Atlantic Graph Theory
Seminar Series

The Localization Number of a Graph

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(he/him/his)
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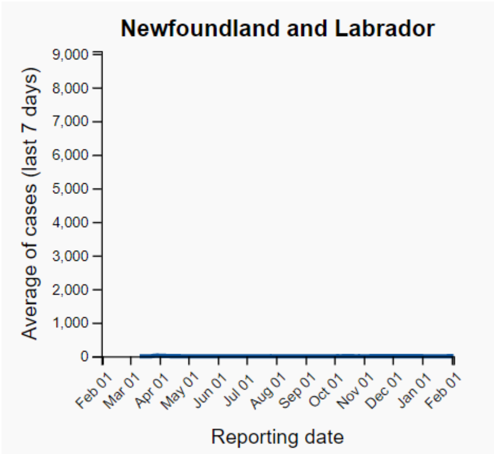
Land acknowledgement

We acknowledge the privilege of working on the traditional territory of the Haudensaunee, Mississauga and Anishnaabeg peoples, and within the lands protected by the Dish With One Spoon Wampum agreement.


Newfoundland and Labrador
Average of cases (last 7 days)
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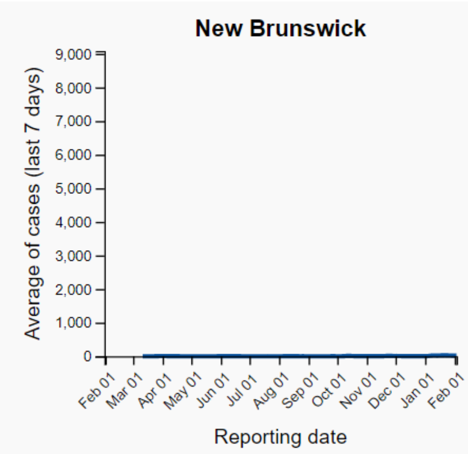
New Brunswick
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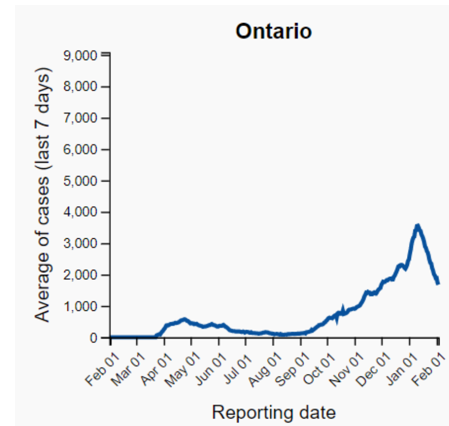
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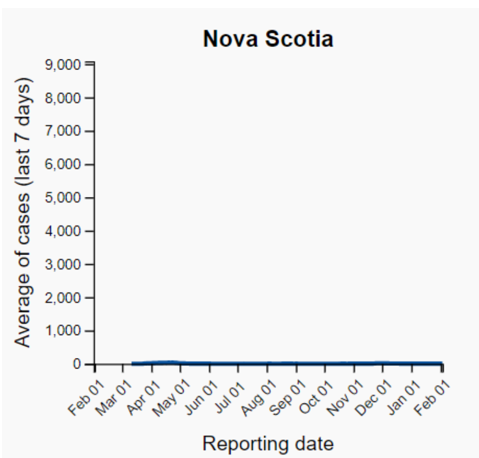
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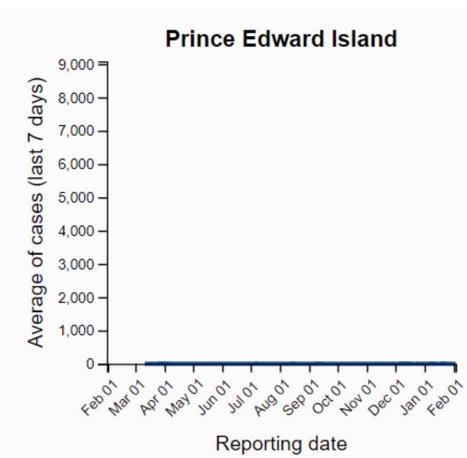
Prince Edward Island
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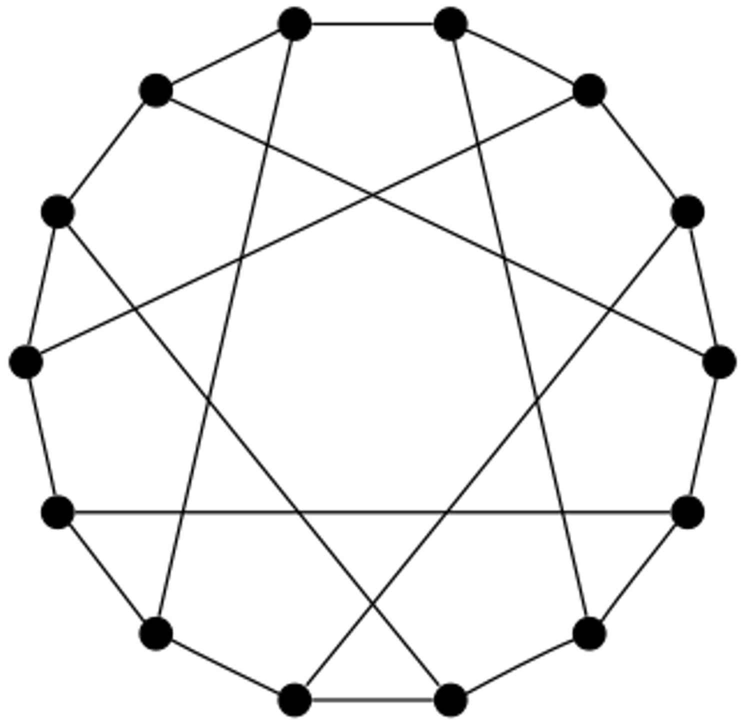

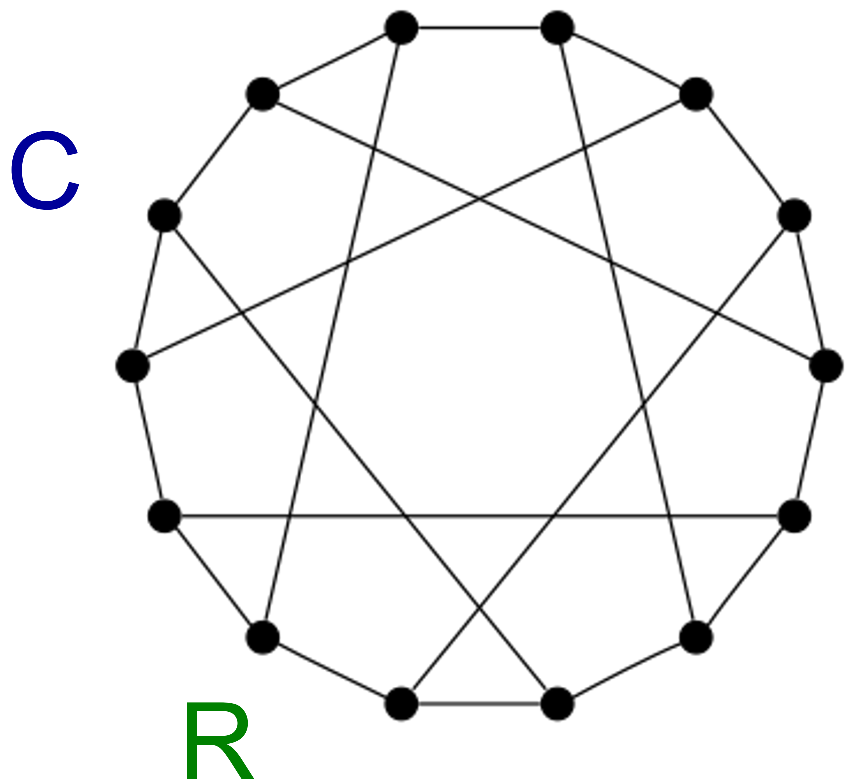
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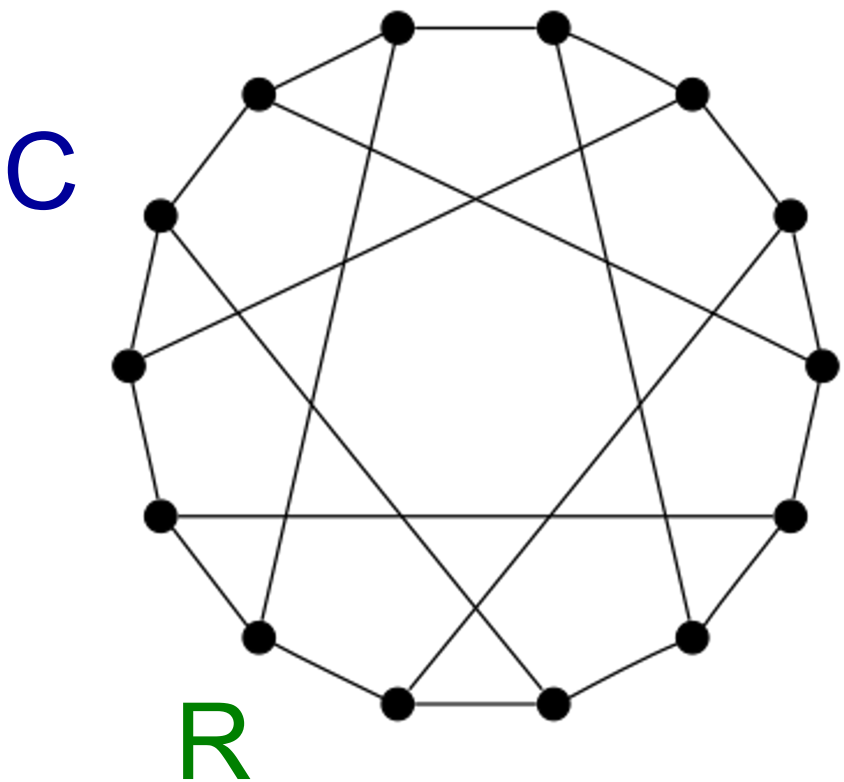
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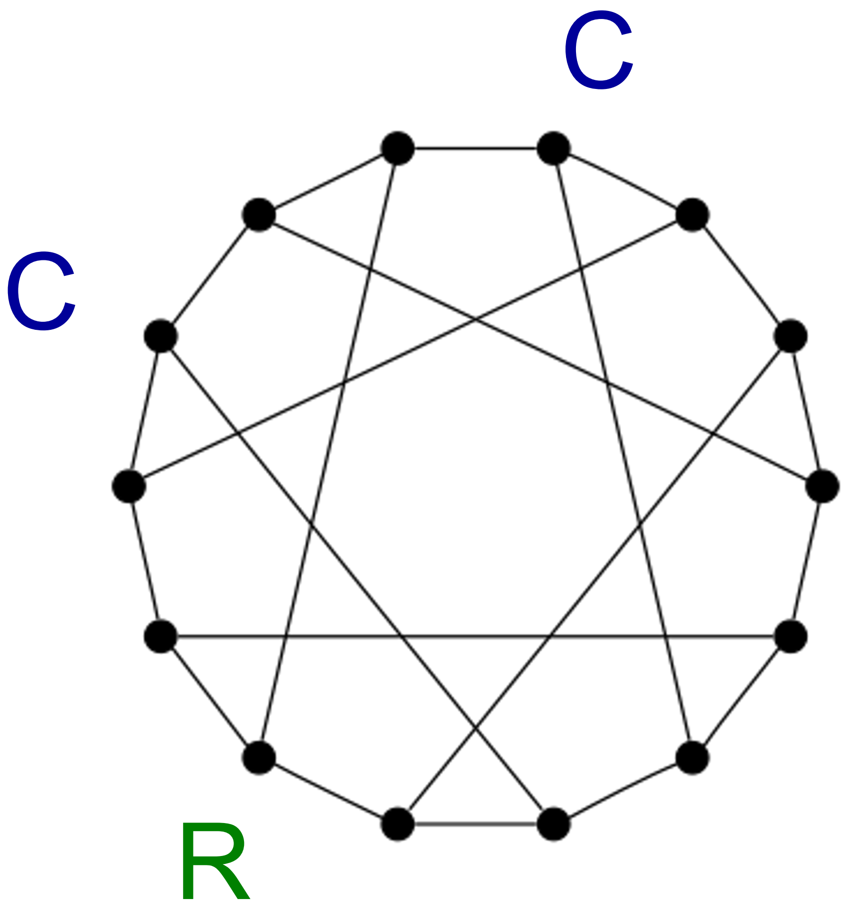
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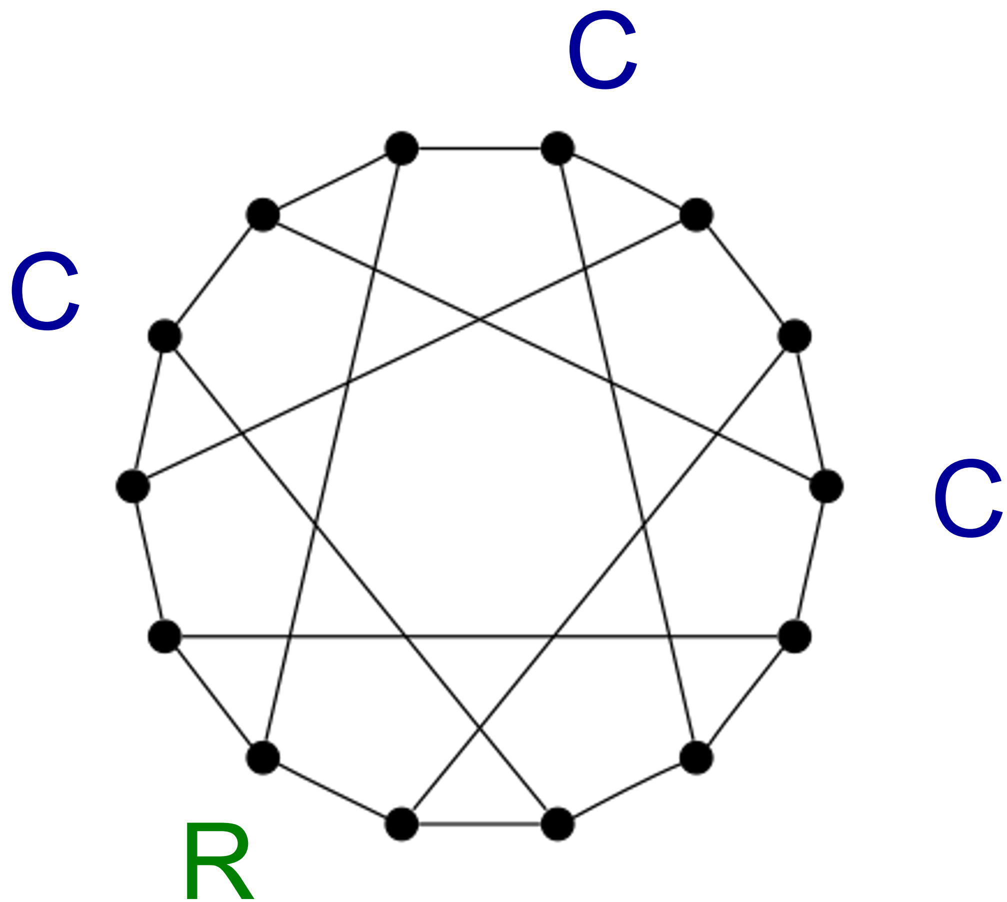
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Game of Cops and Robbers (Nowakowski,Winkler,83), (Quilliot,78)

- two players Cops C and robber R play at alternate time-steps; cops go first
- players move to vertices along edges; may move to neighbors or pass
- cops try to capture the robber by landing on them, while robber tries to evade capture

Cop number (Aigner,Fromme,84)

- minimum number of cops needed to capture the robber is the cop number $c(G)$
  - well-defined as $c(G) \leq |V(G)|$
  - better: $c(G) \leq \gamma(G)$

How big can the cop number be?

Meyniel's Conjecture: If G is connected of order n, then $c(G) = O(n^{1/2})$.

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North-Holland

NOTE

COPS AND ROBBERS IN GRAPHS WITH LARGE GIRTH AND CAYLEY GRAPHS

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It is shown that if a graph has girth at least $8t-3$ and minimum degree greater than d, then more than d^t cops are needed to catch a robber. Some upper bounds, in particular for Cayley graphs of groups, are also obtained.

1. Introduction

In [1] Aigner and Fromme and in [10] Quilliot studied the following game, called cops and robbers. There is a finite, connected, undirected graph $G=(V,E)$, m cops and one robber. First the cops choose one vertex each as initial position. Next the robber makes his choice. Afterwards they move alternately (first the cops, then the robber) along the edges of the graph or stay. Denote by $c(G)$ the minimum value of m for which m cops have a winning strategy, i.e., they have an algorithm to catch

Note that for $t=1$ one obtains the bound of Aigner and Fromme. As to upper bounds, let us mention that Meyniel [8] conjectures $c(G)=O(\sqrt{|V|})$, which would be best possible.

We could only prove:

Proposition 1.2. $c(G)=o(|V|)$.

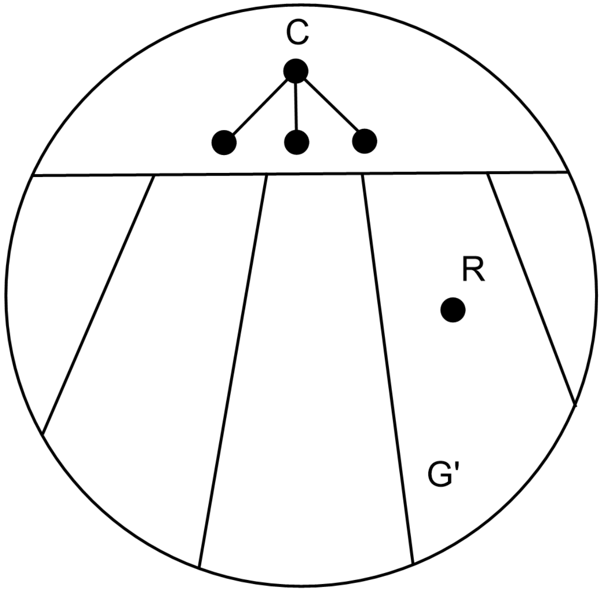
Frankl's bound

Theorem (Frankl,87) If G is connected, then

$$c(\mathrm{G}) = O\left(n \frac{\log\log n}{\log n}\right).$$

Sketch of Frankl's proof

- *Moore bound*: $n \le 1 + \Delta \sum_{i=0}^{D-1} (\Delta - 1)^i = O(\Delta^D)$
- there is either an isometric path or closed neighbor set of order
$$\frac{\log n}{\log \log n}$$
- either subgraph can be 1-guarded (via a *retraction*)
  - guarding the subgraph costs one cop
- Induction. ■



State-of-the-art

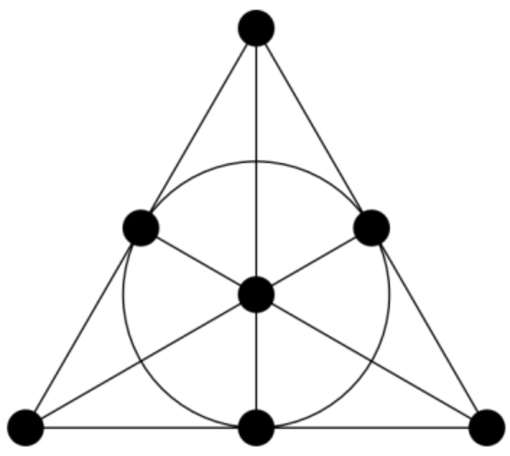
- (Lu,Peng,13), (Scott,Sudakov,11), (Frieze,Krivelevich,Loh,12) proved that

$$c(G) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right) = n^{1-o(1)}$$

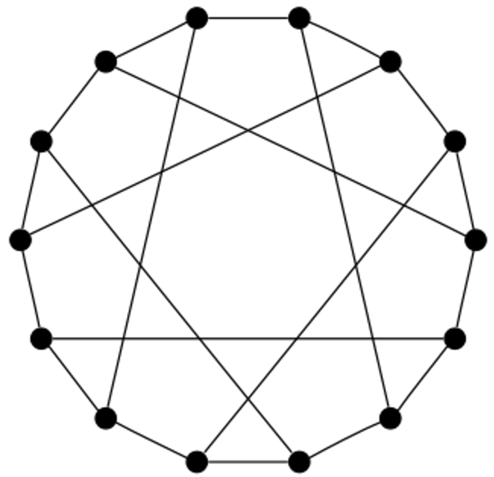
Soft Meyniel's conjecture: for some $\varepsilon > 0$,

$$c(G) = O(n^{1-\varepsilon}).$$

Incidence graphs of projective planes



Fano plane



Heawood graph

- properties: bipartite, order/size $2(q^2+q+1)$, $(q+1)$-regular, diameter 3, girth 6, vertices have at most one common neighbor

Meyniel extremal families

- a family of connected graphs $(G_n: n \geq 1)$ is Meyniel extremal if there is a constant $d > 0$, such that for all $n \geq 1$, $c(G_n) \geq dn^{1/2}$

- incidence graphs of projective planes:
  - order $2(q^2+q+1)$, cop number $q+1$
  - Meyniel extremal (fill in non-prime orders)

Complexity

- (Berrarducci,Intrigila, 93), (Hahn,MacGillivray, 06), (B,Chiniforooshan, 09), (B,MacGillivray,17)

  "$c(G) \leq k$?" k fixed: in **P**; running time $O(n^{2k+1})$, $n = |V(G)|$

- (Fomin,Golovach,Kratochvíl,Nisse,Suchan,08):

  if k not fixed, then computing the cop number is **NP**-hard

EXPTIME-Completeness

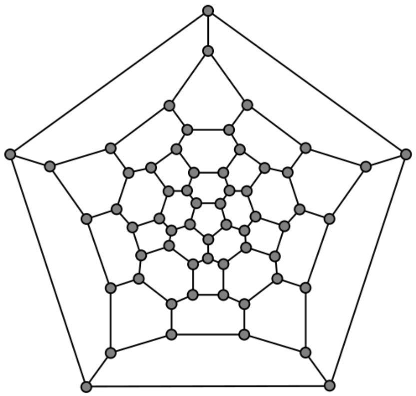
Goldstein, Reingold Conjecture: if k is not fixed, then computing the cop number is **EXPTIME**-complete.

- same complexity as say, generalized chess

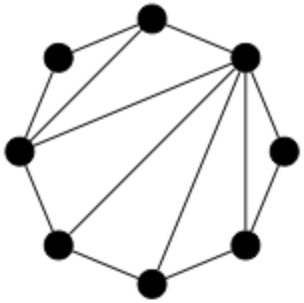
- settled by (Kinnersley,15)

Genus

- (Aigner, Fromme, 84) planar graphs (genus 0) have cop number ≤ 3.



- (Clarke, 02) outerplanar graphs have cop number ≤ 2.



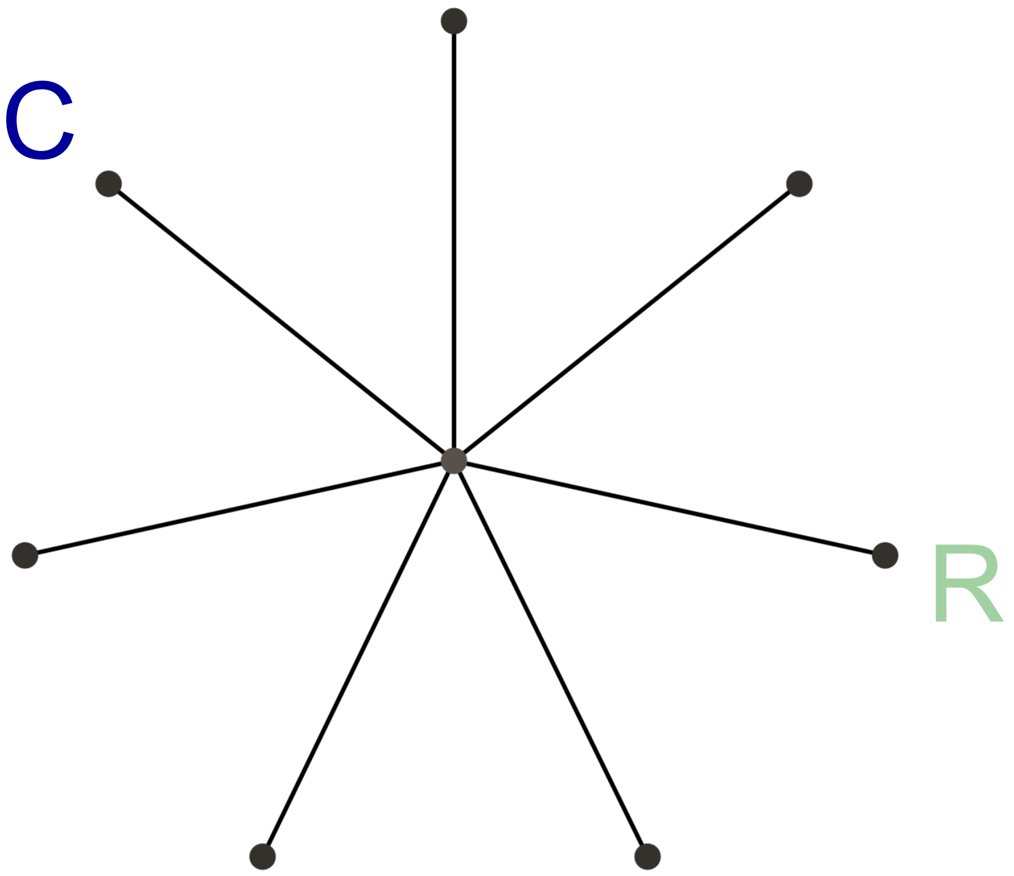
Higher genus

Schroeder's Conjecture: If G has genus k, then

$$c(G) \leq k + 3.$$

- true for $k = 0,1$
- (Schroeder,01): $c(G) \leq \lfloor 3k/2 \rfloor + 3$.
- (Bowler,Erde,Lehner,Pitz,21+): $c(G) \leq \lfloor 4k/3 \rfloor + 10/3$
- (Lehner,21+): If G has genus 1, then $c(G) \leq 3$

- let’s change the rules…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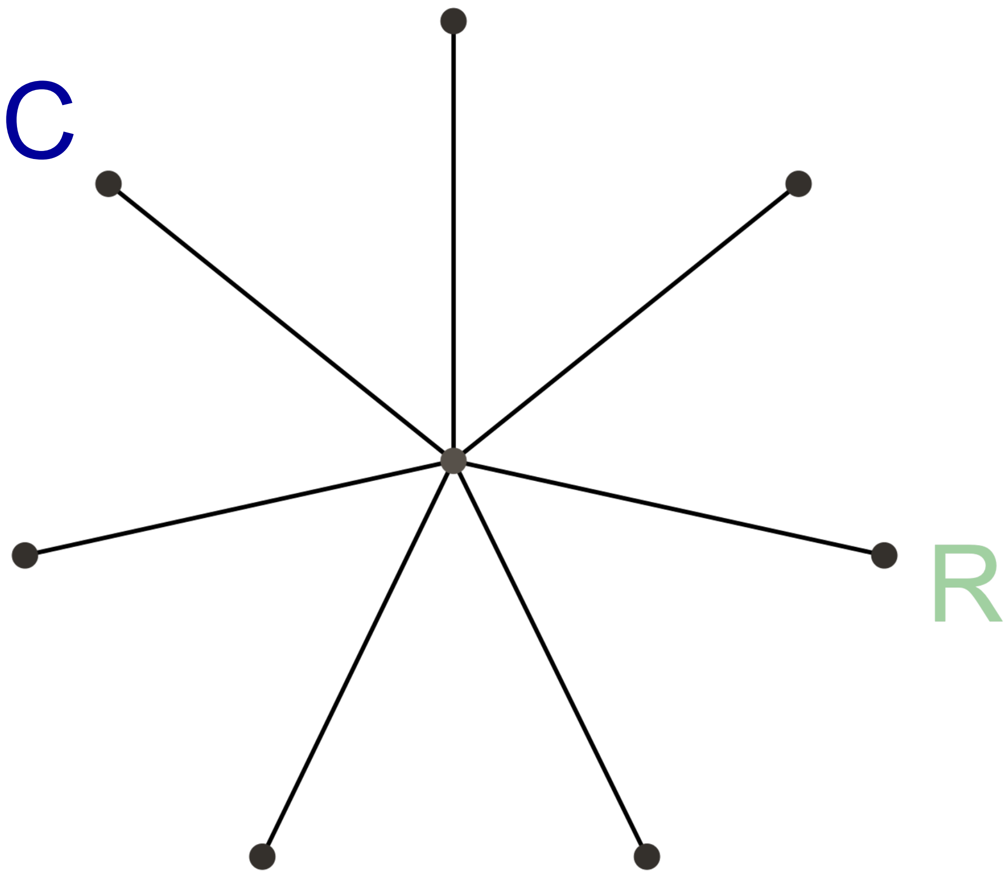
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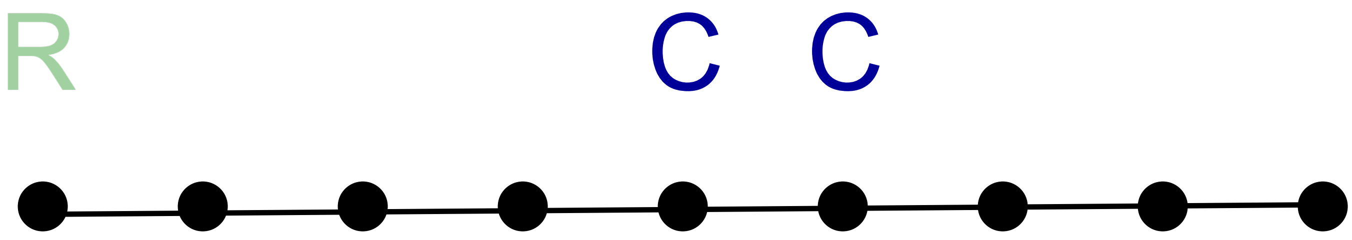
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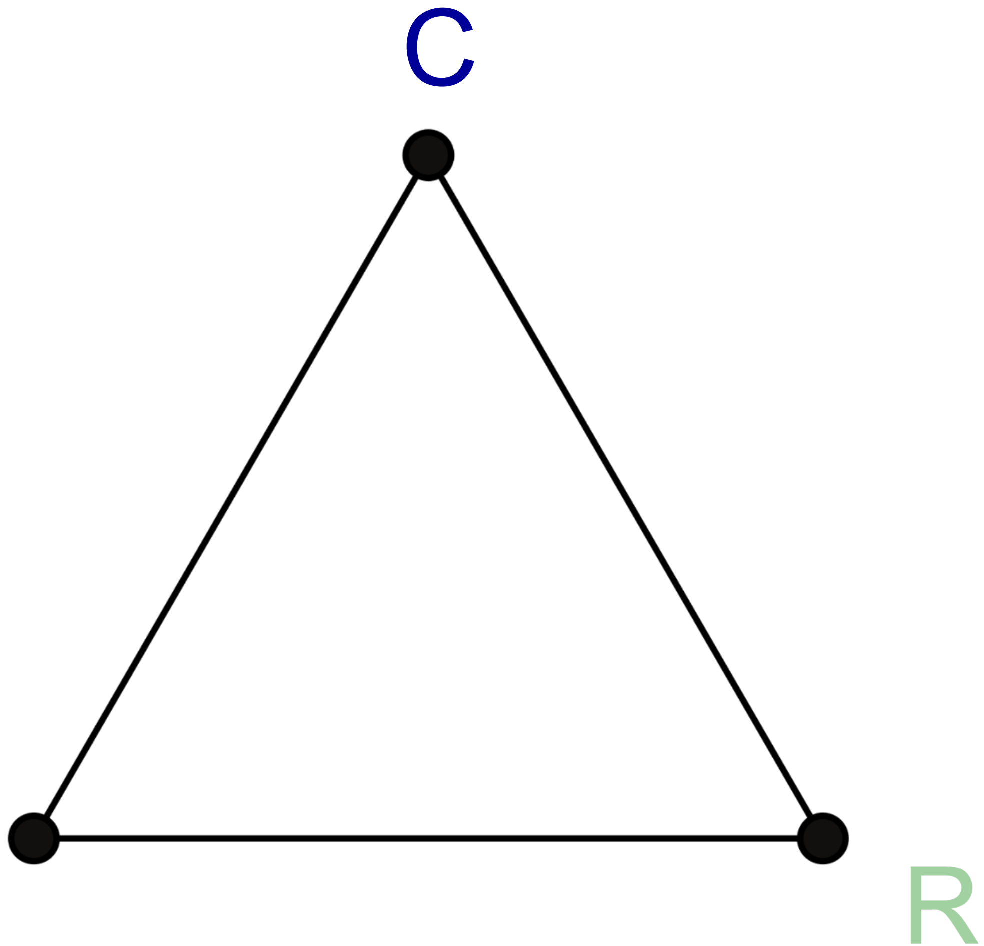
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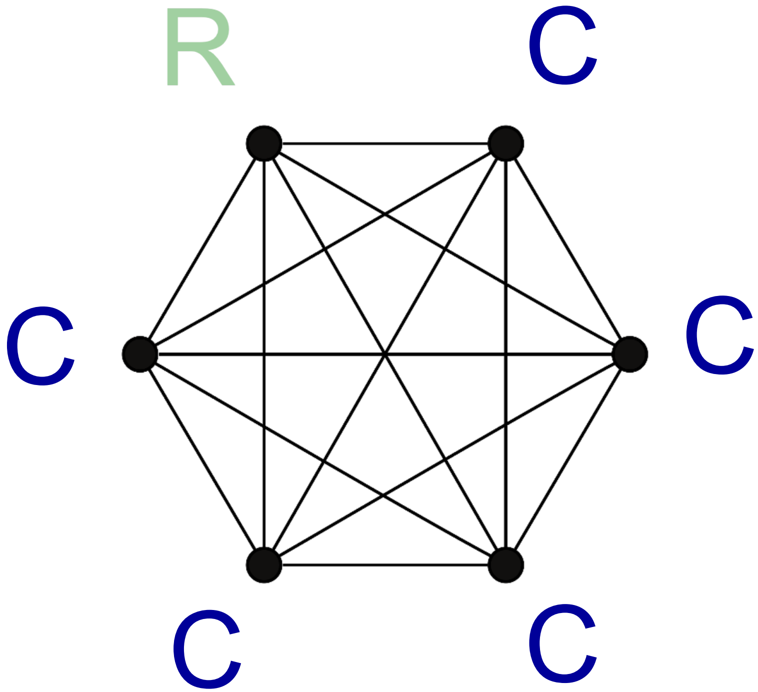
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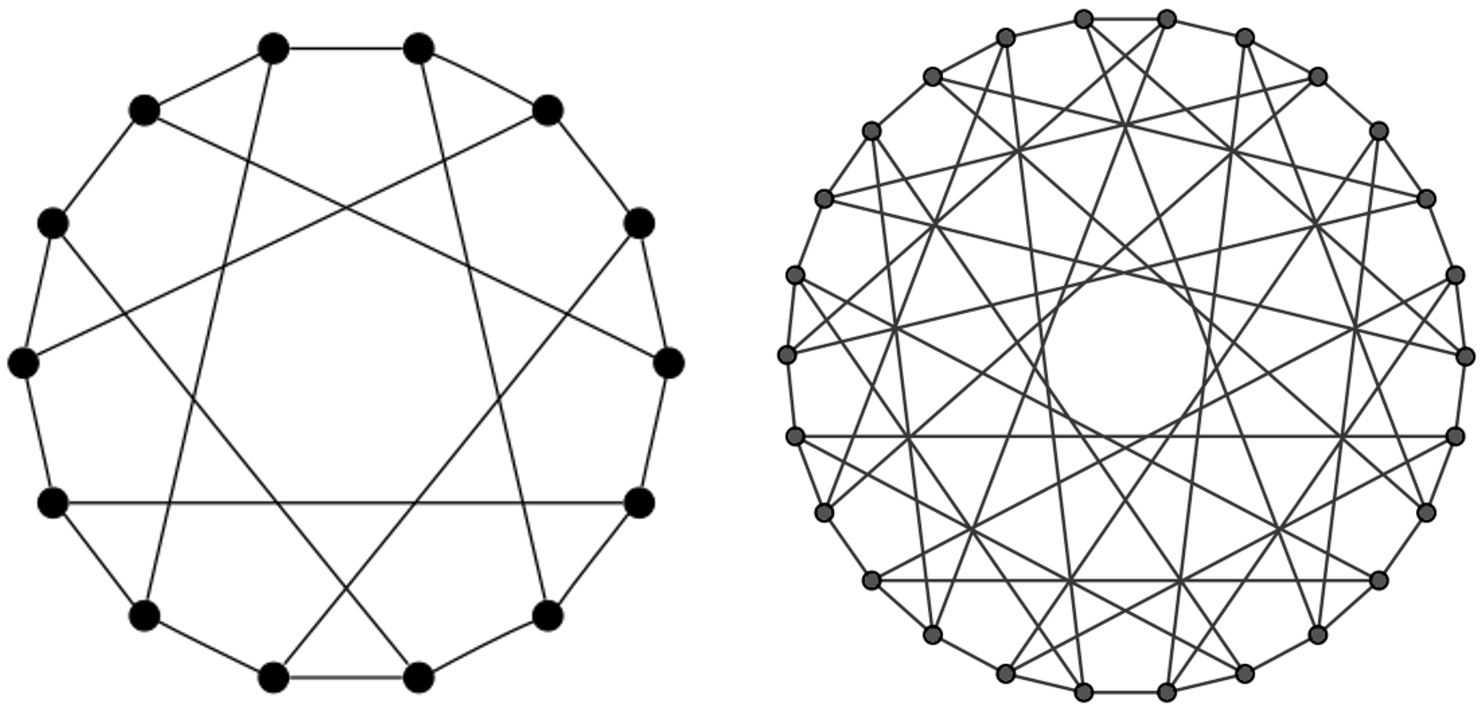
How many cops are needed to win?



The localization game

- *robber*:
  - invisible, moves first
  - move to neighbors or pass

- *cop*:
  - move to <u>any</u> vertex
  - alternate moves with the robber

Localization game

- cops send out distance probes, giving the distance to the robber
- the cops capture the robber if they know which vertex they occupy
- robber is *omniscient*: knows all future moves of the cops
  - avoids a random capture by the cops

Localization number

$\zeta(G)$ = minimum number of cops needed to capture the robber

- well-defined as $\zeta(G) \leq |V(G)| - 1$

Metric dimension

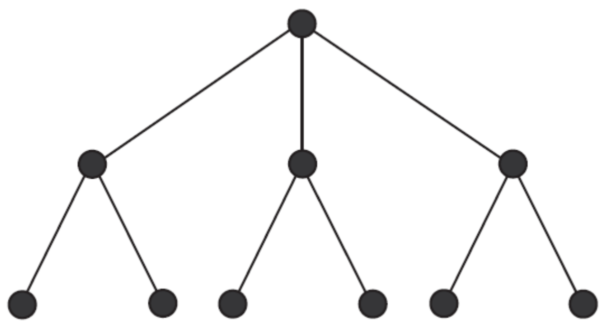
- (Slater,75), (Harary,Melter,76)
  the metric dimension $\beta(G)$ is the minimum number of cops needed to capture the robber in *one* round
- $\zeta(G) \leq \beta(G)$
  - analogy with $c(G) \leq \gamma(G)$

Origins

- (Seager,12) introduced localization game with one cop
  - *no-backtrack rule*: R cannot visit a vertex occupied by a cop in previous round

- (Carraher,Choi,Delcourt,Erickson,West,12)
  - present version with one cop

- (Seager,14) trees satisfy $\zeta = 2$ exactly when they contain:



Knowns

- (Haslegrave,Johnson,Koch,17)

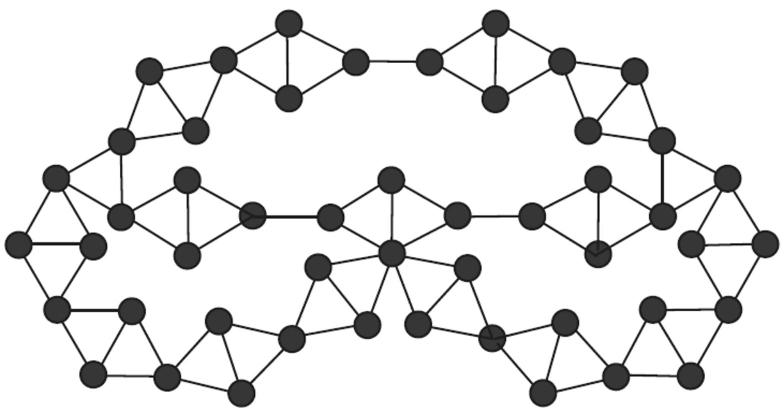
$$\zeta(G) \leq \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor + 1$$

- (Bosek, et al,18) x 2
  - $\zeta(G)$ unbounded on planar graphs
  - $\zeta(G) \leq pw(G)$
  - computing $\zeta(G) \leq k$ is **NP**-complete
  - $\zeta(G) \leq 3$ for outerplanar graphs

- the localization number was studied for binomial random graphs

Localization and coloring

- (Johnson, Koch,17): assuming no backtracking, if one cop wins, then $\chi(G) \leq 4$
- graph showing this bound is tight:



Chromatic number

Conjecture: (Bosek et al,18)

There is an integer-valued function f such that $\zeta(G) \leq k$ implies that $\chi(G) \leq f(k)$.

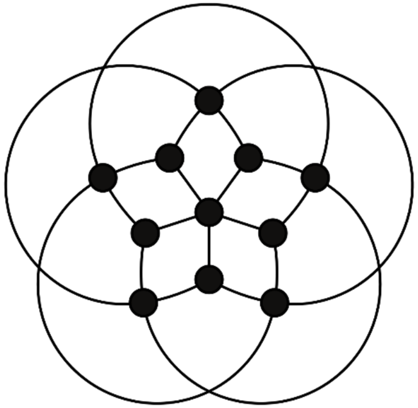
Conjecture solved

Theorem (B,Kinnersley,20)

For a graph G,

$$\chi(G) \leq 3^{\zeta(G)}.$$

- for eg, the robber wins against a cop in:



Graph degeneracy

- the degeneracy of a graph G, dg(G)
  - the least k so that V(G) can be linearly ordered so that each vertex is adjacent to at most k vertices that follow it

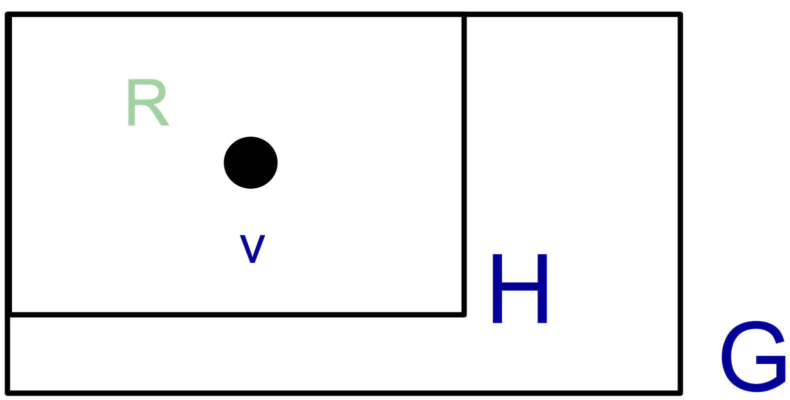
  $\leftrightarrow$ the maximum, over all subgraphs H of G, of δ(H)

- if dg(G) = k, then χ(G) $\leq$ k+1

Proof sketch

- *Claim*: if dg(G) = k, then $\zeta(G) \geq \log_3(k+1)$

- suppose there are m cops, with $m < \log_3(k+1)$
- fix a subgraph H with dg(G) = $\delta(H)$ = k
- keep the robber in H say on v

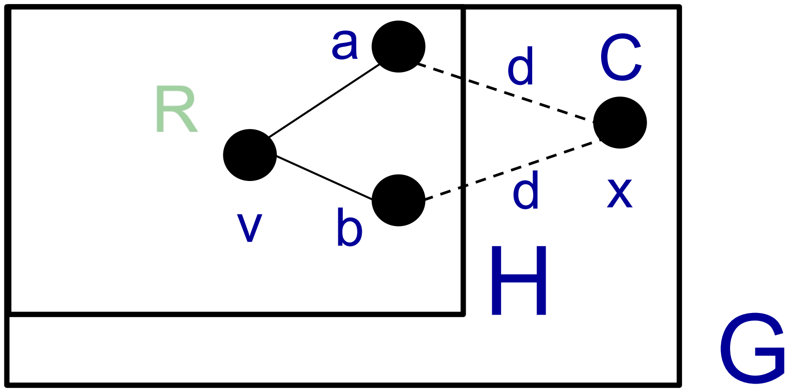


Proof sketch, cont.

- for a cop on x, let $d_x = d_G(x,v)$
- for w in $N_H[v]$,

  $$d_G(x,w) \in \{d_x - 1, d_x, d_x + 1\}$$

- vertices in $N_H[v]$ correspond to at most $3^m < k+1$ distinct distance vectors
- by the *Pigeonhole Principle*, some two vertices in $N_H[v]$, say a and b, share the same distance vector
- then R moves to one of a or b and is safe for another round ∎



Graph families

- (BK,20) if G is outerplanar, then $\zeta(G) \leq 2$.

- (BK,20) for a hypercube Q_n,

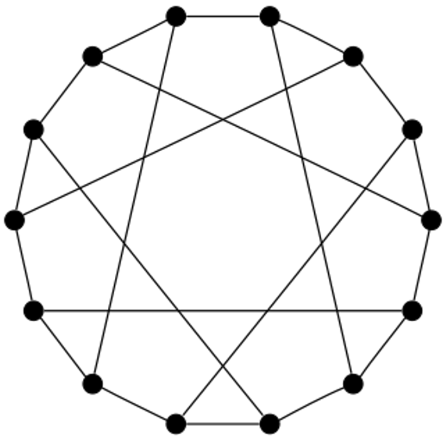
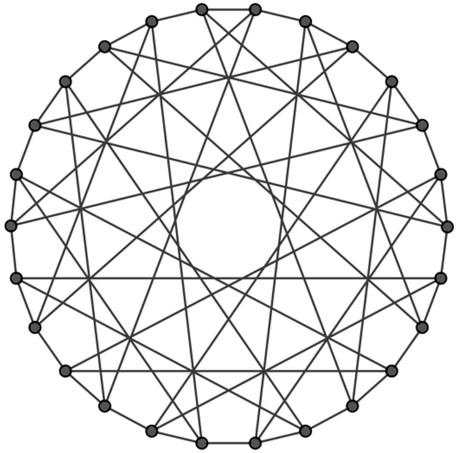
$$\lceil \log_2 n \rceil \leq \zeta(Qn) \leq \lceil \log_2 n \rceil + 2$$

Localizing projective planes

(B,Huggan,Marbach,21+):

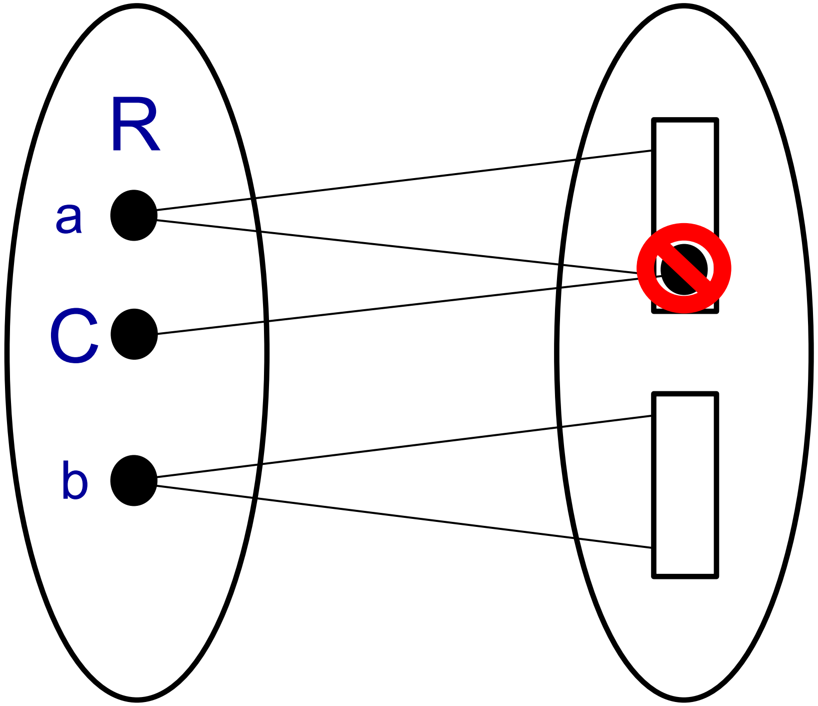
if P is a projective plane of order q with incidence graph G, then

$$\zeta(G) = q+1.$$

- $\zeta(G) = 3$:  $\zeta(G) = 4$: 

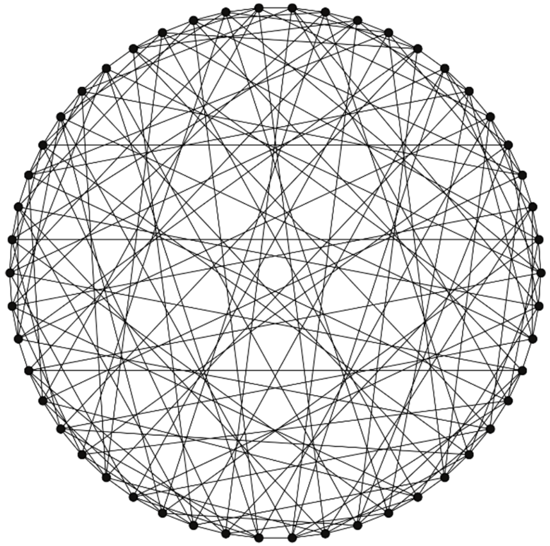
Sketch of lower bound

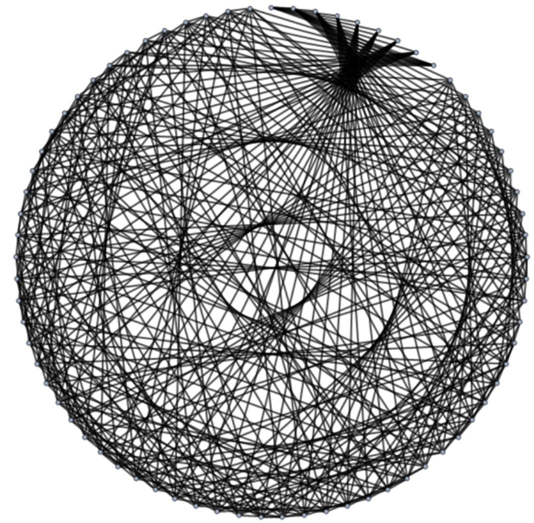
- for a contradiction, suppose that q cops win
- assume R has two points a and b in its *territory*
- a cop must distinguish a and b
- each remaining q-1 cops can uniquely identify at most one vertex in N(R)
- at least two vertices of N(R) remain in robber territory, a contradiction ∎



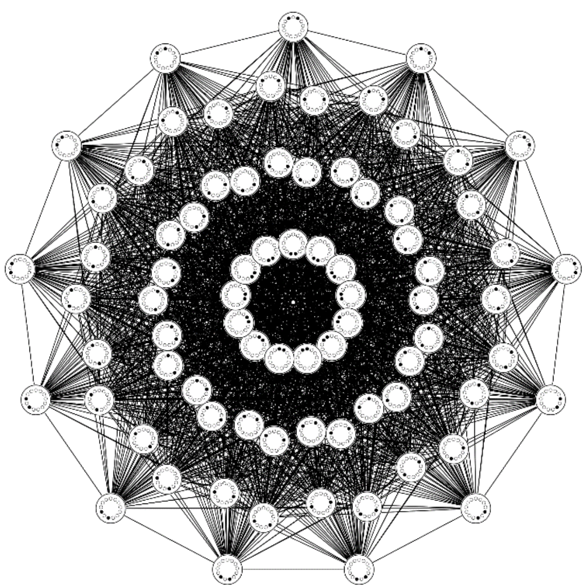
Other designs

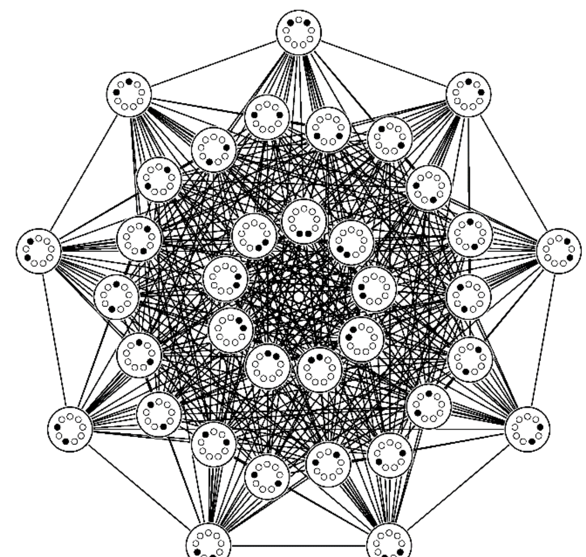
Design	**Bounds or values**
BIBD(v,b,r,k,λ), $2 \le \lambda \le r-1$	$\zeta(G) \le f(G)+r+1$
BIBD$(v,b,r,k,1)$	$\zeta(G) \le 2r+k-3$
BIBD$(v,b,r,k,1)$, $k<r$	$d < \zeta(G)$
Symmetric BIBD$(v,b,r,k,1)$	$\zeta(G) = k$
Projective plane of order q	$\zeta(G) = q+1$
BIBD$(k^2,k^2+k,k+1,k,1)$, $k \ge 3$	$\zeta(G) = k$ or $k+1$
Affine plane of order q	$\zeta(G) = q$
STS(v), $v>9$	$\lfloor \frac{v-2}{8} \rfloor \le \zeta(G) \le \frac{v+1}{2}$
STS(v)	$\zeta(G) \le (1+o(1))v/3$
$S(3,4,v)$, $v \ge 6$	$\zeta(G) \le v-3$
$S(t,k,v)$	$\zeta(G) \le (1+o(1))v/k$
$TD(k,n)$	$\zeta(G) \le n+k-4$





Diameter 2 case

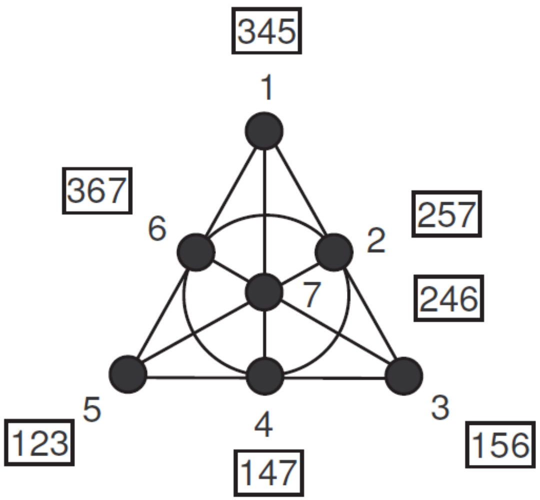


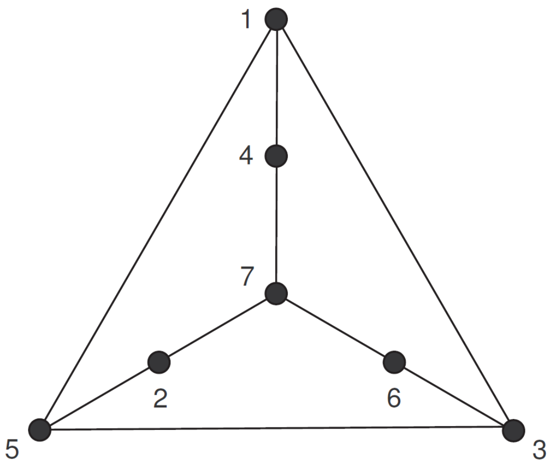


Polarity graphs

- suppose PG(2,q) has points P and lines L. A polarity is an involutory function π: P→ L that preserves incidence

- polarity graph: vertices are points, distinct x and y adjacent if x ϵ π(y)





Properties of polarity graphs

- order q^2+q+1, size $q(q+1)^2/2$
- $(q,q+1)$-regular
- C_4-free
- diameter 2

Localizing polarity graphs

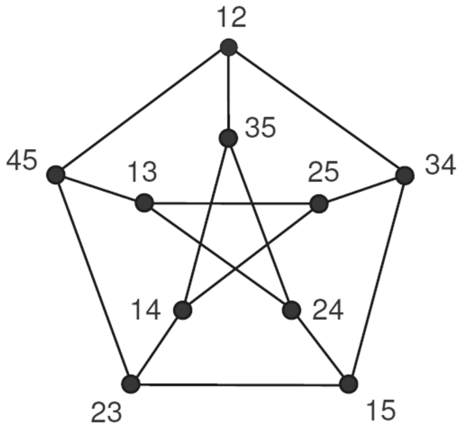
(B,Huggan,Marbach,21+):

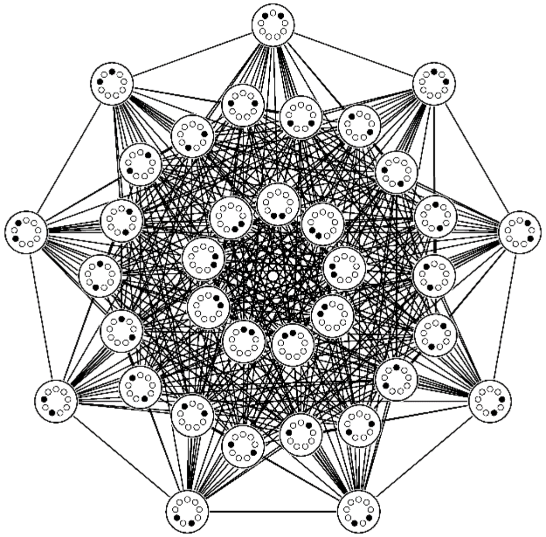
If G is a polarity graph of order q^2+q+1, then

$$(2q-5)/3 \leq \zeta(G) \leq 2q-1.$$

Kneser graphs

- non-intersection graphs K(k,n)
  - the Petersen graph is K(2,5)





- the graph K(k,n) is diameter 2 if $n \geq 3k$

Localization of Kneser graphs (BHM,21+)

- if k is even and $n \geq 3k$, then

$$\zeta(K(k,n)) = n/2 + n/k + O(1)$$

- if k is odd and $n \geq 3k$, then

$$n/2 + n/k - k/2 - 1 \leq \zeta(K(k,n)) \leq n/2 + (3/2)n/k + O(1)$$

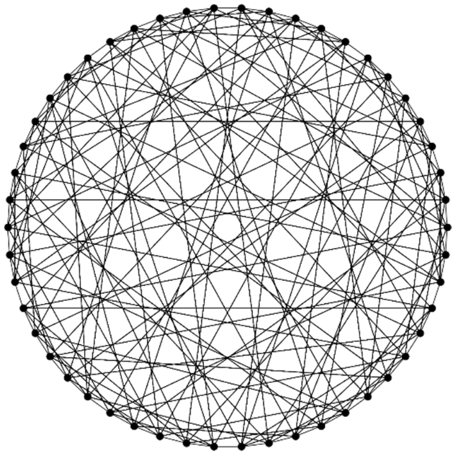
  – proved using a new notion of *hypergraph detection*

- for even $k \geq 6$, $\beta(K(k,n)) = n/2+n/k$ for infinitely many n
  – improves on upper bound $\left\lceil \frac{n}{2k-1} \right\rceil \left(\binom{2k-1}{k} - 1 \right)$

(Bailey et al, 2013)

Moore graphs

- diameter d and girth 2d+1
- diameter 2 case:
  - 5-cycle
  - Petersen graph
  - Hoffman-Singleton graph
  - hypothetical graph of order 3,250 and 57-regular



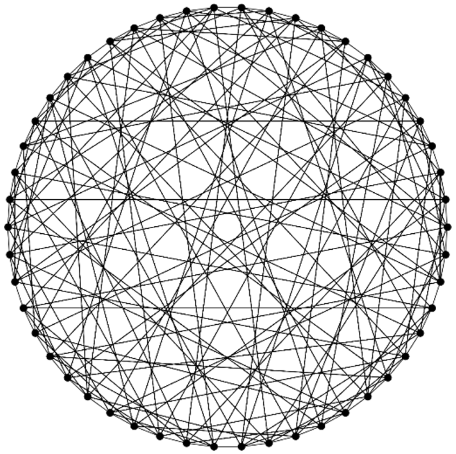
Localization number of Moore graphs of diameter 2

(BHM,21+):

Moore graph G	**ζ(G)**
Petersen graph	3
Hoffman-Singleton graph	6 or 7
hypothetical graph of order 3,250	56 or 57

Unknowns

- exact values on hypercubes, polarity graphs, Kneser graphs?
- is computing ζ **EXPTIME**-complete?
- Hoffman-Singleton graph?
  - $\zeta = 6$ or 7 (!)



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