Permutations of finite subsets of $\mathbb{R}^2$ generated by Euclidean distances.

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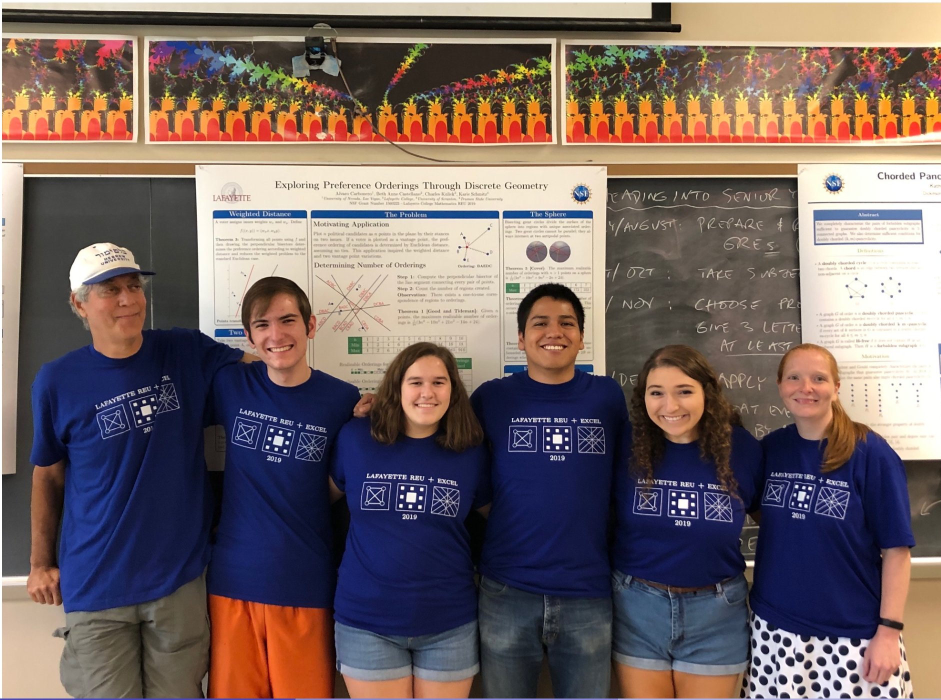
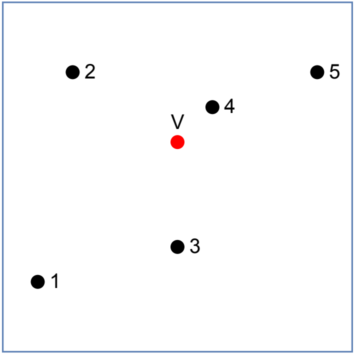
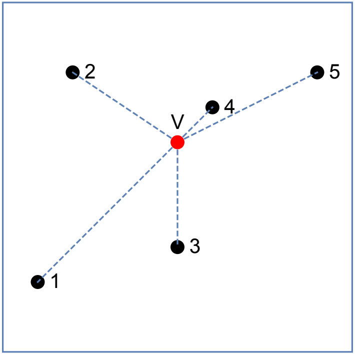

Exploring Preference Orderings Through Discrete Geometry
LAFAYETTE
Weighted Distance
The Problem
The Sphere
Motivating Application
Determining Number of Orderings
NSF
LAFAYETTE REU + EXCEL
2019

Points in the plane



Problem: Given a collection of points in the plane, and a vantage point V, order the points of S from closest to farthest.

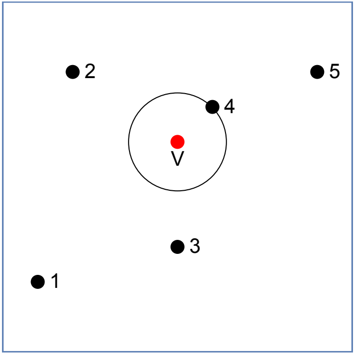
Points in the plane



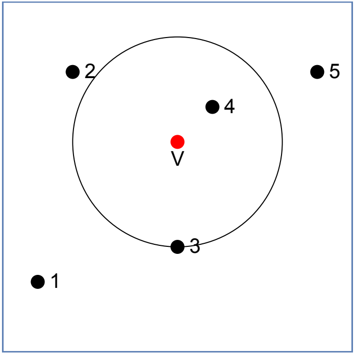
Induced permutation 43251

Problem: Given a collection of points in the plane, and a vantage point V, order the points of S from closest to farthest.

Expanding circles



Expanding circles



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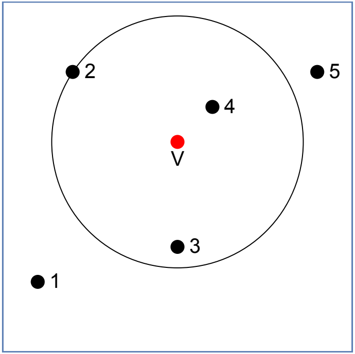

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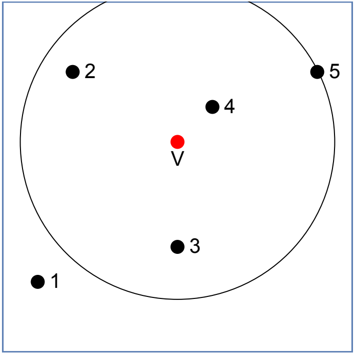
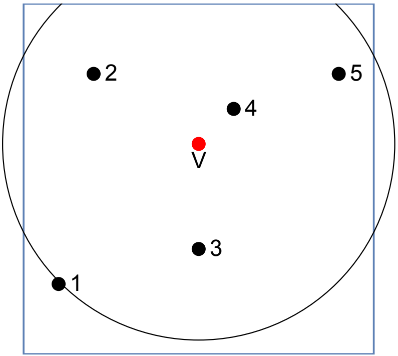
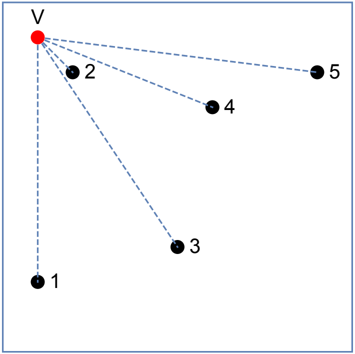


Expanding circles



Move the vantage point



Induced permutation 24135

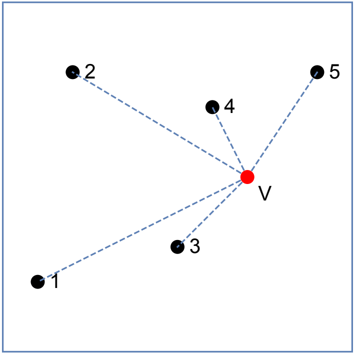
Motivation – social choice theory

- There are n candidates running for office.
- Each candidate is rated 0 – 10 on two independent issues, e.g., baseball and hockey. So each candidate is represented by an ordered pair (a, b), where $0 \leq a, b \leq 10$.
- The voter V also rates herself on the same two issues.
- Then the induced permutation represents the voter's preference list.

Goal: Find the maximum possible number of distinct preference lists.

Maximum

Question: Given *n* points fixed in the plane, how many distinct orderings are possible when the vantage point can roam freely?



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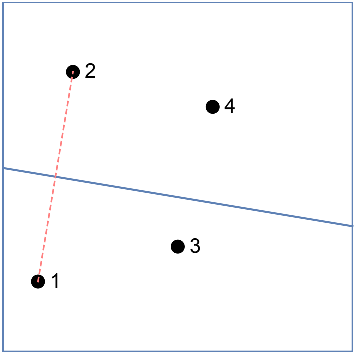
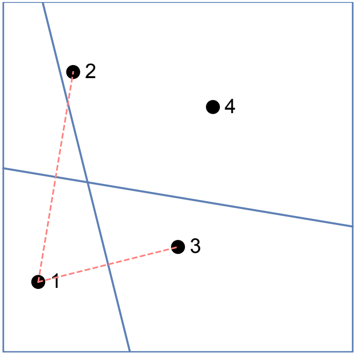
Bisectors

Draw the perpendicular bisector determined by points 1 and 2. If the vantage point V is below the line, then 1 precedes 2 in the induced permutation.



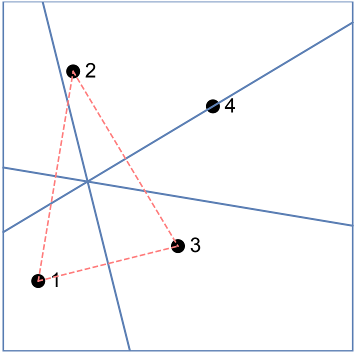
The perpendicular bisector of points 1 and 2.

Bisectors



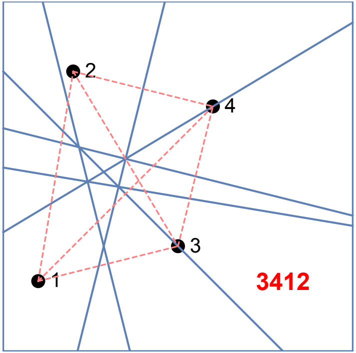
Two perpendicular bisectors.

Bisectors



These three perpendicular bisectors are coincident.

Bisectors

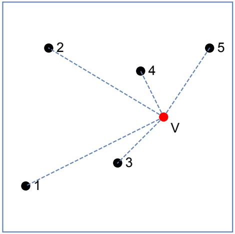


All the perpendicular bisectors.

Fact: The number of achievable permutations = the number of regions determined by all the perpendicular bisectors.

Maximum

Question: Given n points in "free position" the plane, how many distinct orderings are possible?



$$\max = \frac{1}{24}\left(3n^4 - 10n^3 + 21n^2 - 14n + 24\right)$$

n	1	2	3	4	5	6	7	8	9	10
max	1	2	6	18	46	101	197	351	583	916

https://oeis.org/A308305

Theorem

Given $S \subset \mathbb{R}^2$ with $|S| = n$, the maximum number of orderings is

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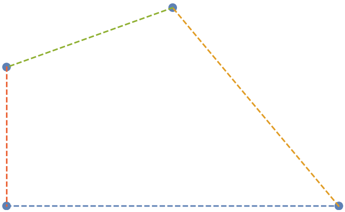
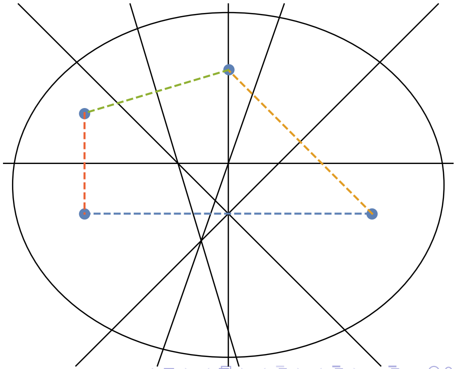
Proof idea: Use $v - e + r = 1$ for an associated graph.

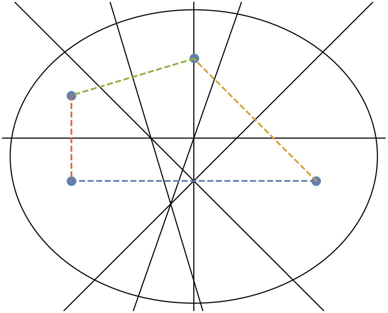
- Make a planar graph using the perpendicular bisectors, and draw a big circle around everything.





$$\max = \frac{1}{24}\left(3n^4 - 10n^3 + 21n^2 - 14n + 24\right).$$

- Next, count the number of vertices of degree 3, 4, and 6.



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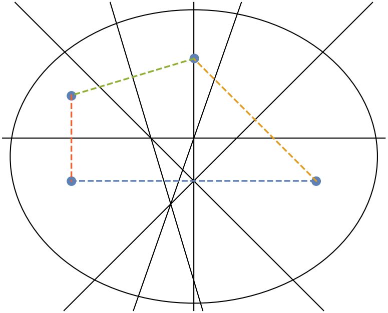

$$v_3 = 2\binom{n}{2}, \quad v_4 = 3\binom{n}{4}, \quad v_6 = \binom{n}{3}$$

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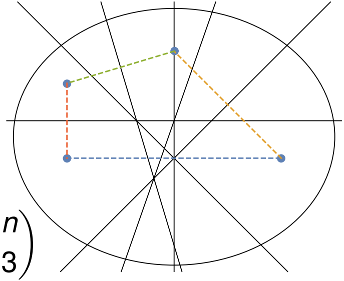
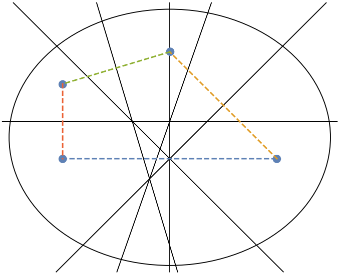


- Number of vertices:

$$\begin{aligned} v &= v_3 + v_4 + v_6 \\ &= 2\binom{n}{2} + 3\binom{n}{4} + \binom{n}{3} \\ &= \frac{1}{24}\left(3n^4 - 14n^3 + 45n^2 - 34n\right). \end{aligned}$$

- Number of edges: $2e = 3v_3 + 4v_4 + 6v_6$.

$$\begin{aligned} e &= (3v_3 + 4v_4 + 6v_6)/2 \\ &= 3\binom{n}{2} + 6\binom{n}{4} + 3\binom{n}{3} \\ &= \frac{1}{4}\left(n^4 - 4n^3 + 11n^2 - 8n\right) \end{aligned}$$



- Then the number of regions is $r = -v + e + 1$.

$$\begin{aligned} r &= -v + e + 1 \\ &= \binom{n}{2} + 3\binom{n}{4} + 2\binom{n}{3} + 1 \\ &= \frac{1}{24}\left(3n^4 - 10n^3 + 21n^2 - 14n + 24\right) \end{aligned}$$

History

- Suppose $S \subset \mathbb{R}^d$, and $|S| = n$. Then the maximum number of regions is

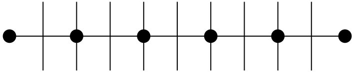
$$s(n,n) + s(n,n-1) + \cdots + s(n,n-d),$$

where $s(n,k)$ is the unsigned Stirling number of the first kind. ($s(n,k)$ counts the number of permutations of $\{1,\ldots,n\}$ with exactly k cycles.)

  - Good and Tideman, "Stirling numbers and a geometric structure from voting theory," J. Combinatorial Theory Ser. A **23** (1977), 34–45.
  - T. Zaslavsky, "Perpendicular dissections of space," Discrete Comput. Geom. **27** (2002), 303–351.

Minimum

Question: Given n points in $\mathbb{R}^d$, what is the minimum possible number of orderings we generate?



Easiest case: $d = 1$.

Dimension 1: The minimum occurs when the n points are equally spaced on a line.

Answer: n equally spaced points generate $2n - 3$ distinct midpoints, so min $= 2n - 2$.

Minimum in all dimensions

Theorem

Let $S \subset \mathbb{R}^n$ with $|S| = n$. Then $\min = 2n - 2$*, and this occurs precisely when the points are collinear and equally spaced (for $n > 4$).*

- $d = 1$: First, prove this for $S \subset \mathbb{R}^1$, i.e., sets of points on the real line.
- $d = 2$: Next, assume $S \subset \mathbb{R}^2$. Then, if the points are not collinear, use Ungar's theorem on slopes:

  Ungar [1980]: n points in the plane (not all on a line) determine at least $n - 1$ distinct slopes.

- $d > 2$: Higher dimensional analogues of Ungar's theorem and projection finish this problem.

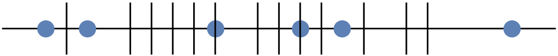
  Pach, Pinchasi, Sharir [2004]: n points in $\mathbb{R}^3$ determine at least $2n - 3$ different directions.

These two papers settled two conjectures of Scott [1970].

Between the min and the max?

Question: Fix n and let k be an integer between the min and the max. Is there a configuration of points that produces exactly k orderings?

- Dimension 1: min $= 2n - 2$ max $= \frac{n^2 - n + 2}{2}$



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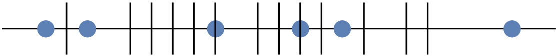

Theorem

Fix $n > 0$. For all k in $[2n-2, \frac{n^2-n+2}{2}]$, there is a configuration $S \subset \mathbb{Z}$ such that S generates exactly k regions.

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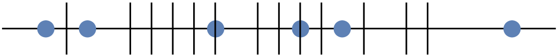


Theorem

Fix $n > 0$. For all k in $[2n-2, \frac{n^2-n+2}{2}]$, there is a configuration $S \subset \mathbb{Z}$ such that S generates exactly k regions.

Sum-set Problem. *Let n be given and let k satisfy $2n - 3 \leq k \leq \frac{n^2-n}{2}$. Then there is a collection of integers $a_1 < a_2 < \cdots < a_n$ such that the number of distinct sums $a_i + a_j$ (where $i \neq j$) is exactly k.*

Between the min and the max?

Question: Fix n and let k be an integer between the min and the max. Is there a configuration of points in the plane that produces exactly k orderings?

$$\min = 2n - 2 \qquad \max = \frac{1}{24}\left(3n^4 - 10n^3 + 21n^2 - 14n + 24\right)$$

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

Filling in the gaps in the plane – computer evidence

n	(Possible) Possible Numbers of Orderings
2	2
3	4, 6
4	6, 7, 8, 10, 12, 16, 17, 18
5	8, 9, 10, 11, 12, 14, 16, 18, 20, 24, 26, 28, 30, 36, 38, 40, 42, 44, 45, 46

$n = 4.$

■■■□■□■□□□■■■

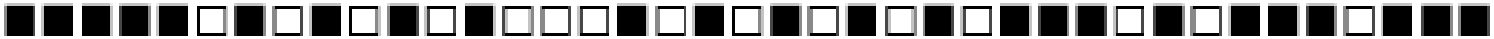
Min $= 6$, Max $= 18$.

Achievable percentage is 61.53%.

Filling in the gaps in the plane – computer evidence

n	(Possible) Possible Numbers of Orderings
2	2
3	4, 6
4	6, 7, 8, 10, 12, 16, 17, 18
5	8, 9, 10, 11, 12, 14, 16, 18, 20, 24, 26, 28, 30, 36, 38, 40, 42, 44, 45, 46

$n = 5.$



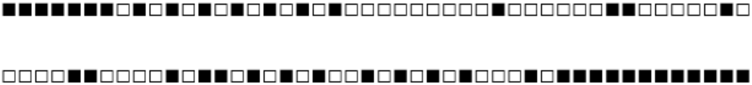
Min $= 8$, Max $= 46$.

Achievable percentage is 61.53%.

Filling in the gaps in the plane – computer evidence

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

$n = 6.$



Min $= 10$, Max $= 101$.

Achievable percentage is 46.74%.

Filling in the gaps in the plane – computer evidence

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

$n = 7.$

Min $=$ 12, Max $=$ 197.

Achievable percentage is 52.15%.

Filling in the gaps in the plane – computer evidence

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

$n = 8.$

Min $= 14$, Max $= 351$.

Achievable percentage is 58.88%.

Percent achievable

n	4	5	6	7	8
min	6	8	10	12	14
max	18	46	101	197	351
%	61.53%	61.53%	46.74%	52.15%	58.88%

$n = 8$.

Min $= 14$, Max $= 351$.

Filling in the gaps in the plane

Theorem

The following number of orderings are achievable by some configuration of n points in the plane.

1. *At the bottom, all k satisfying*

$$\min = 2n - 2 \leq k \leq \frac{1}{2}n^2 - \frac{1}{2}n + 1$$

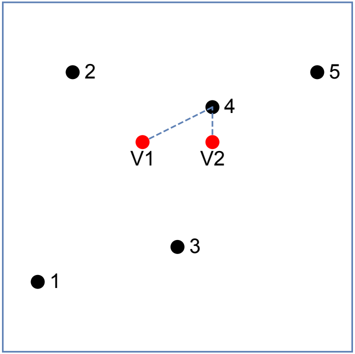
are possible.

2. *At the top, all k satisfying*

$$\max - \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq k \leq \max .$$

Two vantage points

Given n points in the plane, locate two distinct vantage points V_1 and V_2. Compute the average distance from V_1 and V_2 to each point.



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Two vantage points

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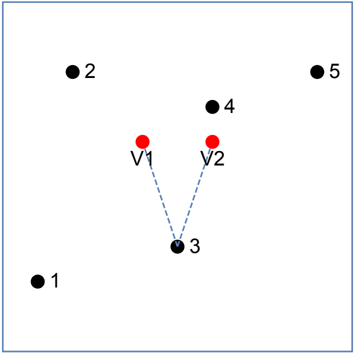

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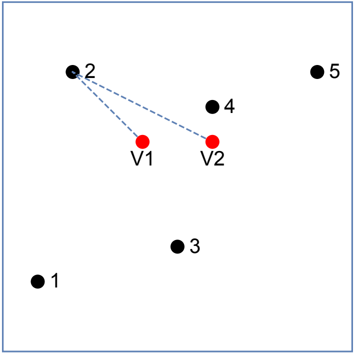
Two vantage points

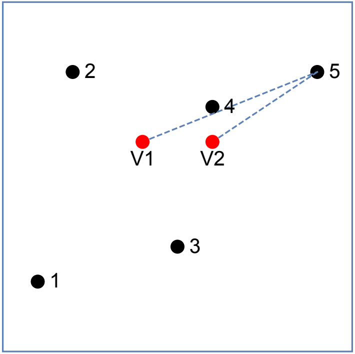
Given n points in the plane, locate two distinct vantage points V_1 and V_2. Compute the average distance from V_1 and V_2 to each point.



432

Two vantage points

Given n points in the plane, locate two distinct vantage points V_1 and V_2. Compute the average distance from V_1 and V_2 to each point.



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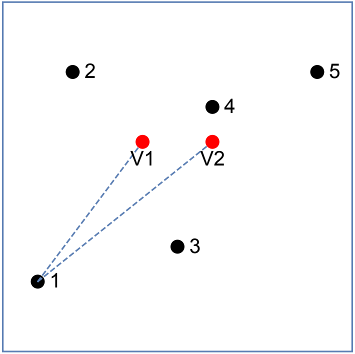
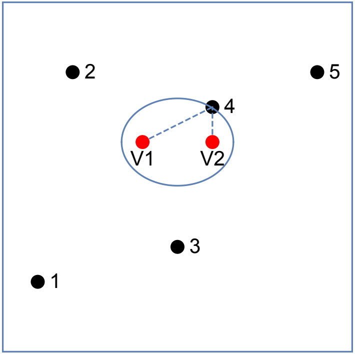
Two vantage points

Given n points in the plane, locate two distinct vantage points V_1 and V_2. Compute the average distance from V_1 and V_2 to each point.



43251

Expanding ellipses



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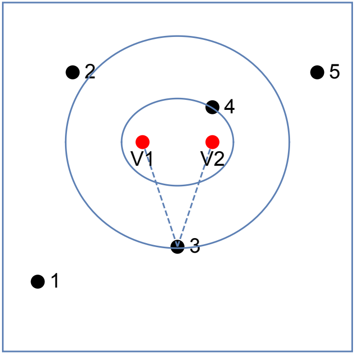
Expanding ellipses



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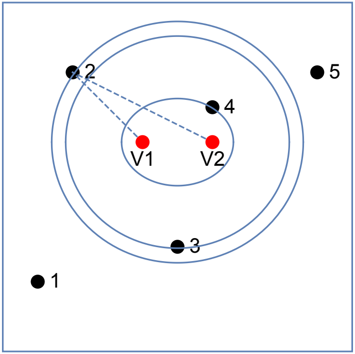
Expanding ellipses



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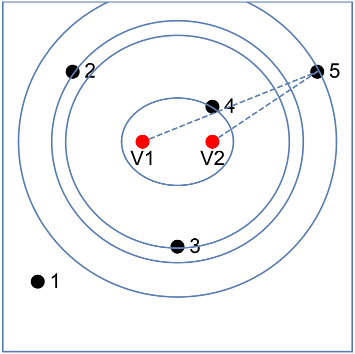
Expanding ellipses



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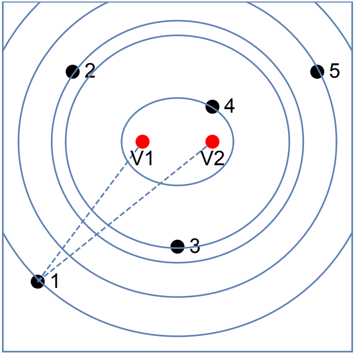
Expanding ellipses



43251

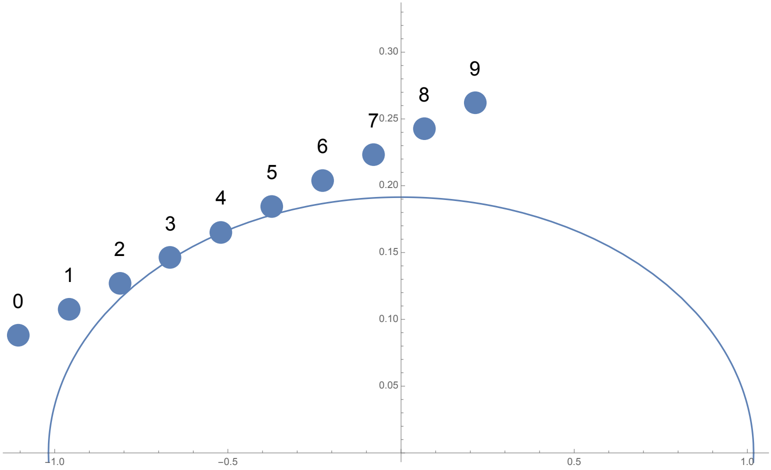
Max and min for two vantage points in the plane

Problem: Determine the max and the min for the number of orderings produced with two vantage points.

n	2	3	4	5	6	7	8	9	10
min	2	4	8	16	30	54	94	160	268
max	2	6	24	120	$\geq 680^*$	?	?	?	?

* Charles Kulick reported that "this took an entire weekend of computation on two rows of Lafayette laptops."

Two vantage points, minimum number of orderings



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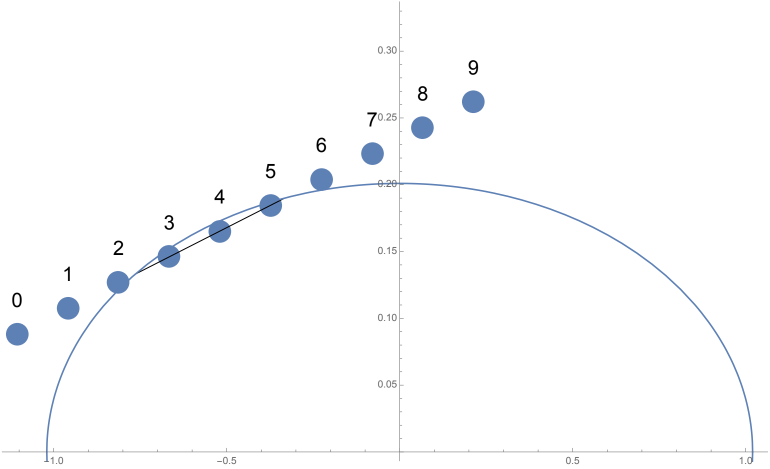
Two vantage points, linear point-sets



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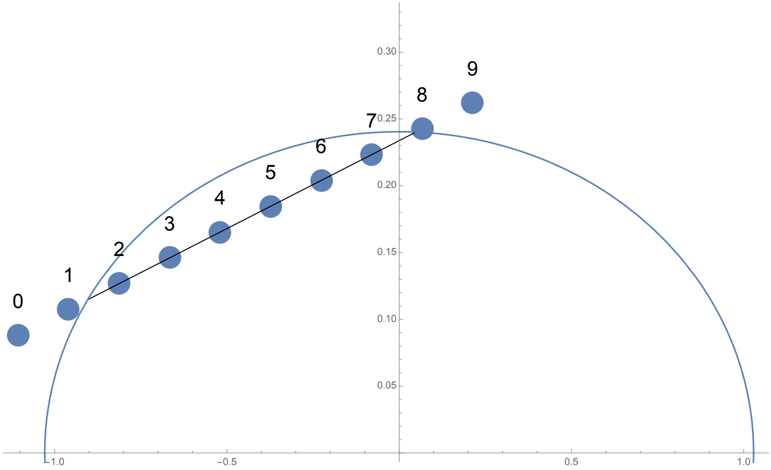
Two vantage points, linear point-sets



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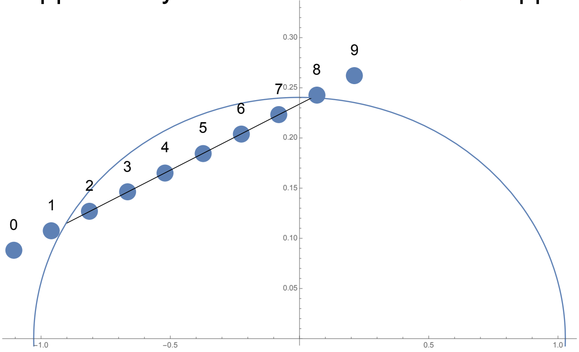
Two vantage points, linear point-sets

Note: Betweenness property of linear sequences: After the first position, i can appear only after either $i-1$ or $i+1$ appears.



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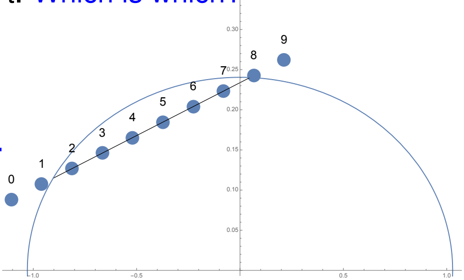
First bound: The minimum number of possible orderings is $\leq 2^{n-1}$.

Two vantage points, linear point-sets

Quiz: Suppose we have 8 points equally spaced on a line. One of these is possible, and the other isn't. Which is which?

1. 4,3,2,5,6,7,1,8
2. 4,3,5,2,6,7,1,8

Both satisfy betweenness property.



Two vantage points, linear point-sets

Quiz: Suppose we have 8 points equally spaced on a line. One of these is possible, and the other isn't. Which is which?

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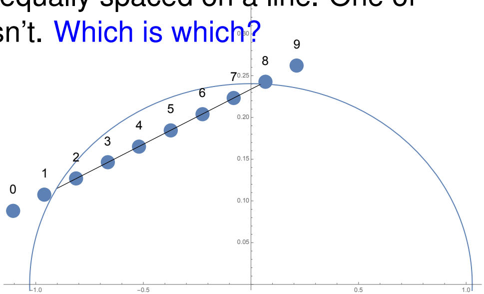

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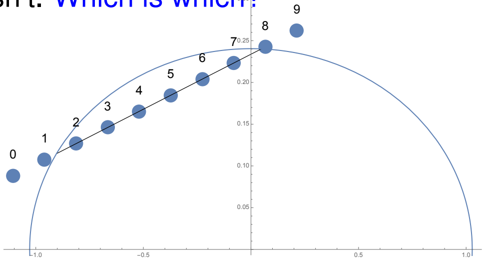
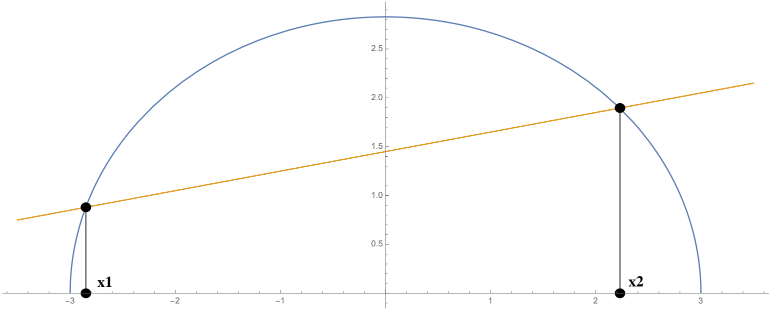


4,3,2,5,6,7,1,8 is impossible because the sequence has two consecutive downs and two consecutive ups:

D D U U U D U

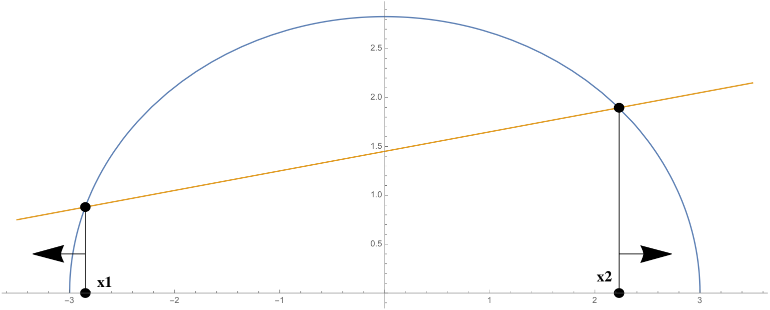
Velocity

- Vantage points are located at $(\pm 1, 0)$.
- Let t represent the (changing) positive x-intercept.
- Then ellipse equation is $\frac{x^2}{t^2} + \frac{y^2}{t^2 - 1} = 1$.
- Finally, the points reside on the line $y = mx + b$ has $m, b > 0$.

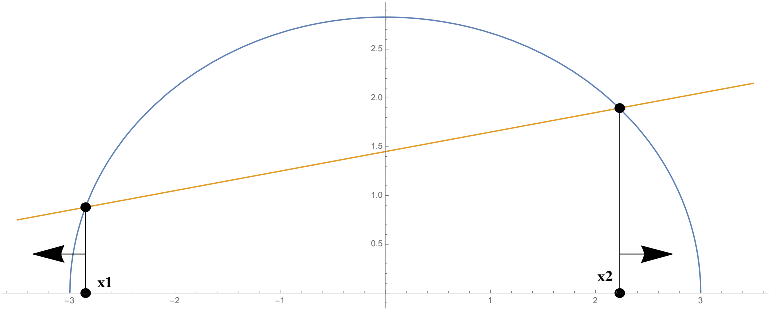


Velocity

- Let $x_1(t)$ and $x_2(t)$ be the x-coordinates of the two intersection points. Let $v_i(t) = \frac{dx}{dt}$ evaluated at $x = x_i$, with $i = 1, 2$.



Velocity



$$|v_2(t)| - |v_1(t)| = \frac{4bmt}{\left(t^2\left(m^2+1\right)-1\right)^2}.$$

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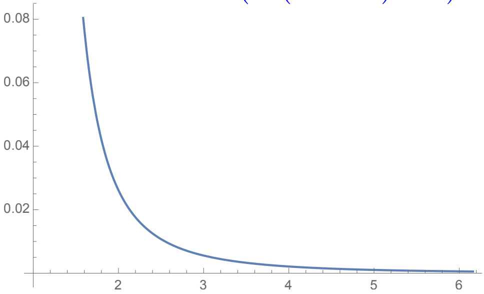

Difference of speeds.

Note: The point on the right moves faster than the point on the left.

Upper bound on minimum

First, some data:

n	1	2	3	4	5	6	7	8	9	10
min	1	2	4	8	16	30	54	94	160	268

This counts the number of sequences of 0's and 1's of length n that have the property that consecutive 0's and consecutive 1's cannot both appear except at the beginning or the end of the sequence.

For instance, the sequence

- 11011101000 is good, but
- but 10001100 is bad.

This is $2a_{n-1}$ where a_n is OEIS sequence A000126.

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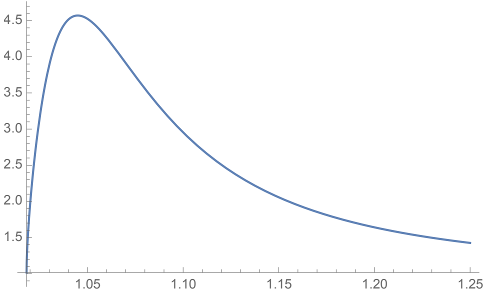
Very cool fact: This bound equals $2f_{n+4} - 2n - 4$, where f_n is the n^{th} Fibonacci number.

Unfortunately, ...

This bound is too big.



Ratio of speeds. There is a unique max.

Conclusion: Assume $b, m > 0$. Suppose the sequence generated has "up-blocks" $a_1, a_2, \ldots$. Then the up blocks form a unimodal sequence, i.e.,

$$a_1 \leq a_2 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_r.$$

Things to do

- For one vantage point:
  - characterize the permutations that appear;
  - fill in more of the gaps.
- For two vantage points:
  - Find any reasonable bounds on the max and the min.
  - extend to higher dimensions.