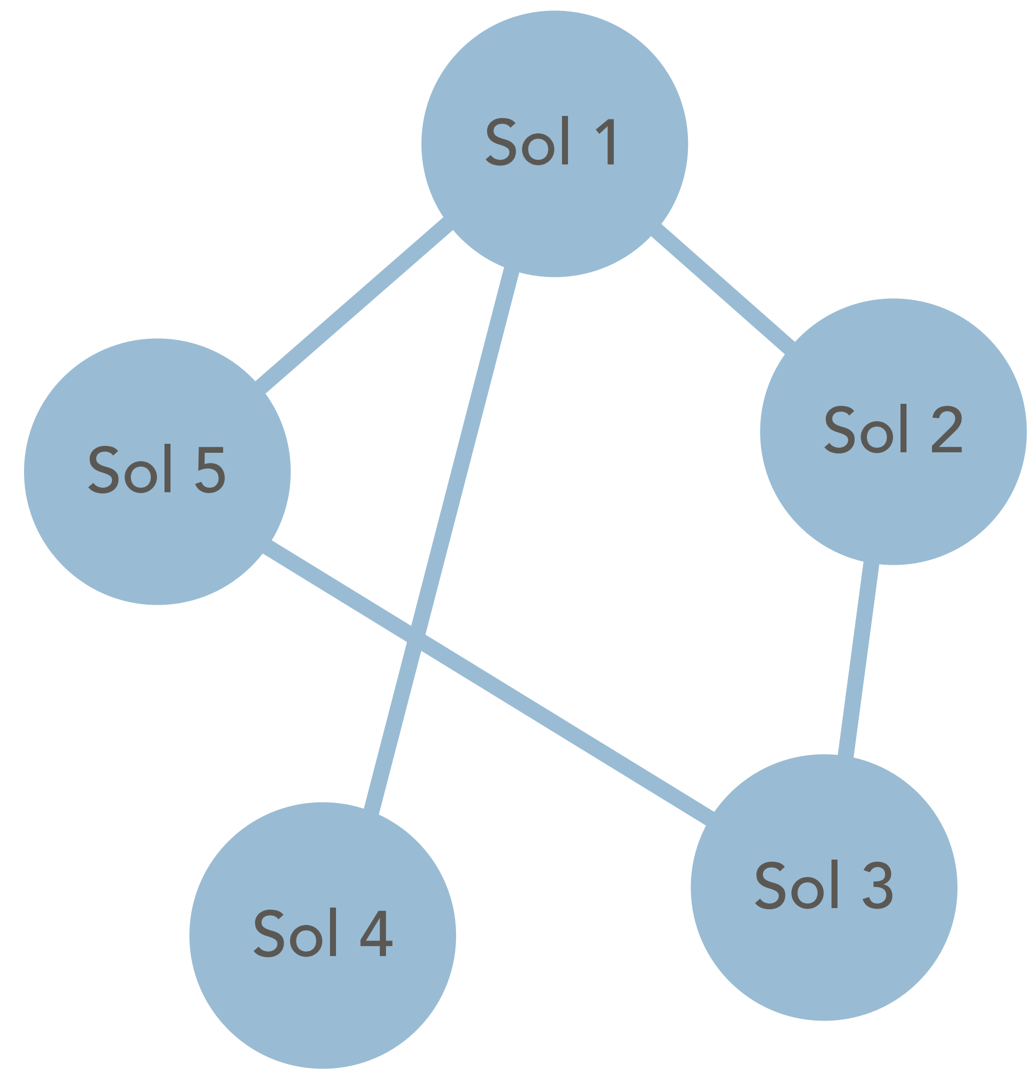
Mount Allison University

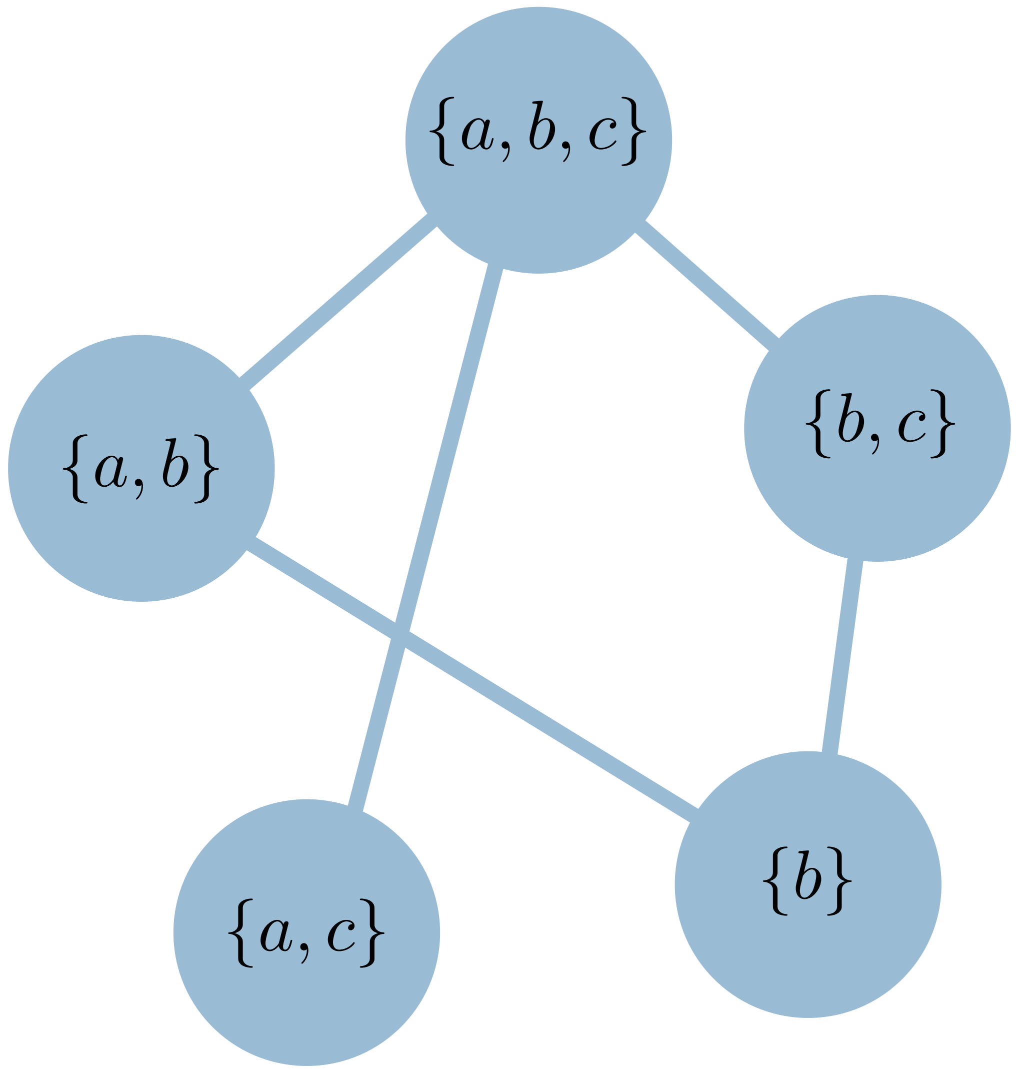
Feb 2022

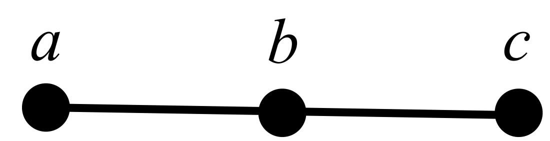
The *reconfiguration problem* involves determining if one feasible solution can be transformed into another by following a predetermined rule.

Any reconfiguration problem can be modelled with a *reconfiguration graph*.



Each dominating set of a graph G is represented by a vertex in the reconfiguration graph





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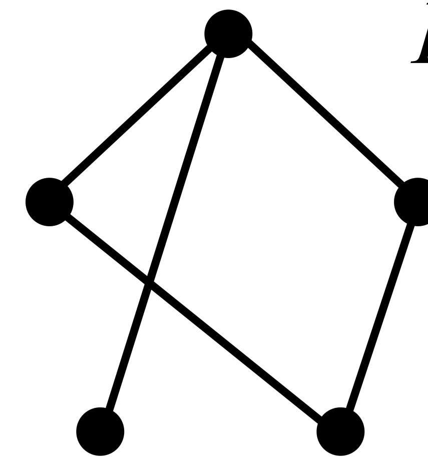
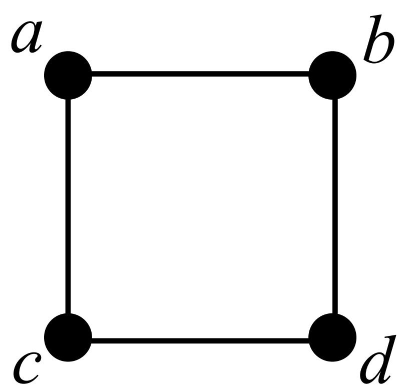


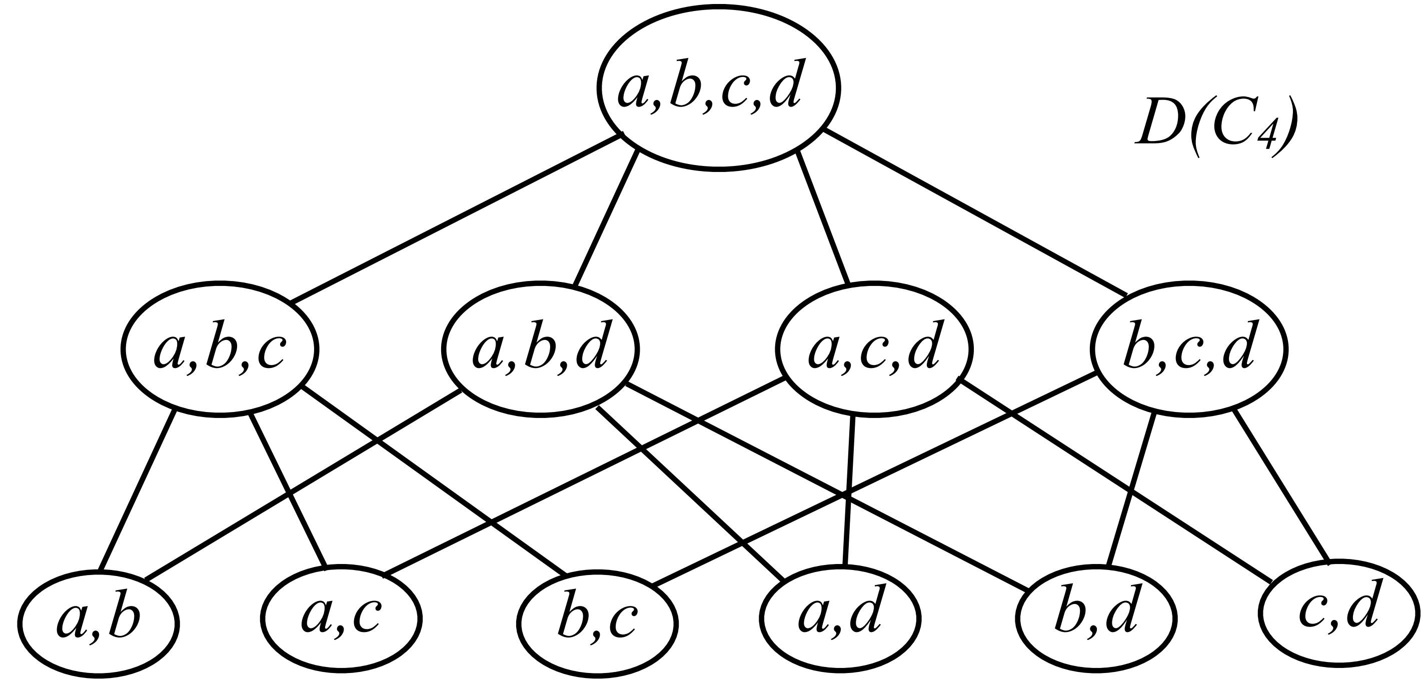
$D(G)$

Let x,y be two nodes in the reconfiguration graph $D(G)$ and let X,Y denote the dominating sets of G represented by x,y.

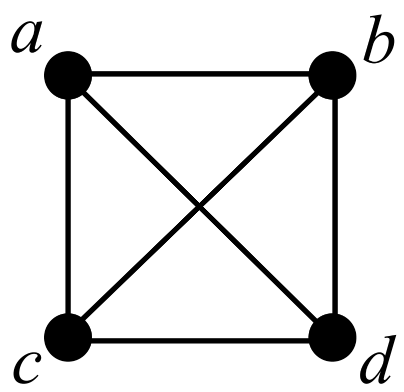
$x \sim y$ in $D(G)$ $\Leftrightarrow$ set X can be obtained by adding a vertex of G to Y or removing a vertex of G from Y.



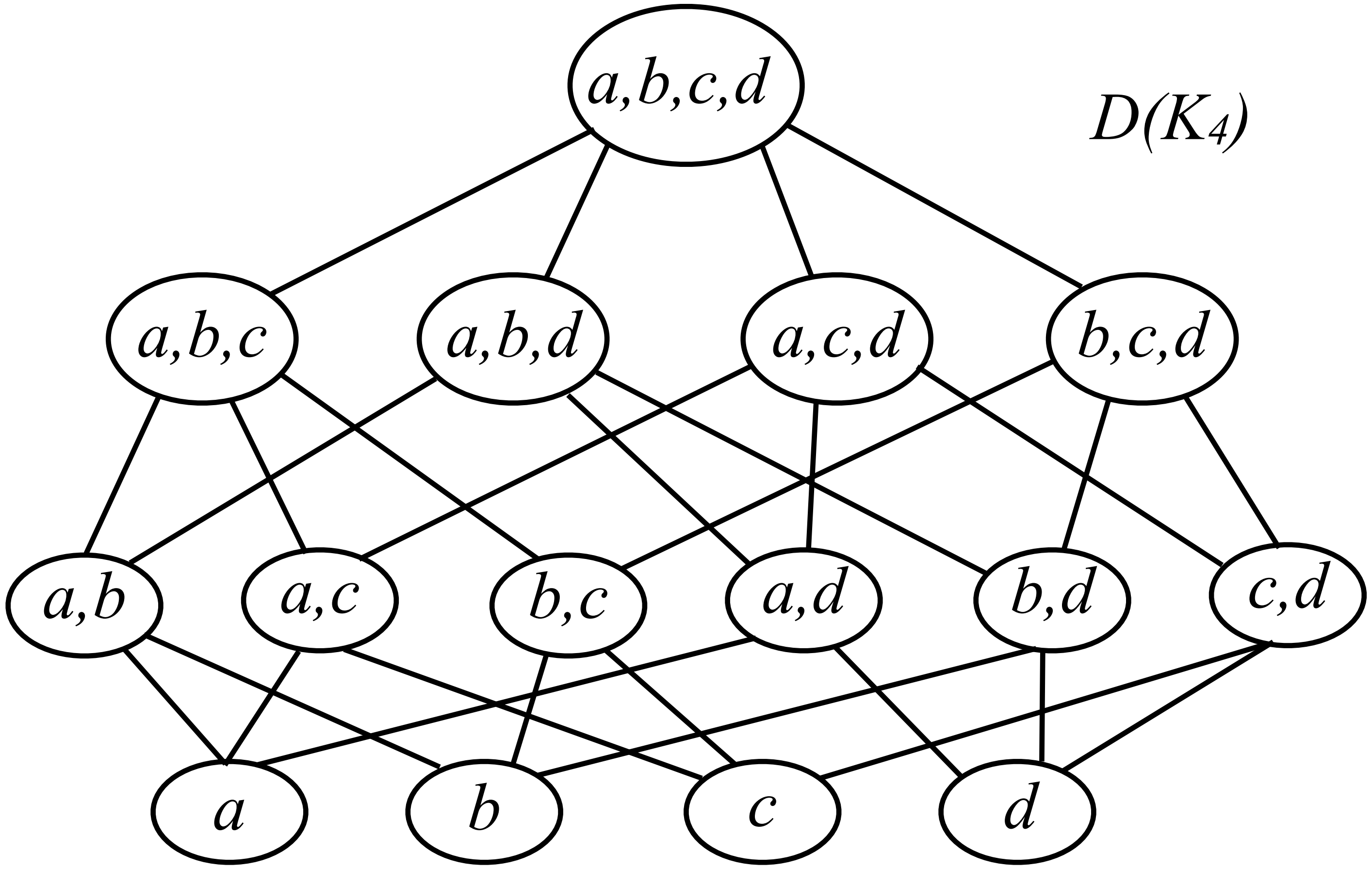
C_4



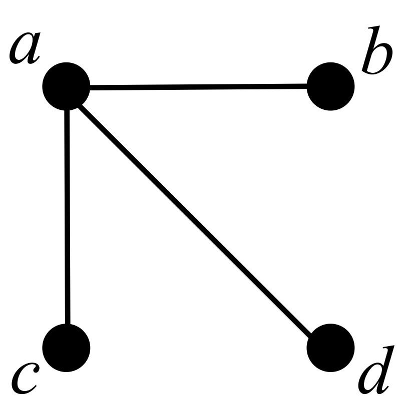
$D(C_4)$



K_4



$D(K_4)$



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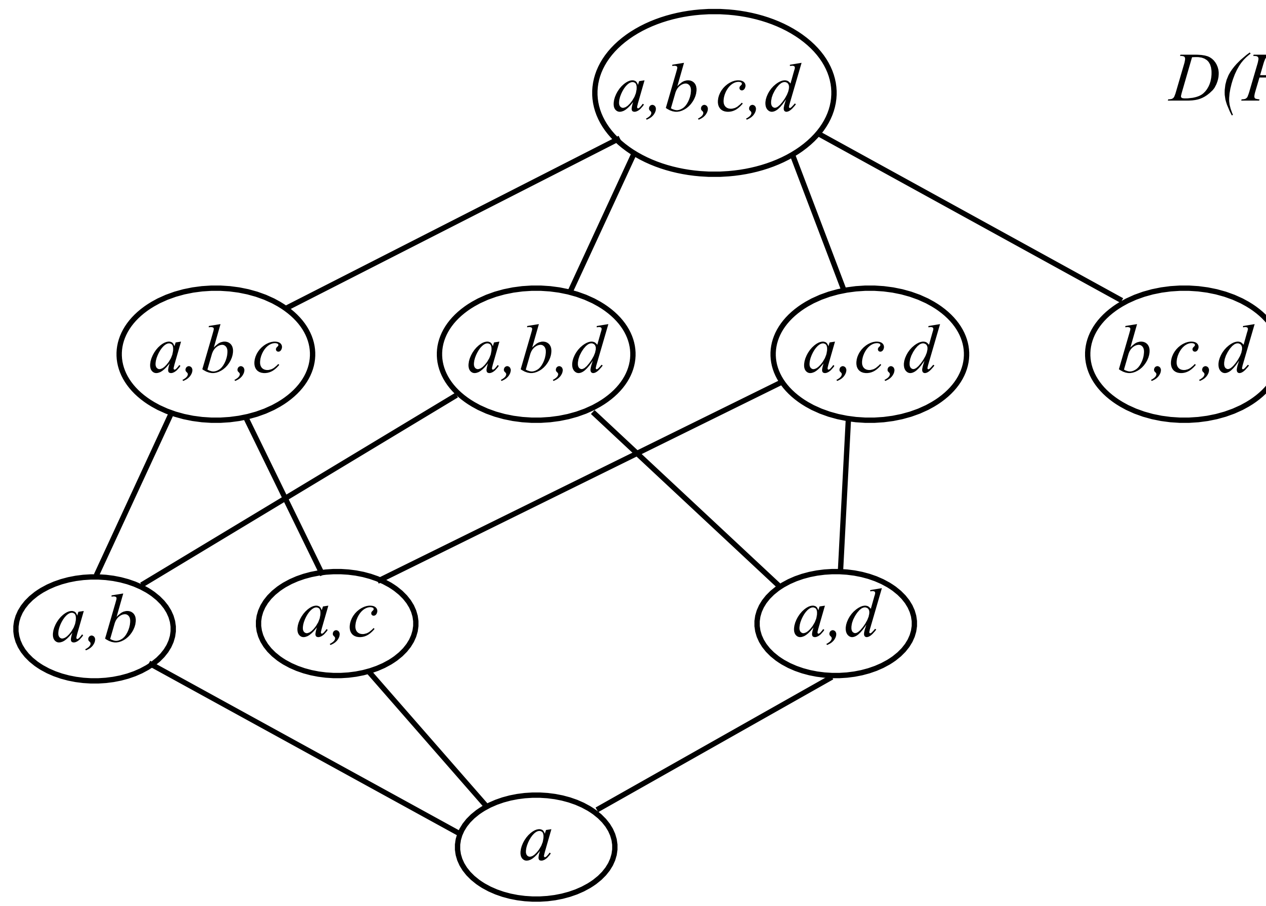


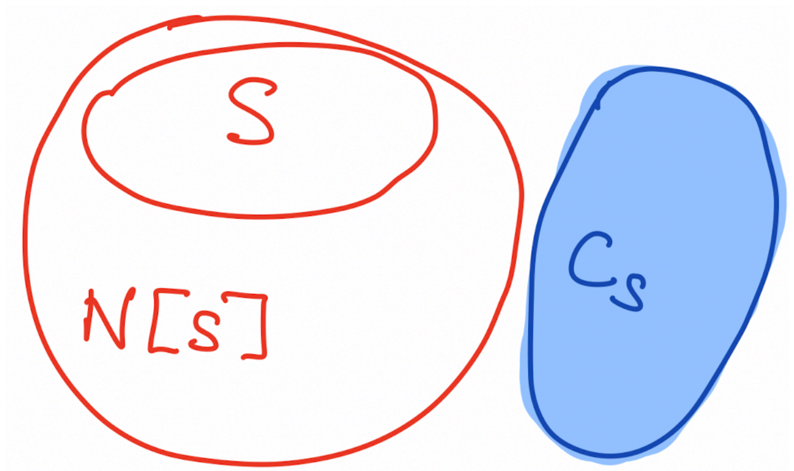
D(H)

Theorem: The number of dominating sets of a finite graph is odd.

[Brouwer, Csorba, Schrijver, unpublished]

Let $\mathcal{A} := \left\{(S,T) : S, T \subseteq V(G), S \cap T = \emptyset, st \notin E(G) \forall s \in S, t \in T\right\}$

Let $S \subseteq V(G)$ and $\mathcal{C}_S = V(G) \backslash N[S]$



$\mathcal{C}_S = \emptyset$ iff $N[S] = V(G)$, i.e. S is a dominating set

$(S,T) \in \mathcal{A}$ iff $T \subseteq \mathcal{C}_S$

$\therefore |\{T : (S,T) \in A\}| = 2^{|\mathcal{C}_S|}$

$\therefore |A|$ and the number of sets S with $\mathcal{C}_S = \emptyset$ have the <u>same parity</u>

$(S,T) = (T,S)$ only when $S = T = \emptyset$. Thus, $|\mathcal{A}|$ is odd.

Summer 2021

For which graphs G, is D(G) Eulerian?

Amanda Porter
Mount Allison University
Reconfiguration Graphs and Dominating Sets

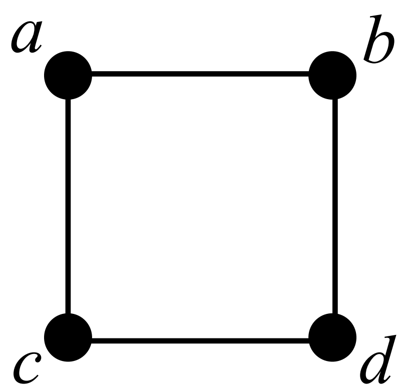
Lemma 1. *For a graph G with m components $G_1, G_2, \ldots, G_m$,*

$$\mathcal{D}(G_1) \,\square\, \mathcal{D}(G_2) \,\square \ldots \mathcal{D}(G_m) \cong \mathcal{D}(G).$$

[MP 2022+]

Theorem 2. *For a graph G with m components $G_1, G_2, \ldots, G_m$. Then $\mathcal{D}(G)$ is Eulerian if and only if $\mathcal{D}(G_i)$ is Eulerian for all $i \in \{1, 2, \ldots, m\}$.*

[MP 2022+]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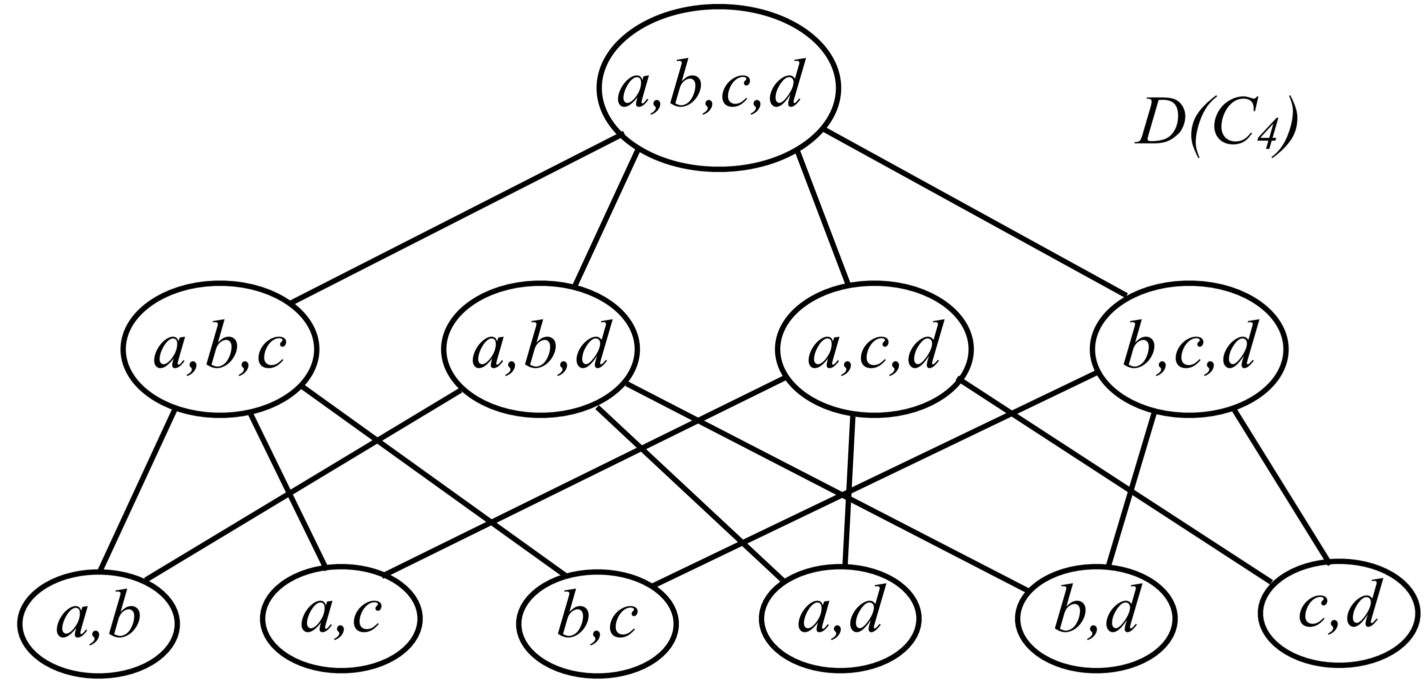
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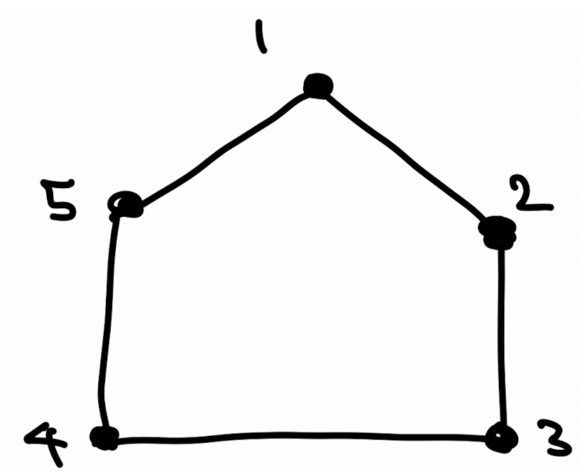
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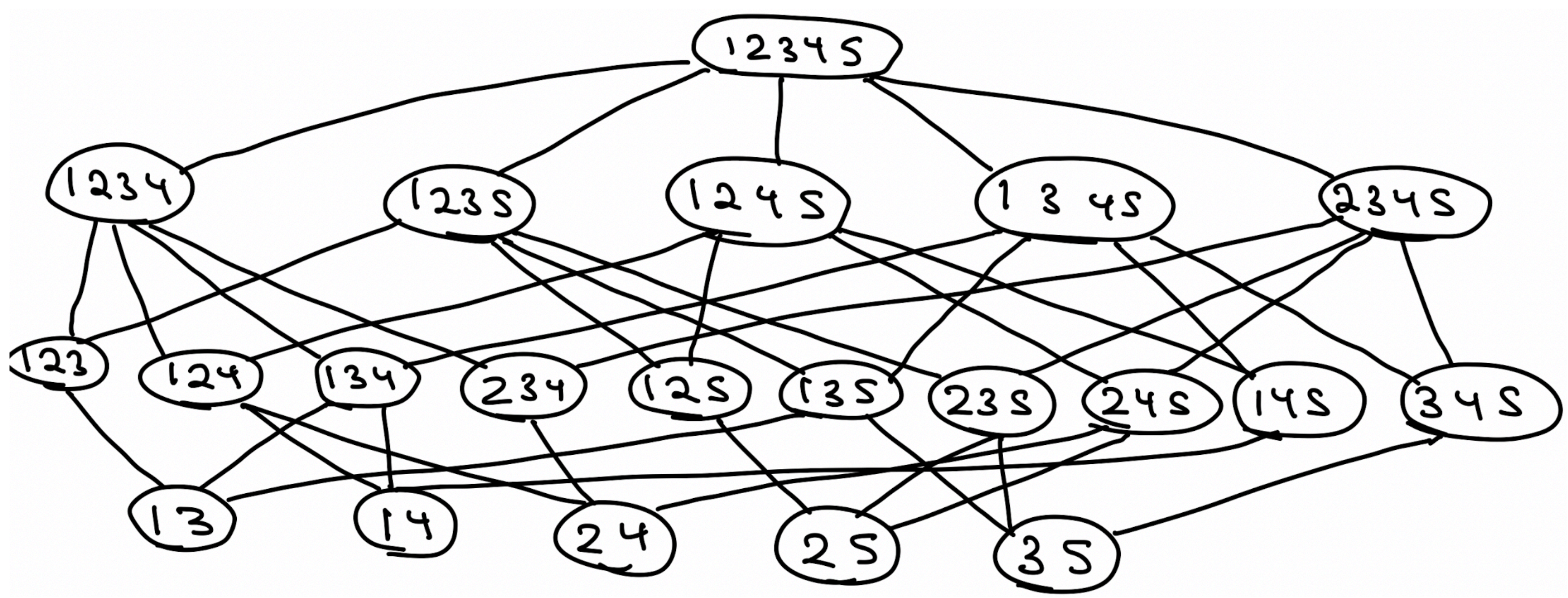
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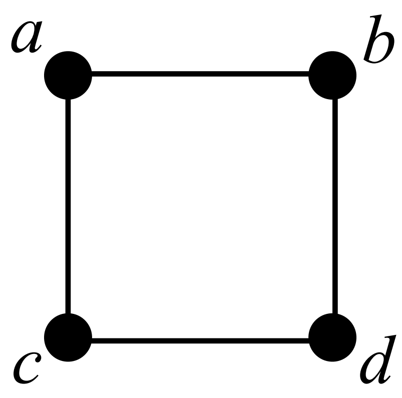

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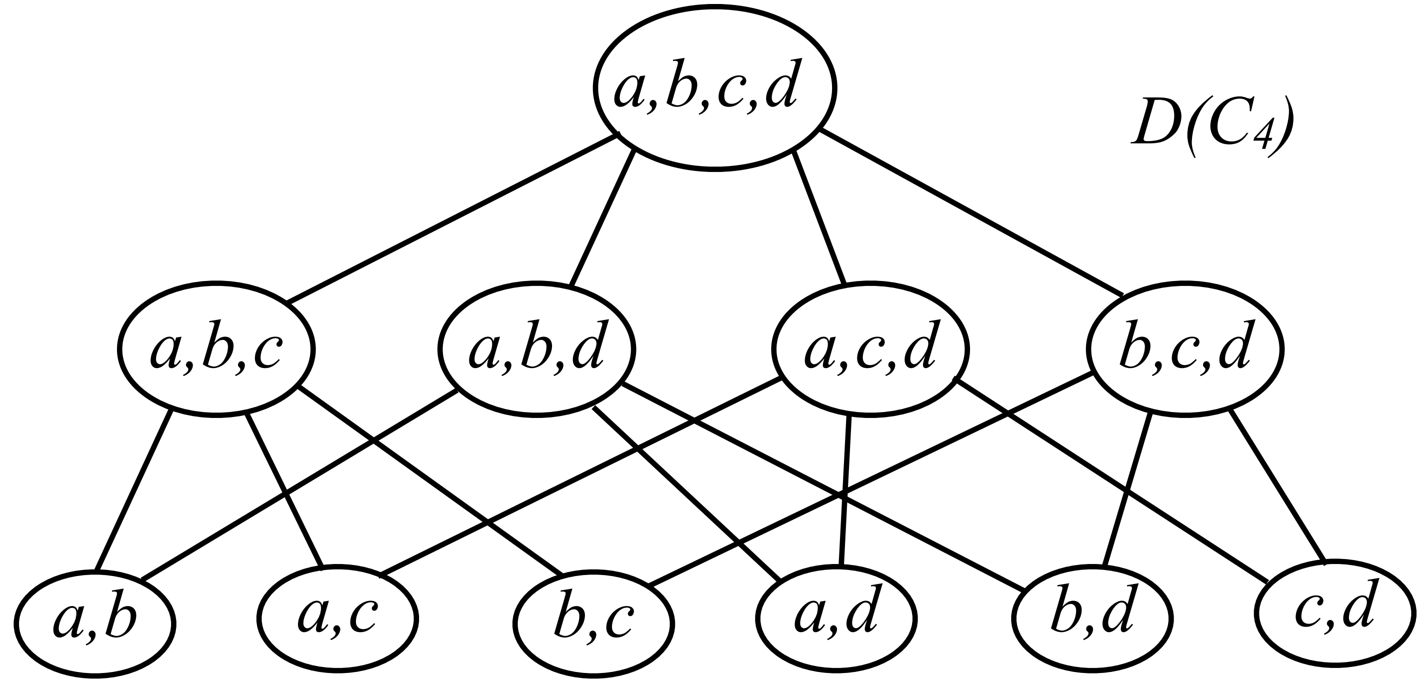


$D(C_4)$

$\mathcal{D}(K_n) \cong Q_n$ minus the "bottom element"

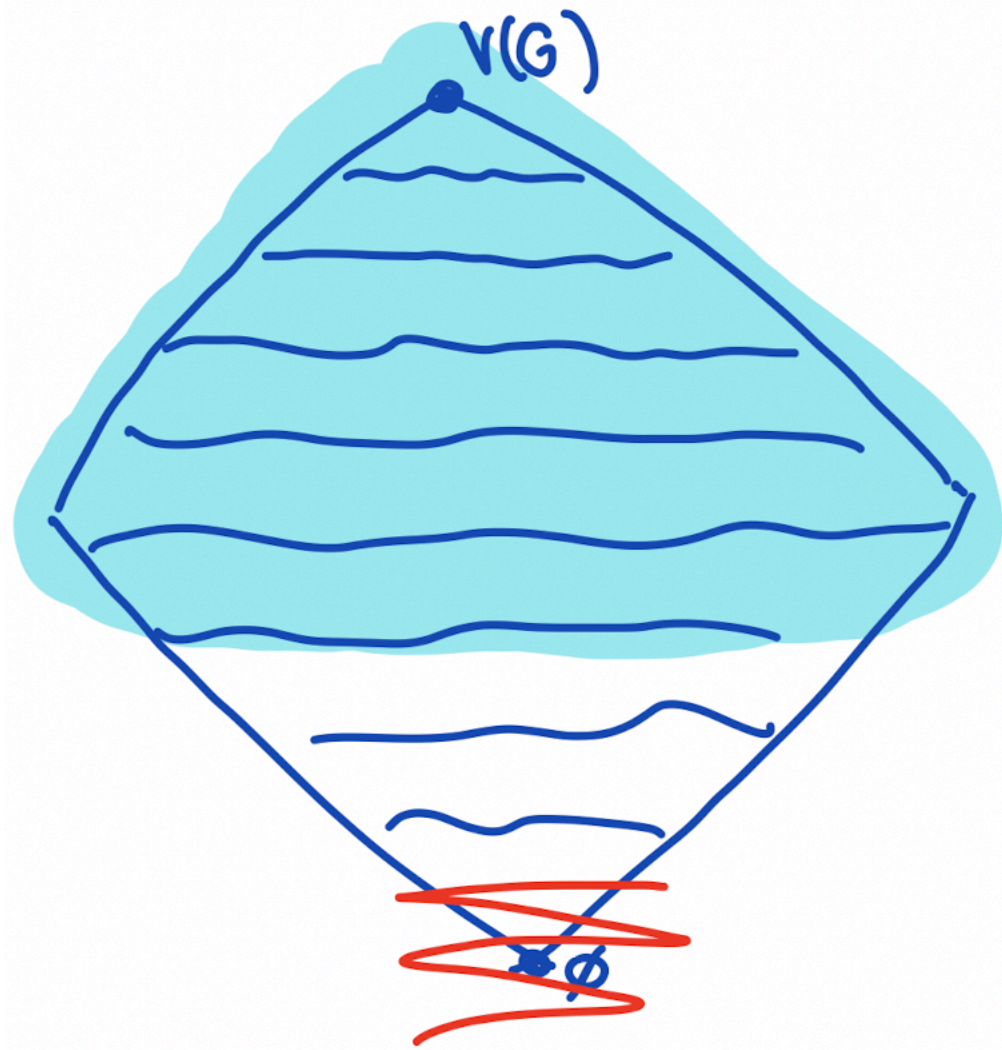
$\mathcal{D}(K_n$ minus a perfect matching$) \cong Q_n$ minus the "bottom two rows"

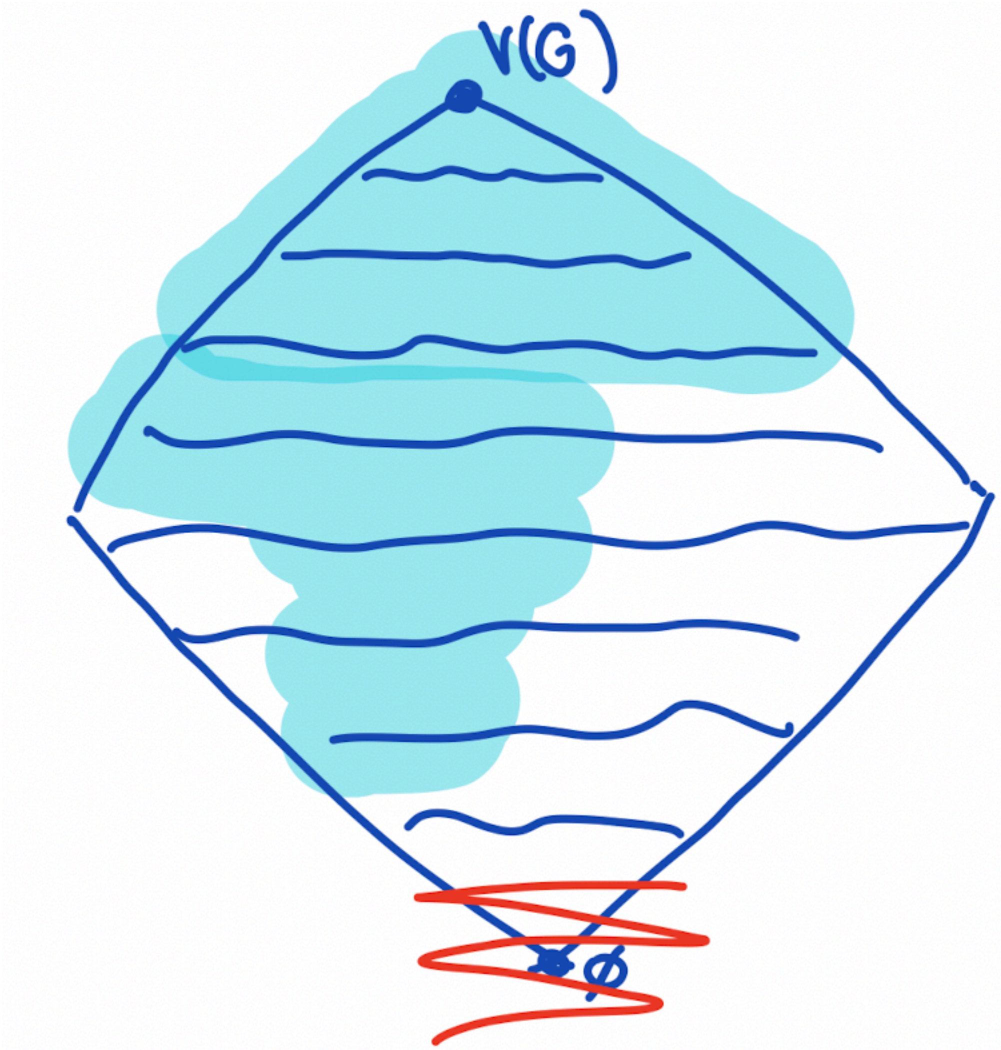
$\Rightarrow n$ is even

$\Rightarrow$ any two vertices dominate the domination reconfiguration graph

Corollary 7. *For a connected graph G, $\mathcal{D}(G)$ is Eulerian if and only if G is an isolated vertex or a complete graph (on $n \geq 4$ vertices) minus a perfect matching.*

[MP 2022+]

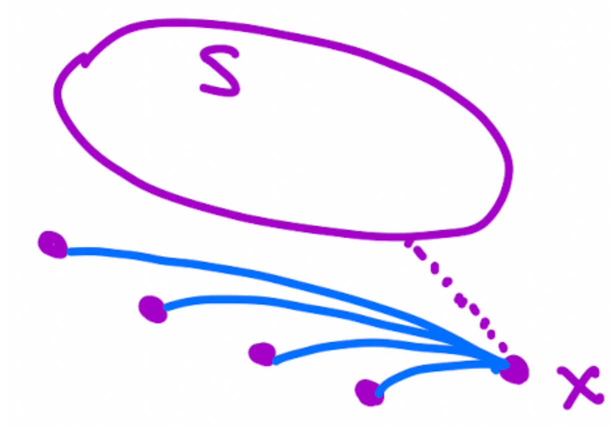


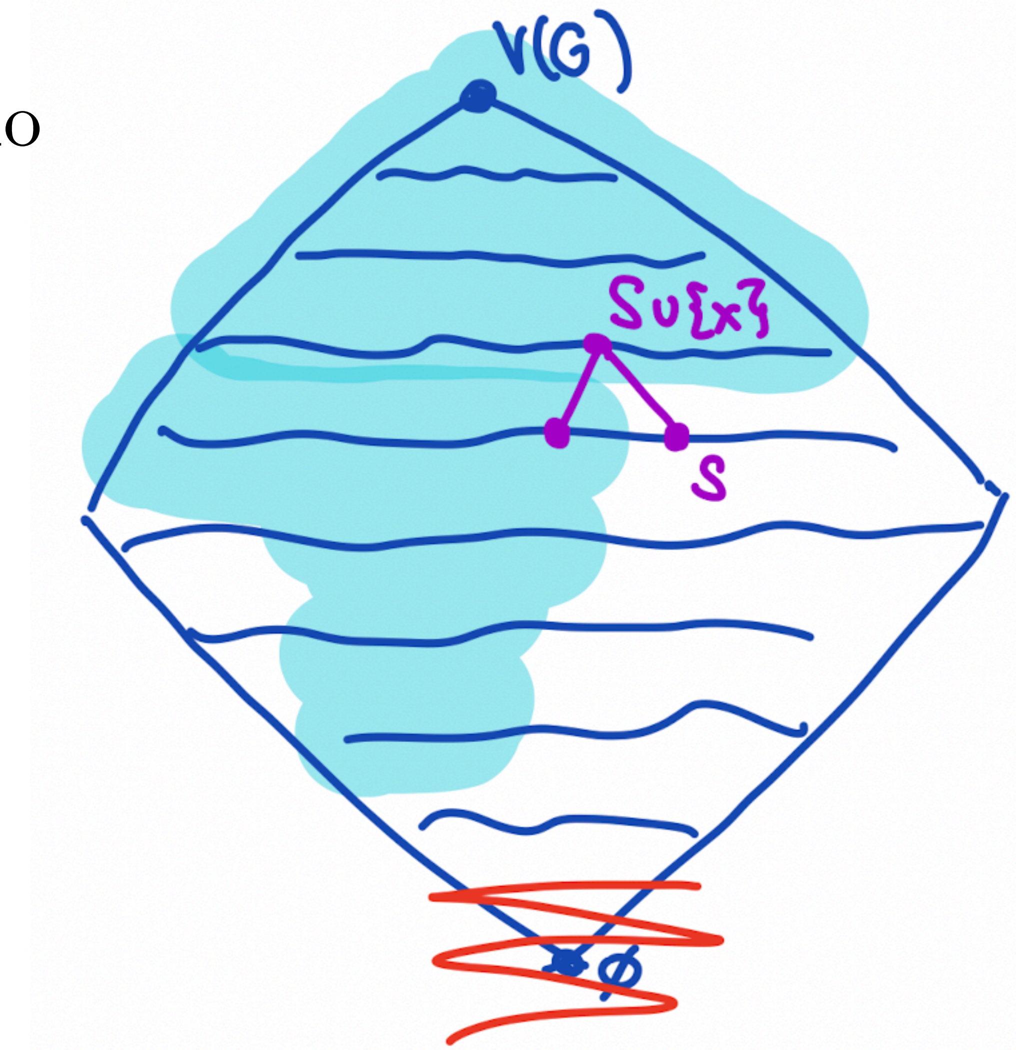


Corollary 7. *For a connected graph G, $\mathcal{D}(G)$ is Eulerian if and only if G is an isolated vertex or a complete graph (on $n \geq 4$ vertices) minus a perfect matching.*

[MP 2022+]

Let S be a subset of k vertices that do not dominate G and suppose every subset of $k+1$ vertices dominates G.





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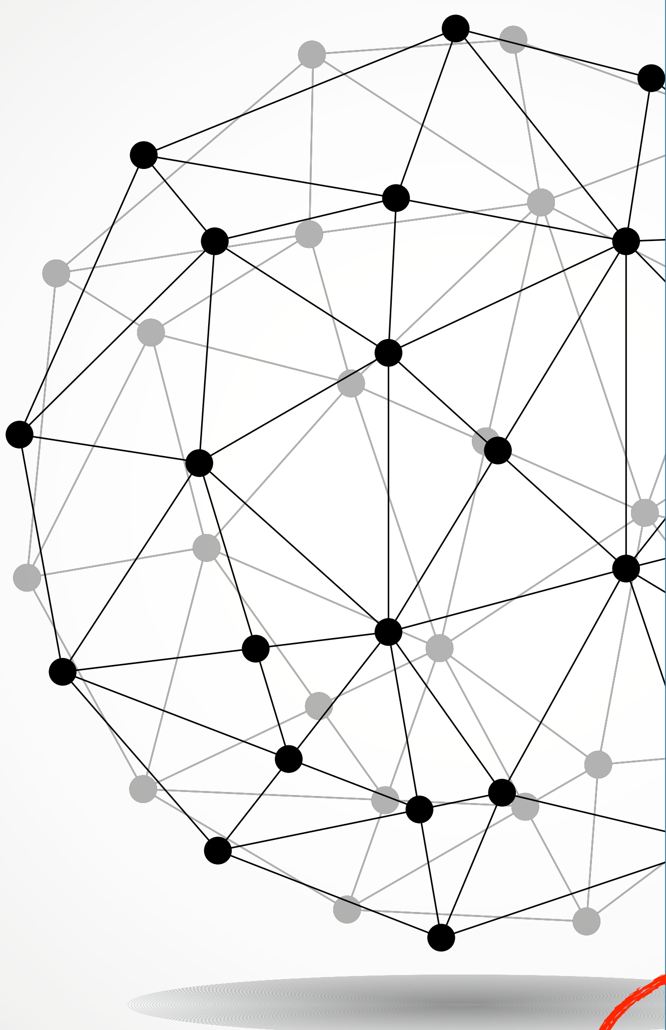

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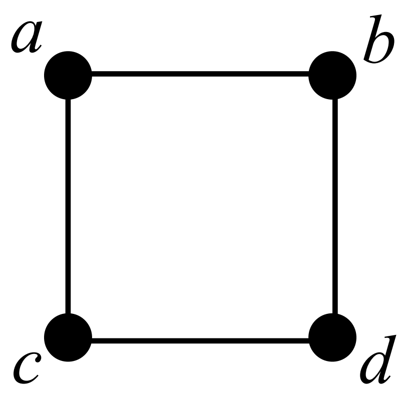
[ABCHMSS 2021] K. Adaricheva, C. Bozeman, N.E. Clarke, R. Haas, M.E. Messinger, K. Seyffarth and H. Smith, Reconfiguration graphs for dominating sets, Chapter 6 in *Research Trends in Graph Theory and Applications*, D. Ferrero, L. Hogben, S.R. Kingan and G.L. Matthews Eds., Springer International Publishing (2021).

[ABCHMSS 2022+] K. Adaricheva, C. Bozeman, N.E. Clarke, R. Haas, M.E. Messinger, K. Seyffarth, H. Smith, Hamilton Paths in Dominating Graphs of Trees and Cycles, submitted.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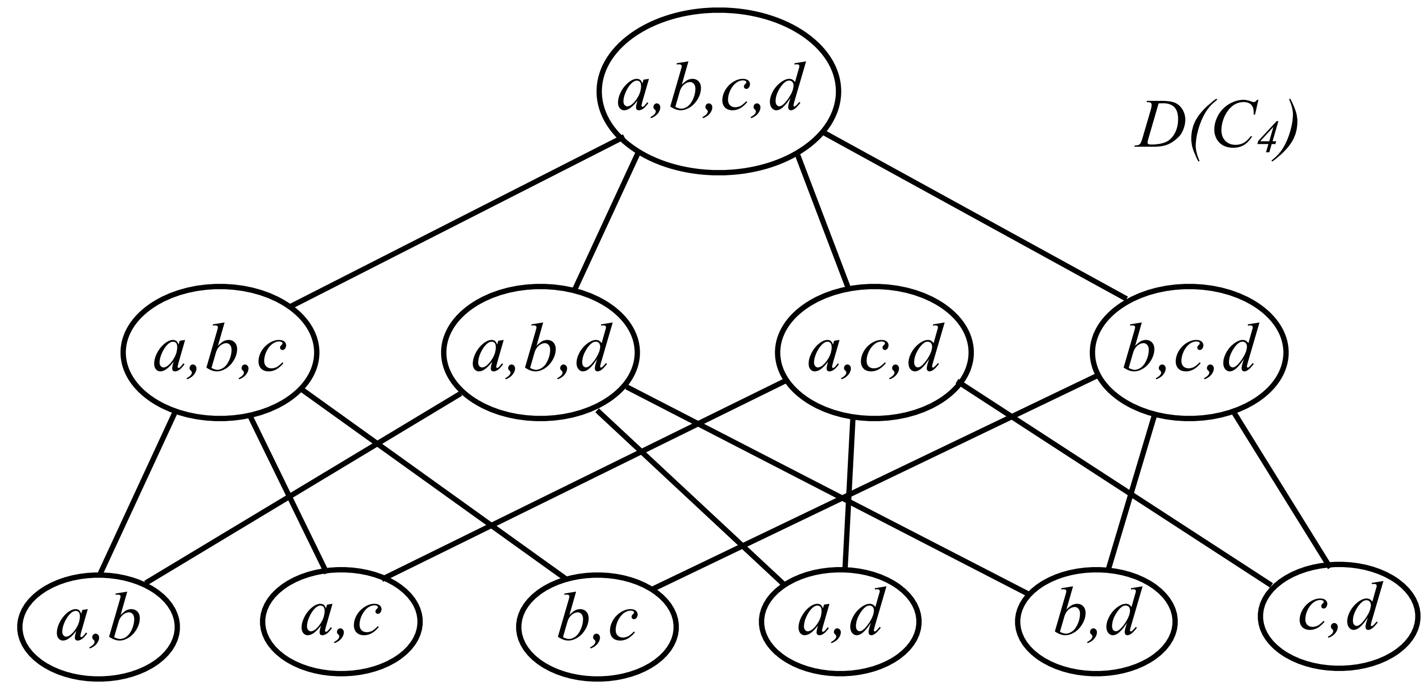
C_4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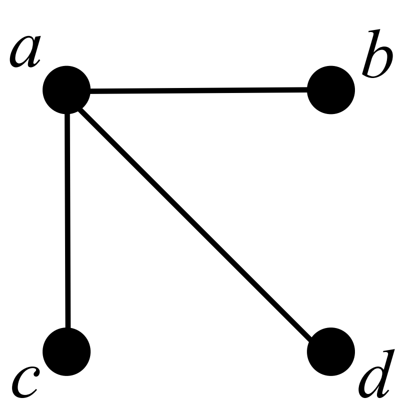
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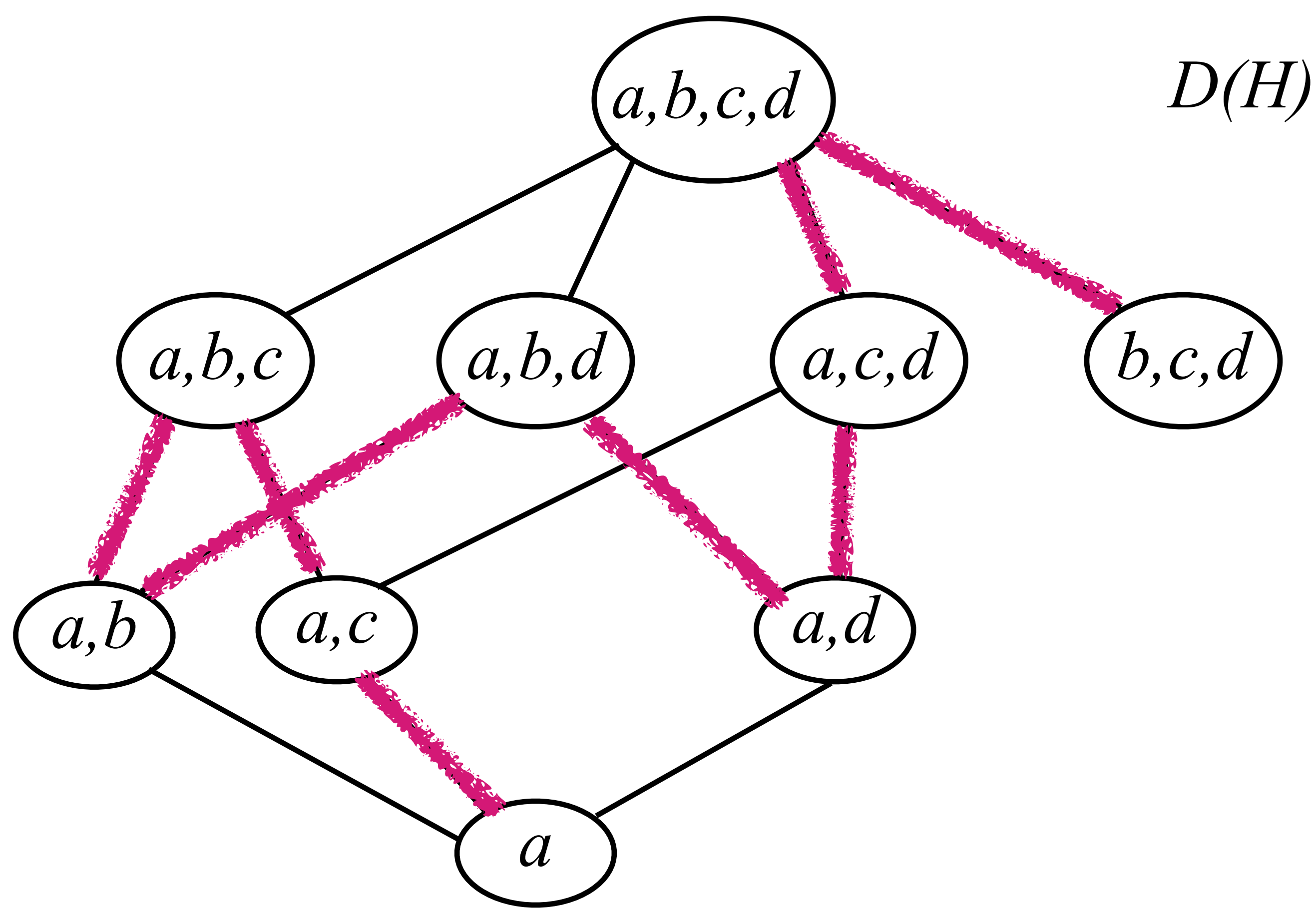
$D(H)$
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a,b,c
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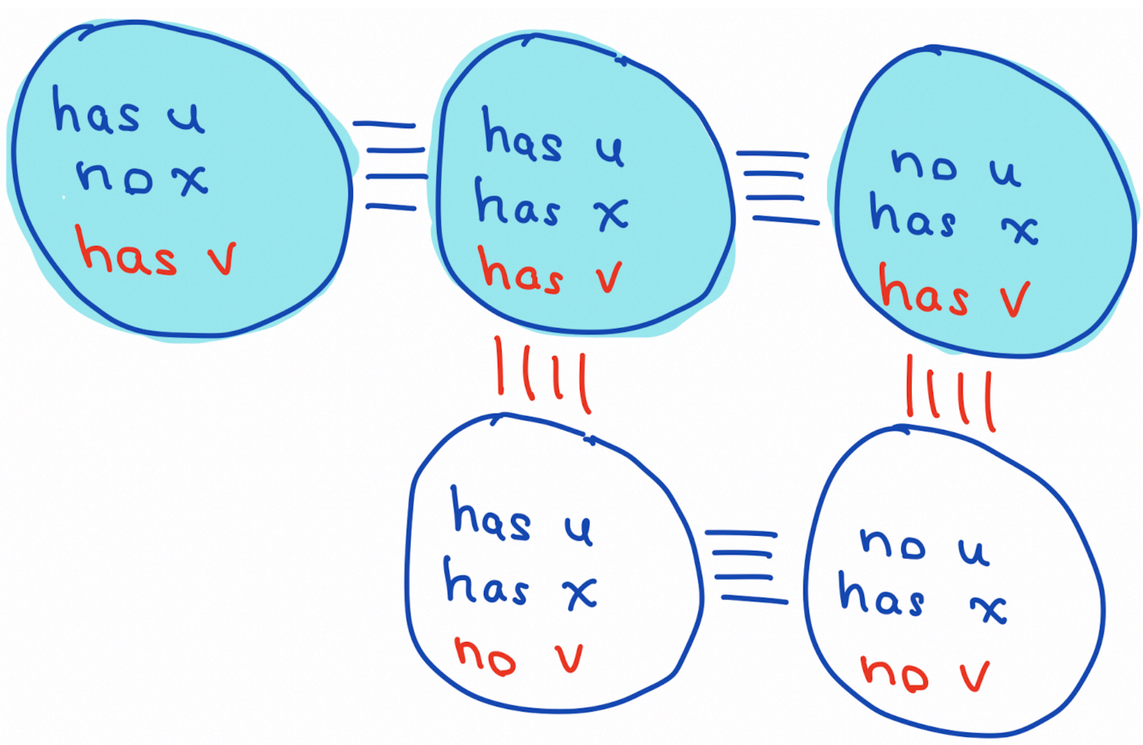
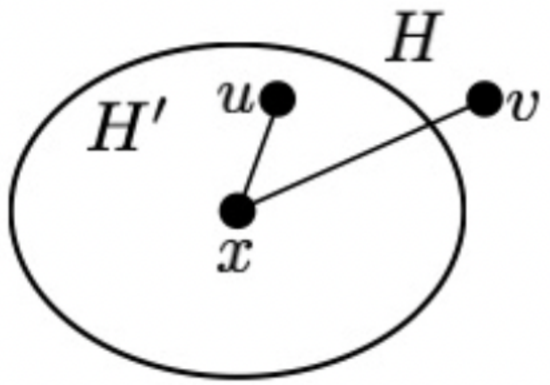
a

Operation I. *Let H be a graph with vertices u, v and x such that $N_H(u) = N_H(v) = \{x\}$. We say that $H' := H - v$ is obtained from H by Operation I.*

Lemma 6. Let H and H' be graphs such that H' is obtained from H by Operation I. If $\mathcal{D}(H')$ has a Hamilton path, then $\mathcal{D}(H)$ has a Hamilton path.

[ABCHMSS 2022+]

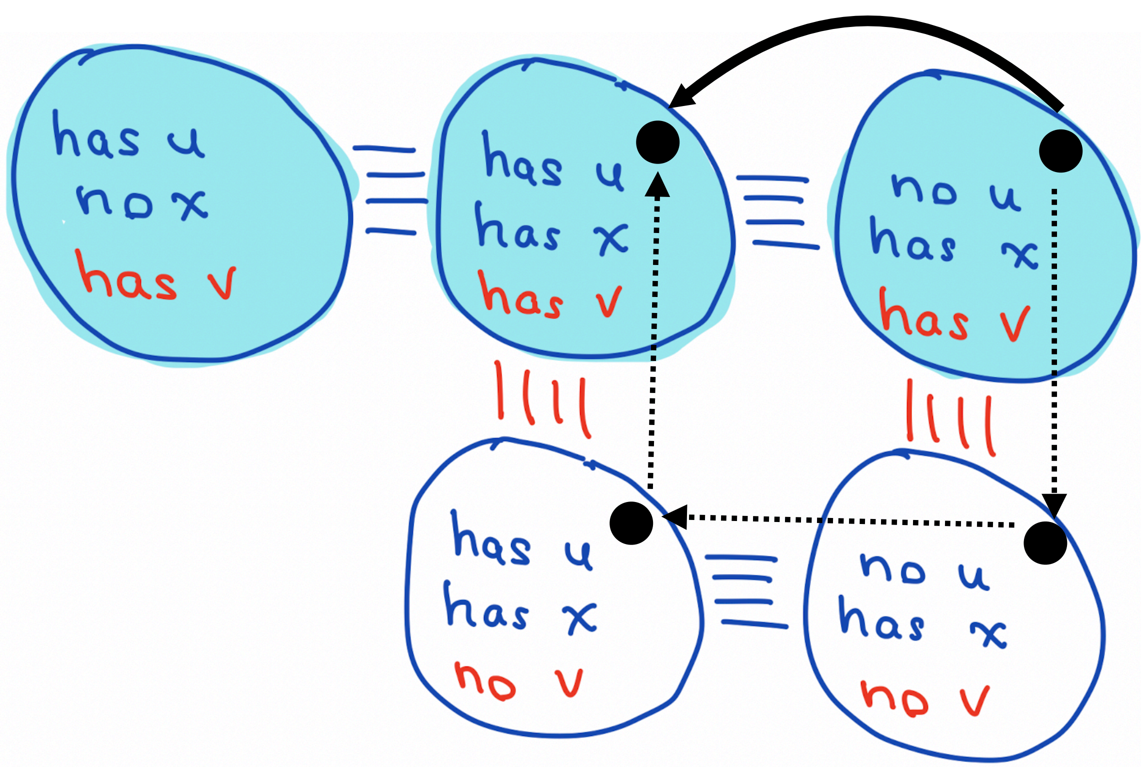


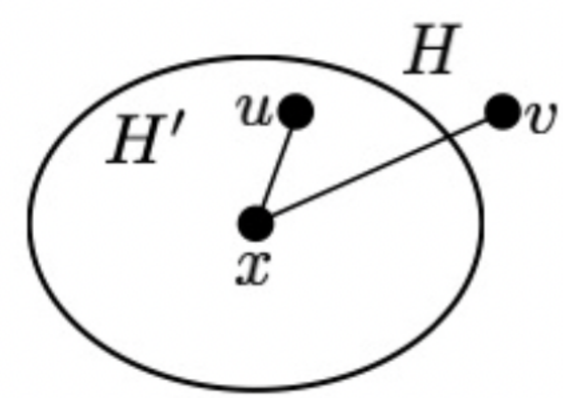


Operation I. *Let H be a graph with vertices u, v and x such that $N_H(u) = N_H(v) = \{x\}$. We say that $H' := H - v$ is obtained from H by Operation I.*

Lemma 6. Let H and H' be graphs such that H' is obtained from H by Operation I. If $\mathcal{D}(H')$ has a Hamilton path, then $\mathcal{D}(H)$ has a Hamilton path.

[ABCHMSS 2022+]

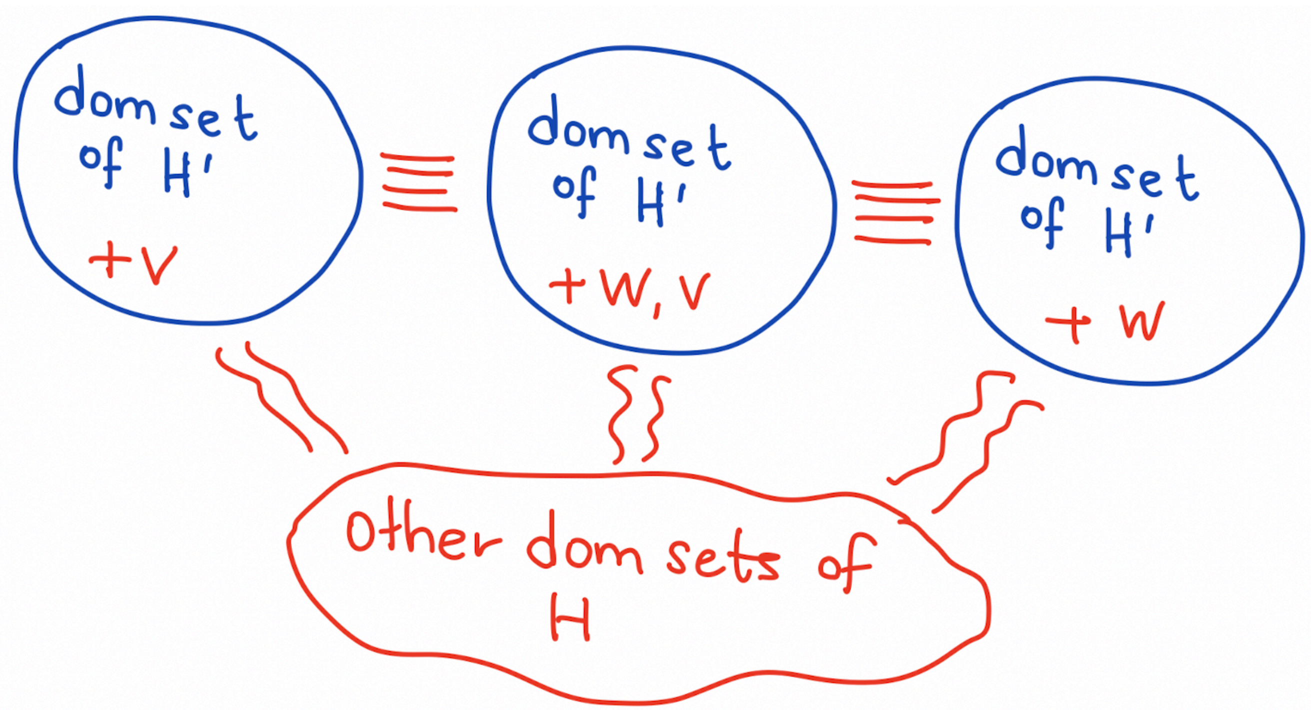


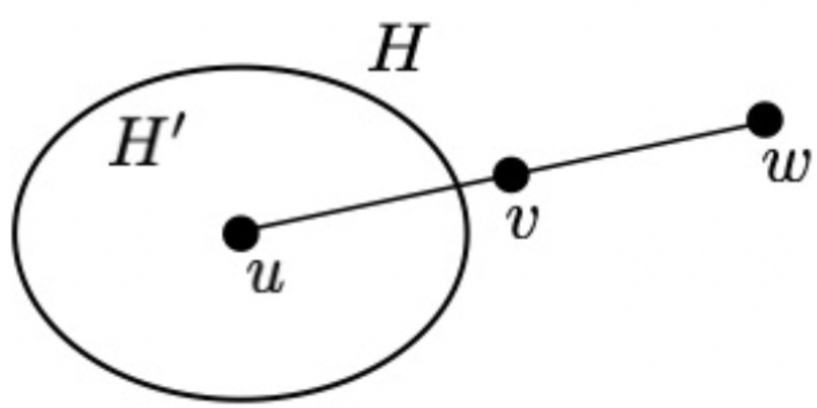


Operation II. *Let H be a graph with vertices u, v and w such that $N_H(v) = \{u, w\}$ and $N_H(w) = \{v\}$. We say that $H' := H - w - v$ is obtained from H by Operation II.*

Lemma 7. Let H and H' be graphs such that H' is obtained from H by Operation II. If $\mathcal{D}(H')$ has a Hamilton path, then $\mathcal{D}(H)$ has a Hamilton path.

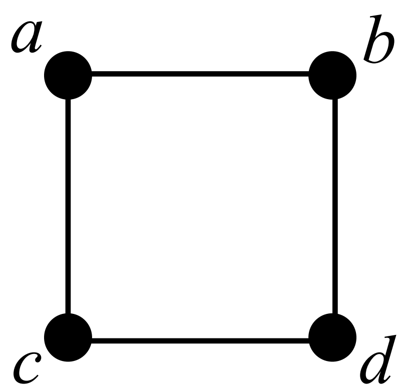
[ABCHMSS 2022+]





Theorem 4. *For any tree T, $\mathcal{D}(T)$ has a Hamilton path.*

[ABCHMSS 2022+]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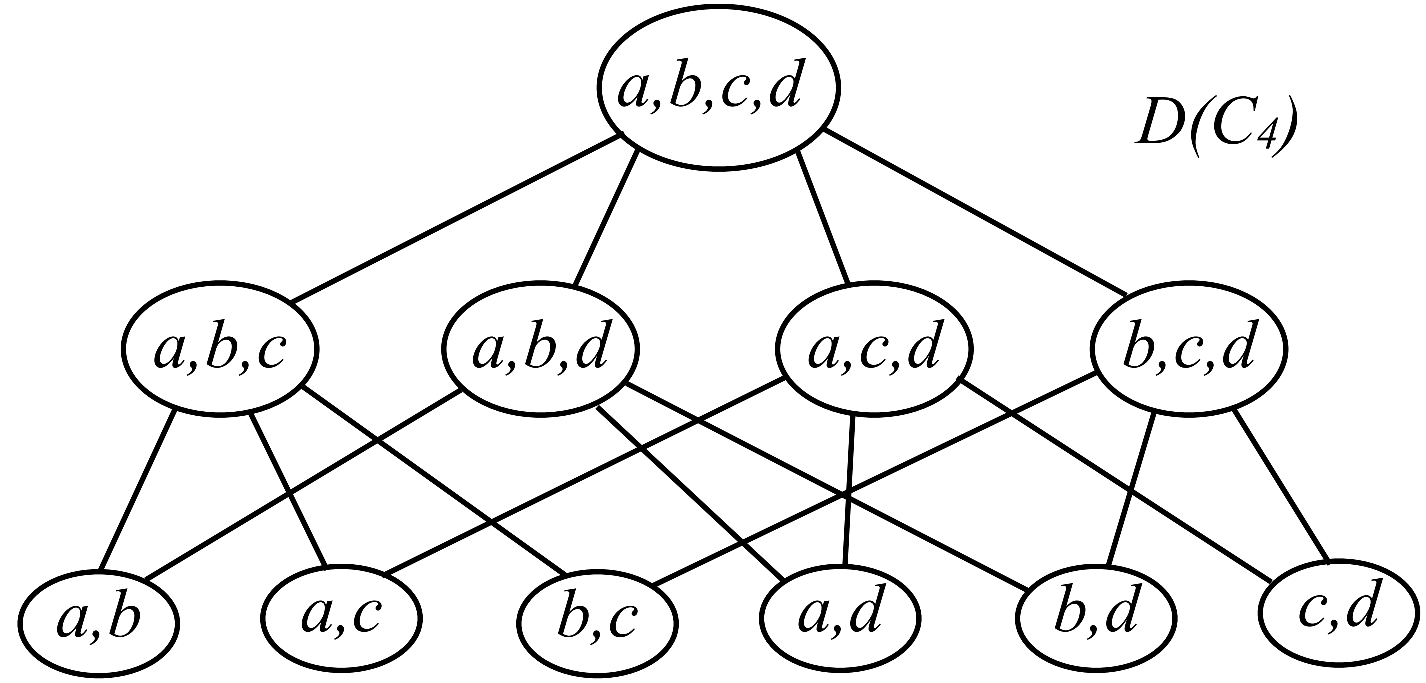
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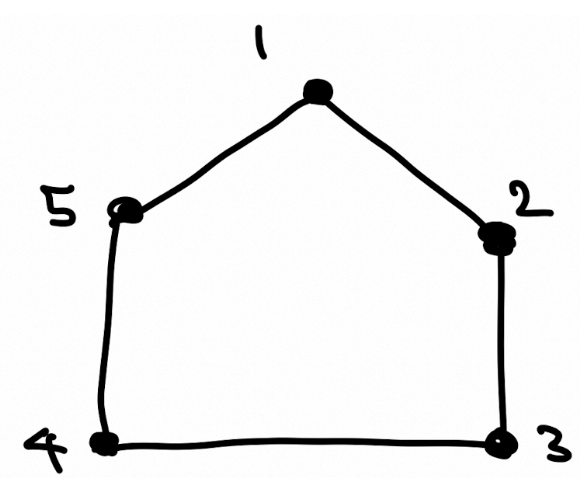
$D(C_4)$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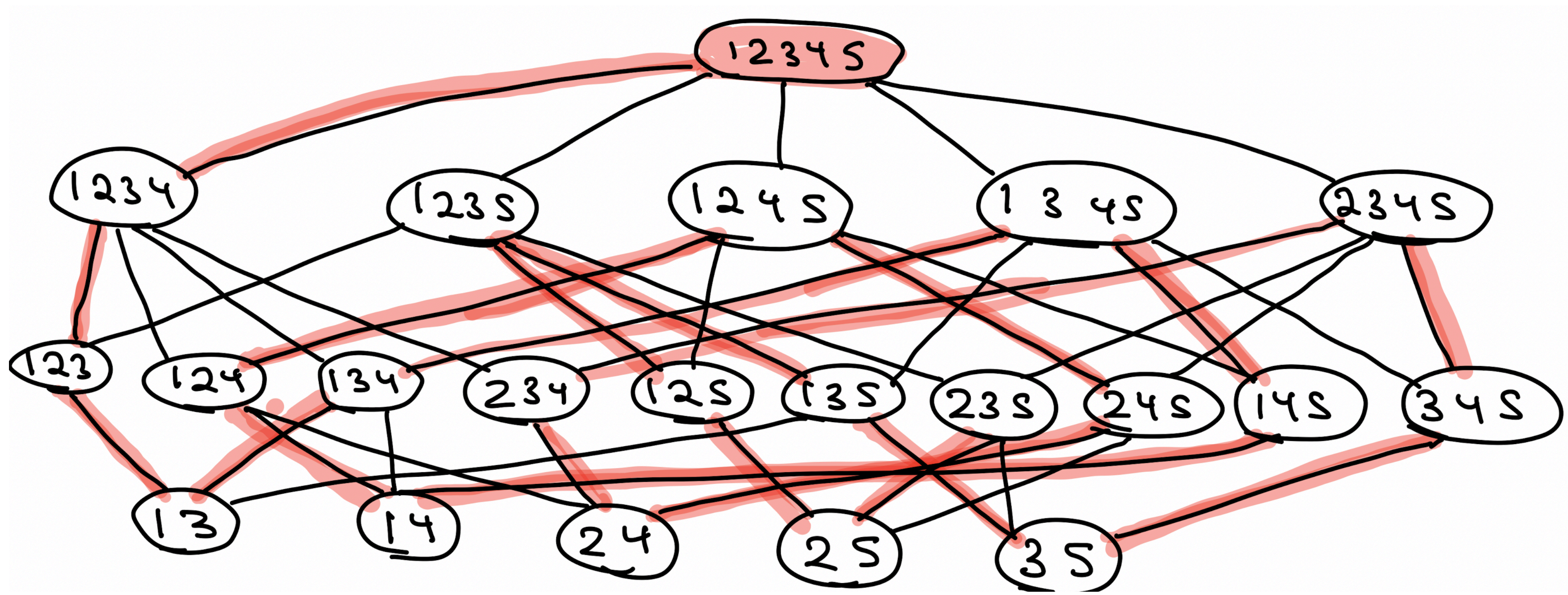
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Theorem 5. *For all integers $n \geq 3$, $\mathcal{D}(C_n)$ has a Hamilton path if and only if $n \not\equiv 0 \pmod 4$.*

[ABCHMSS 2022+]

Thanks!

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