Complex Zeros of Edge-Cover Polynomials of Hypergraphs

Atlantic Graph Theory Seminar 2022

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What about the zeros of graph polynomials?

Investigating and locating the zeros of graph polynomials could lead to

- information about the coefficients (E.g. Newton lemma, Brenti-Royle-Wagner)
- algorithmic aspects of the computation of the polynomial (E.g. Patel-Regts)
- understanding statistical physical models.

Problem

What is the the relation between the location of the zeros of graph polynomials (e.g. largest, smallest modulus, etc.) and their structure? (e.g.$\Delta(G)$, n, etc.)?

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Wagner's approach had success story:

- independence polynomial zeros of line graphs are real [Heilmann, Lieb]
- ferromagnetic Ising model zeros are on the unit circle [Lee, Yang]
- antiferromagnetic Ising model zeros of bounded degree graphs are in a ring
- antiferromagnetic Ising model zeros of line graphs are real [B, Csikvári, Regts]
- edge cover zeros in a "nice region" [B, Csikvári, Regts]

Edge covers

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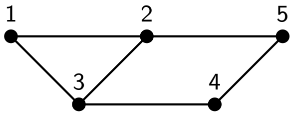

$$\mathcal{E}(G,x) = z^6 + 6z^5 + 12z^4 + 7z^3$$

Observations, facts

- If G contains an isolated vertex, then $\mathcal{E}(G,z) \equiv 0$.
- The largest degree of $\mathcal{E}(G,z)$ is $|E|$.
- Multiplicity of 0 is $\rho(G)$, that is $n - \nu(G)$.
- $\mathcal{E}(C_n, x)$ is a transform of the first Chebyshev polynomial. Therefore the zeros of $\mathcal{E}(C_n, x)$ are dense in $(-4, 0]$.

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Theorem (Csikvári, Oboudi)

All the zeros of $\mathcal{E}(G,z)$ are contained in

$$\left\{ z \mid |z| < \frac{(2+\sqrt{3})^2}{(1+\sqrt{3})} \approx 5.099 \right\}$$

Moreover, if n large enough and $\delta(G) > \sqrt{2n \ln n}$, then $\{z \mid |z| < 4\}$.

Result on graphs

Theorem (B, Csikvári, Regts)

All the zeros of $\mathcal{E}(G, z)$ are contained in

$$\{z \mid |z| \leq 4\}.$$

Moreover -4 never a zero.

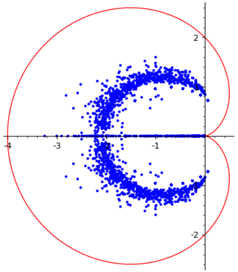

Figure: Zeros of the edge cover polynomial of some graphs on 10 vertices

Result on graphs

Theorem (B, Csikvári, Regts)

All the zeros of $\mathcal{E}(G,z)$ *are contained in*

$$\{-(1-\alpha)^2 \mid |\alpha| \leq 1\} \subseteq \{z \mid |z| \leq 4\}.$$

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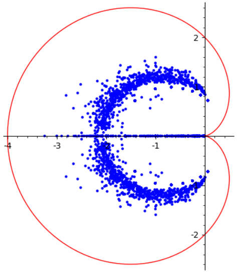

Figure: Zeros of the edge cover polynomial of some graphs on 10 vertices

A “related” graph polynomial

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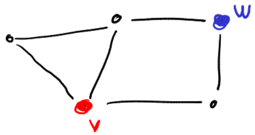
A vertex set $A \subseteq V$ is a dominating set of V, if $\forall v \in V$ there is a vertex $a \in A$ such that $v \in N_G[a]$.

$$D(G,x) = \sum_{A \text{ dominating}} x^{|A|}$$

E.g.



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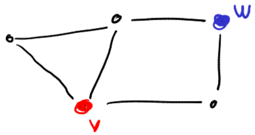

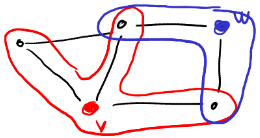

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- All the complex zeros are dense in $\mathbb{C}$ [Brown, Tufts]
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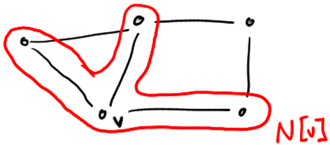
The Relation:
If $\mathcal{H}_G$ is the hypergraph on $V(G)$ with hyperedges $E = \{N_G[v]\}_{v \in V(G)}$



Then A is a dominating set if and only if the hyperedges $\{N_G[v]\}_{v \in A}$ cover all the vertices.

Generalization/relaxation helps

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For a hypergraph $\mathcal{H} = (V, E)$ define the *edge cover polynomial* as

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where the sum goes through those $F \subseteq E$, such that $\cup F \supseteq V$.

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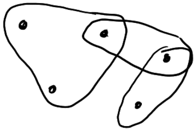

$z^3 + z^2$

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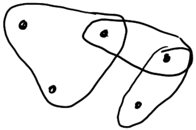

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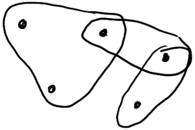

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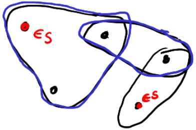

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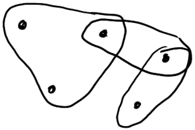

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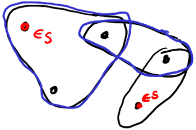

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Observation: For any $F \subseteq E$ to be an edge cover can be decided only from the degree sequence.

$$F \subseteq E \text{ s.t } \cup F \supseteq V \setminus S \iff \forall v \in V \setminus S : \quad \deg_F(v) \geq 1$$

We will use Wagner's subgraph counting polynomial in hypergraph settings.

Wagner's "subhypergraph" counting polynomial

Strategy: express the model as a half-edge model, then use Asano-contraction to make the choice of half-edges of an edge to be consistent and keep following the change of the zero-free region!

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For each v and for any subset of the edges $N_v = \{v \in e\}$ establish a weight and encode it into a polynomial as follows:

$$K^{(v)}(z) = \sum_{F \subseteq \{e \in E \ | \ u \in e\}} u_{|F|}^{(v)} z^{|F|}$$

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E.g: Edge cover relaxed over S:

- If $v \in S$, then $u^{(v)}_{|F|} \equiv 1$, thus

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- If $v \notin S$, then $K^{(v)}(z) = (1+z)^{\deg(v)} - 1$

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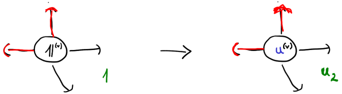

What is the change, when we are revealing v?

$$Z_{k-1}(z_1, \ldots, z_n) = \sum_{i=0}^{\deg_E(v)} P_i(z_{\neq v}) \cdot z_v^i \longrightarrow$$

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This is the "Schur-Szegő composition" of $Z_{k-1}(z)$ and $K^{(v)}(z)$!

Theorem (Wagner's theorem for hypergraphs – univariate)

For a $\mathcal{H}$ hypergraph

$$Z_{\mathcal{W}}(\mathcal{H}; z) = \sum_{F \subseteq E} \prod_{v \in V} u^{(v)}_{\deg_F(v)} z^{\sum_{v \in V} \deg_F(v)}$$

has its complex zeros only in $\{z \mid |z| \leq M\}$, if

1. *$K^{(v)}(z)$ has zeros only in $\{z \mid |z| \leq M\}$.*

Recall: Edge cover relaxed over S:

- If $v \in S$, then $K^{(v)}(z) = (1+z)^{\deg(v)}$, i.e. $z = -1$ is the only zero.

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- If $v \notin S$, then $K^{(v)}(z) = (1+z)^{\deg(v)} - 1$, i.e. the zero has length at most 2.

Corollary (Almost "Edge cover" polynomial)

For any hypergraph $\mathcal{H}$ and $S \subseteq V$

$$Z_{\mathcal{W}}(\mathcal{H}, z) = \sum_{F} z^{\sum_{v \in V} \deg_F(v)}$$

has its complex zeros in $\{z \mid |z| \leq 2\}$.

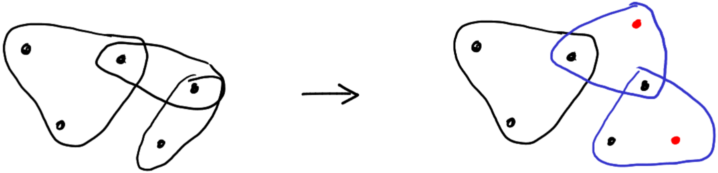
If $\mathcal{H}$ is a b-uniform hypergraph, then for any subhypergraph F

$$\sum_{v \in V} \deg_F(v) = b|F|,$$

thus

$$Z_{\mathcal{W}}(\mathcal{H}, z) = \sum_{F} z^{\sum_{v \in V} \deg_F(v)} = \sum_{F} z^{b|F|} = \mathcal{E}(\mathcal{H}, S, z^b).$$

What can we do if $\mathcal{H}$ is not uniform?



Let $\widehat{\mathcal{H}}$ be a b-uniform hypergraph obtained from $\mathcal{H}$ by adding new vertices into each hyperedge.
We also add these new vertices into the relaxed vertex set.
Thus

$$\mathcal{E}(\mathcal{H}, S, z) = \mathcal{E}(\widehat{\mathcal{H}}, S \cup \{\text{new vertices}\}, z)$$

Corollaries

Theorem (B, Csikvári, Regts)

For a hypergraph $\mathcal{H}$ without isolated vertices and with edges of size at most b, the zeros of $\mathcal{E}(\mathcal{H}, z)$ are contained in

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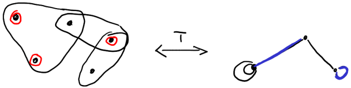
Corollary (Independence polynomail)

For a hypergraph $\mathcal{H}$, the zeros of $\mathcal{I}(\mathcal{H}, z)$ are contained in

$$\{z \mid |z| \geq 2^{-\Delta(\mathcal{H})}\}.$$

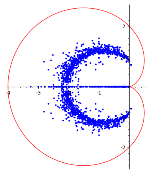


$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Questions

Problem

Is the closure of the zeros of edge cover polynomials of all graphs the cardioid?



Problem

Is -2^b in the closure of the zeros of the edge cover polynomial of hypergraphs of edge size at most b?

Problem

Is $-2^{\Delta(G)+1}$ in the closure of the zeros of the domination polynomial of bounded degree graphs?

Thank you for your attention!

Multivariate version

For every vertex v we encode the allowed local configurations into

$$K^{(v)}(\{z_e\}) = \sum_{F \subseteq \{v \in e\}} u_F^{(v)} \prod_{e \in F} z_e .$$

E.g. Edge cover

- If $v \in S$, then $K^{(v)}(z) = \prod_{e \in N_v} (1 + z_e)$
- If $v \notin S$, then $K^{(v)}(z) = \prod_{e \in N_v} (1 + z_e) - 1$
- each case $K^{(v)}(z)$ is $\{z \mid |z+1| > 1\} \times \cdots \times \{z \mid |z+1| > 1\}$ nonvanishing.

Theorem (Wagner's theorem for hypergraphs – multivariate)

For a $\mathcal{H}$ hypergraph

$$Z_{\mathcal{W}}(\mathcal{H}; z) = \sum_{F \subseteq E} \prod_{v \in V} u_{F \cap N_v}^{(v)} \prod_{e \in F} z_e$$

is $\left(\mathbb{C} \setminus (-1)^{|e_1|+1} Q^{|e_1|}\right) \times \cdots \times \left(\mathbb{C} \setminus (-1)^{|e_m|+1} Q^{|e_m|}\right)$-nonvanishing, if for every vertex v the multivariate polynomial $K^{(v)}(z)$ is $(\mathbb{C} \setminus Q) \times \cdots \times (\mathbb{C} \setminus Q)$-nonvanishing.