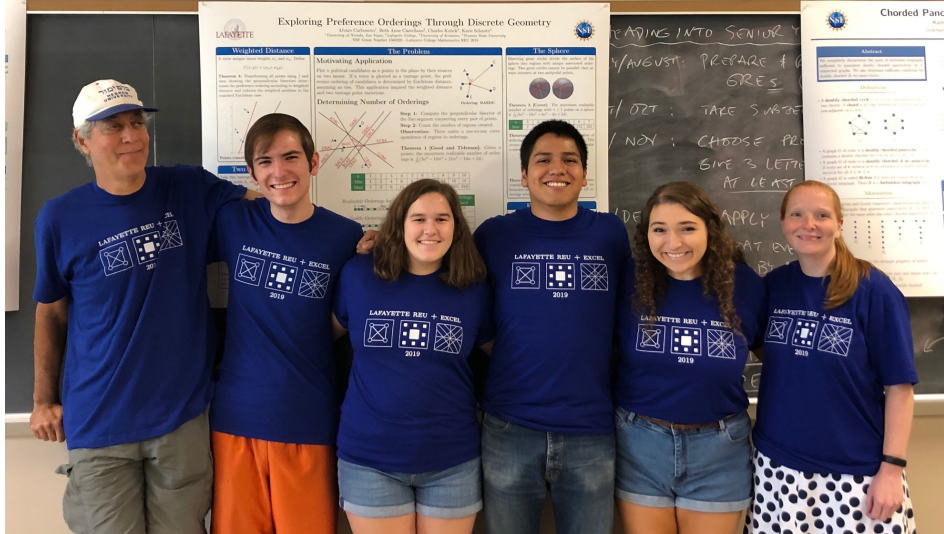


Permutations of finite subsets of \mathbb{R}^2 generated by Euclidean distances.

Gary Gordon – Lafayette College

Brittany Shelton (Albright College), Students: Alvaro Carbonero (UNLV), Beth Anne Castellano (Lafayette), Charles Kulick (U. Scranton), Karie Schmitz (Truman State)



Exploring Preference Orderings Through Discrete Geometry

Murali Chidambaram, Arjun Kumar, Karthik Suresh, Chaitan Mishra, Arjun Muralidharan
University of Florida, San Jose State University, University of Houston, Princeton Park University, West Virginia University, Oklahoma State University, MIT, IITM

Weighted Distance

A point P is said to be closer to a point A than to a point B if $d(P, A) < d(P, B)$. A point P is said to be equidistant to two points A and B if $d(P, A) = d(P, B)$. The set of all points equidistant to two points A and B is called the perpendicular bisector of the line segment AB . The set of all points equidistant to three points A , B , and C is called the circumcenter of the triangle ABC . The set of all points equidistant to four points A , B , C , and D is called the circumcenter of the tetrahedron $ABCD$.

The Problem

Given a set of points P_1, P_2, \dots, P_n in the plane, find the number of points P in the plane such that $d(P, P_1) < d(P, P_2) < \dots < d(P, P_n)$.

The Sphere

Given a set of points P_1, P_2, \dots, P_n on the surface of a sphere, find the number of points P on the surface of the sphere such that $d(P, P_1) < d(P, P_2) < \dots < d(P, P_n)$.

Determining Number of Orderings

Step 1: Consider the perpendicular bisectors of the line segments connecting every pair of points.
Step 2: Count the number of regions formed.
Observation: These regions are non-overlapping and their number is independent of the order of the points.

Lemma 1 (Klein and Tikhonov) Given n points, the maximum number of regions is $\frac{n^2 - 3n + 4}{2}$.

READING INTO SENIOR YEAR

4/AUGUST: PREPARE FOR GRE'S

7/OCT: TAKE SUBJECTS

1/NOV: CHOOSE PROGRAMS TO APPLY TO

3/DEC: APPLY TO COLLEGES

5/FEB: TAKE SAT

7/MAY: TAKE ACT

9/AUG: TAKE GRE

11/NOV: TAKE GRE

1/2020: TAKE GRE

3/2020: TAKE GRE

5/2020: TAKE GRE

7/2020: TAKE GRE

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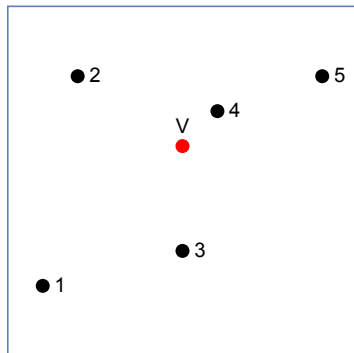
Chorded Pancake

Murali Chidambaram, Arjun Kumar, Karthik Suresh, Chaitan Mishra, Arjun Muralidharan
University of Florida, San Jose State University, University of Houston, Princeton Park University, West Virginia University, Oklahoma State University, MIT, IITM

Abstract

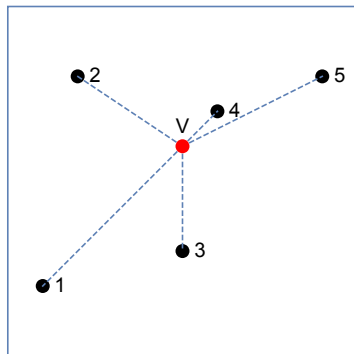
A chorded pancake graph CP_n is a graph with $n!$ vertices, each representing a permutation of $\{1, 2, \dots, n\}$. Two vertices are adjacent if they differ by a transposition of two adjacent elements. A chorded pancake graph CP_n is a graph with $n!$ vertices, each representing a permutation of $\{1, 2, \dots, n\}$. Two vertices are adjacent if they differ by a transposition of two adjacent elements. A chorded pancake graph CP_n is a graph with $n!$ vertices, each representing a permutation of $\{1, 2, \dots, n\}$. Two vertices are adjacent if they differ by a transposition of two adjacent elements.

Points in the plane



Problem: Given a collection of points in the plane, and a vantage point V , order the points of S from closest to farthest.

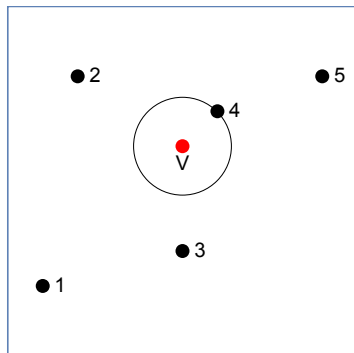
Points in the plane



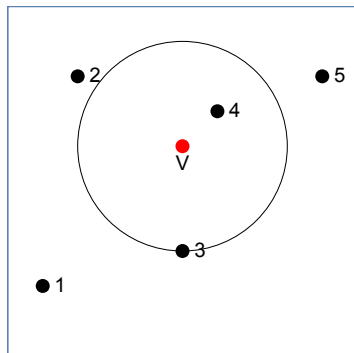
Induced permutation **43251**

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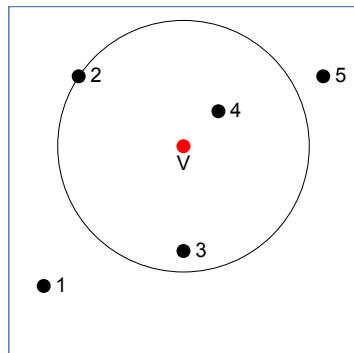
Expanding circles



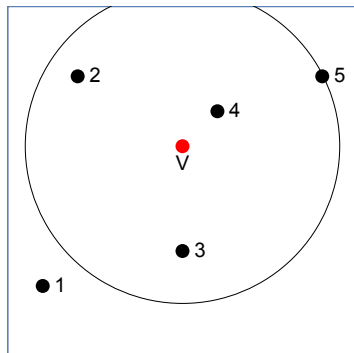
Expanding circles



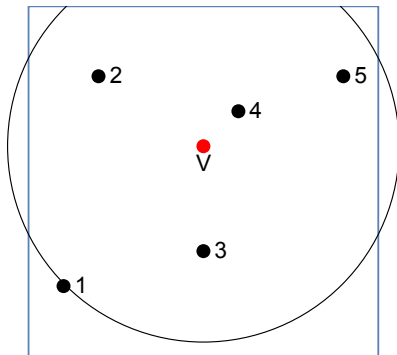
Expanding circles



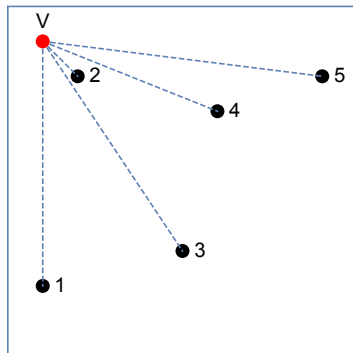
Expanding circles



Expanding circles



Move the vantage point



Induced permutation **24135**

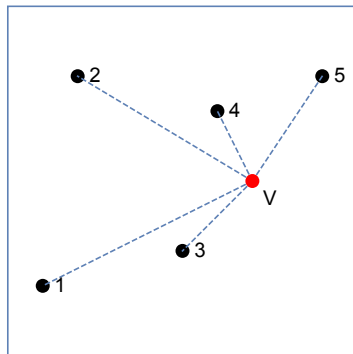
Motivation – social choice theory

- There are n candidates running for office.
- Each candidate is rated 0 – 10 on two independent issues, e.g., baseball and hockey. So each candidate is represented by an ordered pair (a, b) , where $0 \leq a, b \leq 10$.
- The voter V also rates herself on the same two issues.
- Then the induced permutation represents the voter's preference list.

Goal: Find the maximum possible number of distinct preference lists.

Maximum

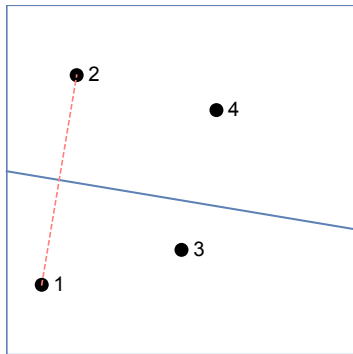
Question: Given n points fixed in the plane, how many distinct orderings are possible when the vantage point can roam freely?



43521

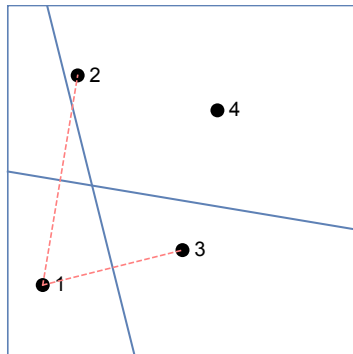
Bisectors

Draw the perpendicular bisector determined by points 1 and 2. If the vantage point \checkmark is below the line, then 1 precedes 2 in the induced permutation.



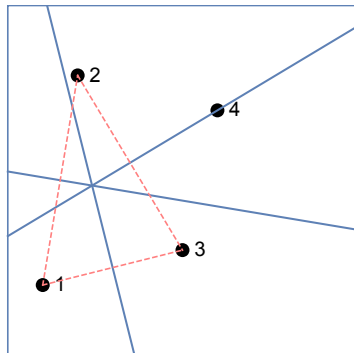
The perpendicular bisector of points 1 and 2.

Bisectors



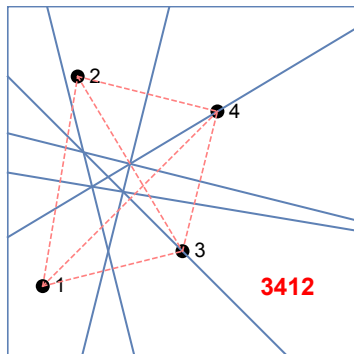
Two perpendicular bisectors.

Bisectors



These three perpendicular bisectors are coincident.

Bisectors

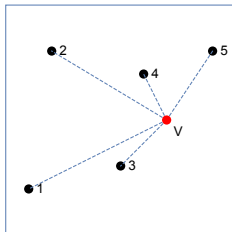


All the perpendicular bisectors.

Fact: The number of achievable permutations = the number of regions determined by all the perpendicular bisectors.

Maximum

Question: Given n points in “free position” the plane, how many distinct orderings are possible?



$$\max = \frac{1}{24} (3n^4 - 10n^3 + 21n^2 - 14n + 24)$$

n	1	2	3	4	5	6	7	8	9	10
max	1	2	6	18	46	101	197	351	583	916

<https://oeis.org/A308305>

Theorem

Given $S \subset \mathbb{R}^2$ with $|S| = n$, the maximum number of orderings is

$$\frac{1}{24} (3n^4 - 10n^3 + 21n^2 - 14n + 24).$$

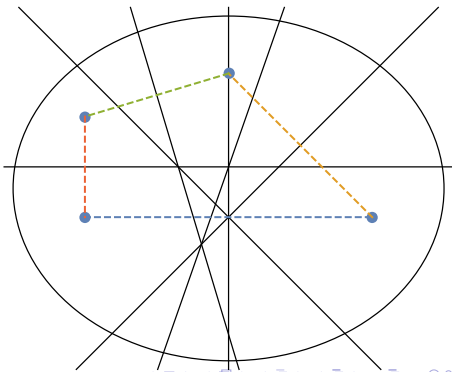
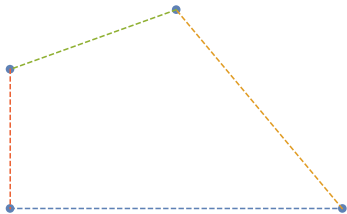
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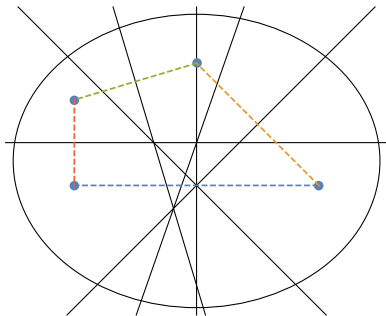
Proof idea: Use $v - e + r = 1$ for an associated graph.

- Make a planar graph using the perpendicular bisectors, and draw a big circle around everything.



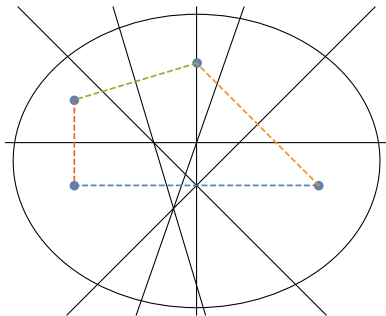
$$\max = \frac{1}{24} (3n^4 - 10n^3 + 21n^2 - 14n + 24).$$

- Next, count the number of vertices of degree 3, 4, and 6.



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- Next, count the number of vertices of degree 3, 4, and 6.

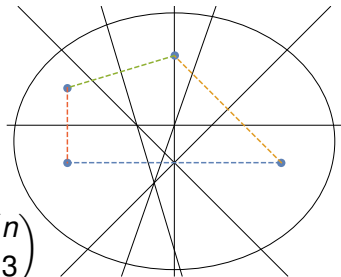


$$v_3 = 2 \binom{n}{2}, \quad v_4 = 3 \binom{n}{4}, \quad v_6 = \binom{n}{3}$$

$$v_3 = 2 \binom{n}{2}, \quad v_4 = 3 \binom{n}{4}, \quad v_6 = \binom{n}{3}$$

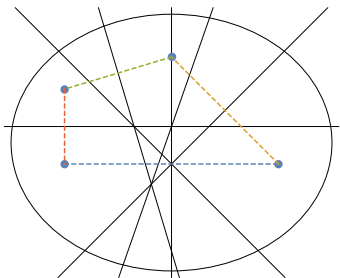
- Number of vertices:

$$\begin{aligned} v &= v_3 + v_4 + v_6 \\ &= 2 \binom{n}{2} + 3 \binom{n}{4} + \binom{n}{3} \\ &= \frac{1}{24} (3n^4 - 14n^3 + 45n^2 - 34n). \end{aligned}$$



- Number of edges: $2e = 3v_3 + 4v_4 + 6v_6$.

$$\begin{aligned} e &= (3v_3 + 4v_4 + 6v_6)/2 \\ &= 3 \binom{n}{2} + 6 \binom{n}{4} + 3 \binom{n}{3} \\ &= \frac{1}{4} (n^4 - 4n^3 + 11n^2 - 8n) \end{aligned}$$



- Then the number of regions is $r = -v + e + 1$.

$$\begin{aligned}
 r &= -v + e + 1 \\
 &= \binom{n}{2} + 3\binom{n}{4} + 2\binom{n}{3} + 1 \\
 &= \frac{1}{24} (3n^4 - 10n^3 + 21n^2 - 14n + 24)
 \end{aligned}$$

History

- Suppose $S \subset \mathbb{R}^d$, and $|S| = n$. Then the maximum number of regions is

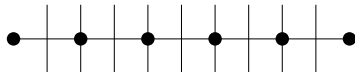
$$s(n, n) + s(n, n - 1) + \cdots + s(n, n - d),$$

where $s(n, k)$ is the unsigned **Stirling number of the first kind**. ($s(n, k)$ counts the number of permutations of $\{1, \dots, n\}$ with exactly k cycles.)

- ▶ **Good and Tideman**, “Stirling numbers and a geometric structure from voting theory,” J. Combinatorial Theory Ser. A **23** (1977), 34–45.
- ▶ **T. Zaslavsky**, “Perpendicular dissections of space,” Discrete Comput. Geom. **27** (2002), 303–351.

Minimum

Question: Given n points in \mathbb{R}^d , what is the minimum possible number of orderings we generate?



Easiest case: $d = 1$.

Dimension 1: The minimum occurs when the n points are **equally spaced on a line**.

Answer: n equally spaced points generate $2n - 3$ distinct midpoints, so **$\min = 2n - 2$** .

Minimum in all dimensions

Theorem

Let $S \subset \mathbb{R}^n$ with $|S| = n$. Then $\min = 2n - 2$, and this occurs precisely when the points are collinear and equally spaced (for $n > 4$).

- $d = 1$: First, prove this for $S \subset \mathbb{R}^1$, i.e., sets of points on the real line.
- $d = 2$: Next, assume $S \subset \mathbb{R}^2$. Then, if the points are not collinear, use Ungar's theorem on slopes:

Ungar [1980]: n points in the plane (not all on a line) determine at least $n - 1$ distinct slopes.

- $d > 2$: Higher dimensional analogues of Ungar's theorem and projection finish this problem.

Pach, Pinchasi, Sharir [2004]: n points in \mathbb{R}^3 determine at least $2n - 3$ different directions.

These two papers settled two conjectures of Scott [1970].

Between the min and the max?

Question: Fix n and let k be an integer between the min and the max. Is there a configuration of points that produces exactly k orderings?

- Dimension 1: $\min = 2n - 2$ $\max = \frac{n^2 - n + 2}{2}$



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- Dimension 1: $\min = 2n - 2$ $\max = \frac{n^2 - n + 2}{2}$



Theorem

Fix $n > 0$. For all k in $[2n - 2, \frac{n^2 - n + 2}{2}]$, there is a configuration $S \subset \mathbb{Z}$ such that S generates exactly k regions.

Between the min and the max?

Question: Fix n and let k be an integer between the min and the max. Is there a configuration of points that produces exactly k orderings?

- Dimension 1: $\min = 2n - 2$ $\max = \frac{n^2 - n + 2}{2}$



Theorem

Fix $n > 0$. For all k in $[2n - 2, \frac{n^2 - n + 2}{2}]$, there is a configuration $S \subset \mathbb{Z}$ such that S generates exactly k regions.

Sum-set Problem. Let n be given and let k satisfy $2n - 3 \leq k \leq \frac{n^2 - n}{2}$. Then there is a collection of integers $a_1 < a_2 < \dots < a_n$ such that the number of distinct sums $a_i + a_j$ (where $i \neq j$) is exactly k .

Between the min and the max?

Question: Fix n and let k be an integer between the min and the max. Is there a configuration of points in the plane that produces exactly k orderings?

$$\min = 2n - 2 \qquad \max = \frac{1}{24} (3n^4 - 10n^3 + 21n^2 - 14n + 24)$$

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

Filling in the gaps in the plane – computer evidence

n	(Possible) Possible Numbers of Orderings
2	2
3	4, 6
4	6, 7, 8, 10, 12, 16, 17, 18
5	8, 9, 10, 11, 12, 14, 16, 18, 20, 24, 26, 28, 30, 36, 38, 40, 42, 44, 45, 46

$$n = 4.$$



Min = 6, Max = 18.

Achievable percentage is 61.53%.

Filling in the gaps in the plane – computer evidence

n	(Possible) Possible Numbers of Orderings
2	2
3	4, 6
4	6, 7, 8, 10, 12, 16, 17, 18
5	8, 9, 10, 11, 12, 14, 16, 18, 20, 24, 26, 28, 30, 36, 38, 40, 42, 44, 45, 46

$$n = 5.$$



Min = 8, Max = 46.

Achievable percentage is 61.53%.

Filling in the gaps in the plane – computer evidence

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

$n = 6.$



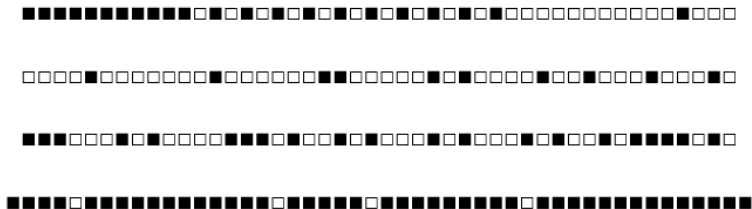
Min = 10, Max = 101.

Achievable percentage is 46.74%.

Filling in the gaps in the plane – computer evidence

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

$n = 7.$



Min = 12, Max = 197.

Achievable percentage is 52.15%.

Filling in the gaps in the plane – computer evidence

n	2	3	4	5	6	7	8	9	10
min	2	4	6	8	10	12	14	16	18
max	2	6	18	46	101	197	351	583	916

$n = 8.$



Min = 14, Max = 351.

Achievable percentage is 58.88%.

Percent achievable

n	4	5	6	7	8
min	6	8	10	12	14
max	18	46	101	197	351
%	61.53%	61.53%	46.74%	52.15%	58.88%

$n = 8.$



Min = 14, Max = 351.

Filling in the gaps in the plane

Theorem

The following number of orderings are achievable by some configuration of n points in the plane.

① *At the bottom, all k satisfying*

$$\min = 2n - 2 \leq k \leq \frac{1}{2}n^2 - \frac{1}{2}n + 1$$

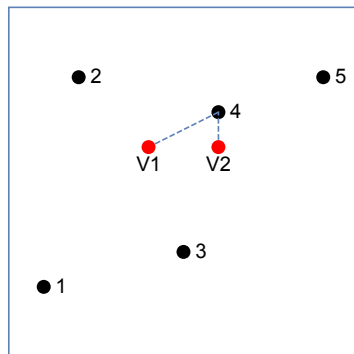
are possible.

② *At the top, all k satisfying*

$$\max - \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq k \leq \max.$$

Two vantage points

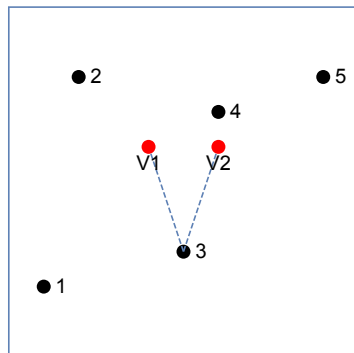
Given n points in the plane, locate **two distinct vantage points** V_1 and V_2 . Compute the **average distance from V_1 and V_2 to each point**.



4

Two vantage points

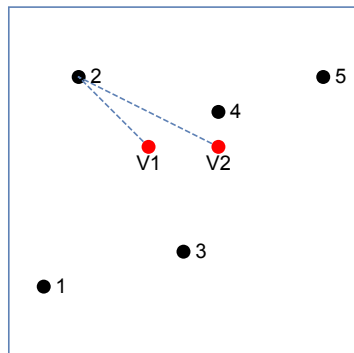
Given n points in the plane, locate **two distinct vantage points** V_1 and V_2 . Compute the **average distance from V_1 and V_2 to each point**.



43

Two vantage points

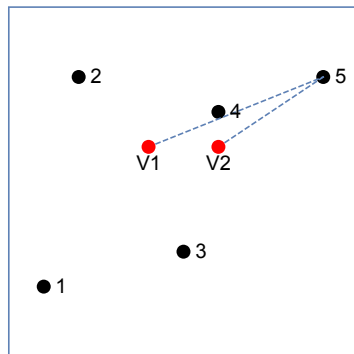
Given n points in the plane, locate **two distinct vantage points** V_1 and V_2 . Compute the **average distance from V_1 and V_2 to each point**.



432

Two vantage points

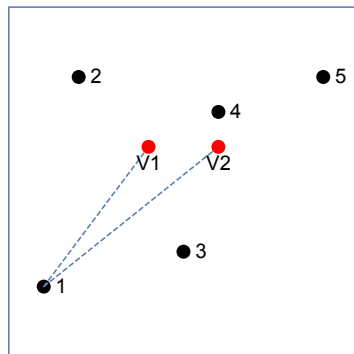
Given n points in the plane, locate **two distinct vantage points** V_1 and V_2 . Compute the **average distance from V_1 and V_2 to each point**.



4325

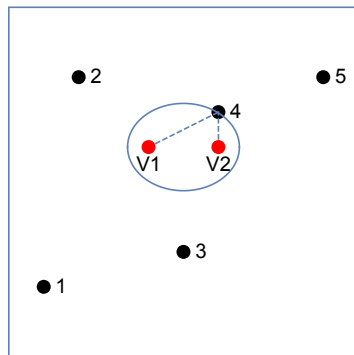
Two vantage points

Given n points in the plane, locate **two distinct vantage points** V_1 and V_2 . Compute the **average distance from V_1 and V_2 to each point**.



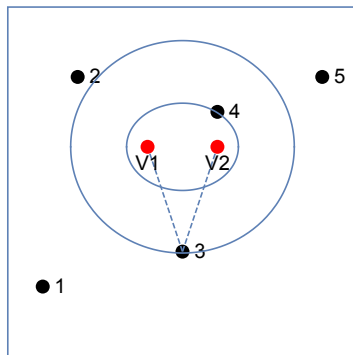
43251

Expanding ellipses



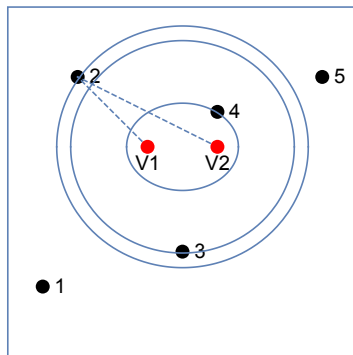
4

Expanding ellipses



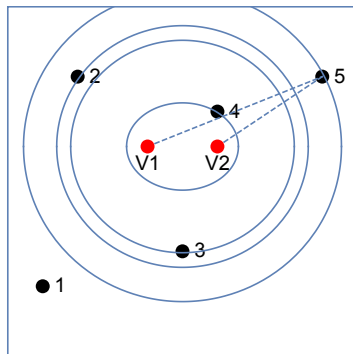
43

Expanding ellipses



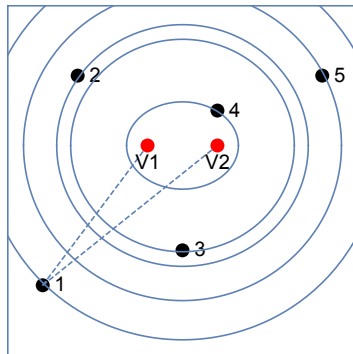
432

Expanding ellipses



4325

Expanding ellipses



43251

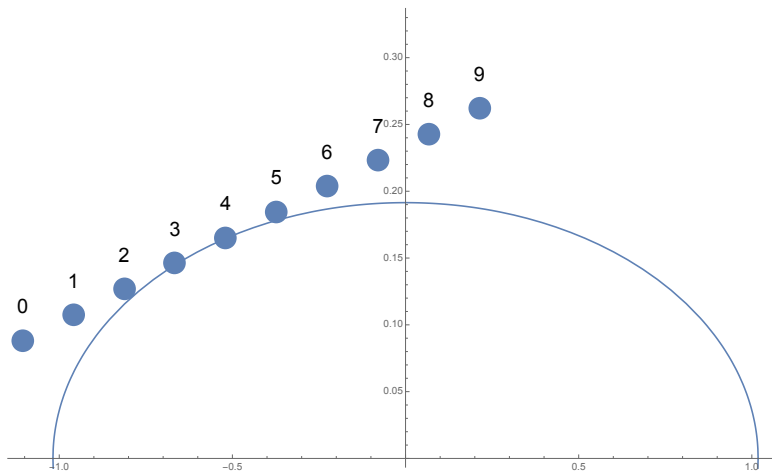
Max and min for two vantage points in the plane

Problem: Determine the max and the min for the number of orderings produced with two vantage points.

n	2	3	4	5	6	7	8	9	10
min	2	4	8	16	30	54	94	160	268
max	2	6	24	120	$\geq 680^*$?	?	?	?

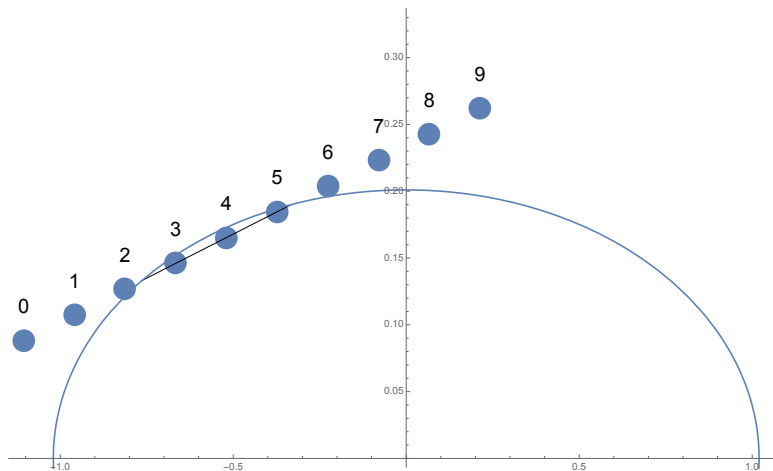
* Charles Kulick reported that “this took an entire weekend of computation on two rows of Lafayette laptops.”

Two vantage points, minimum number of orderings



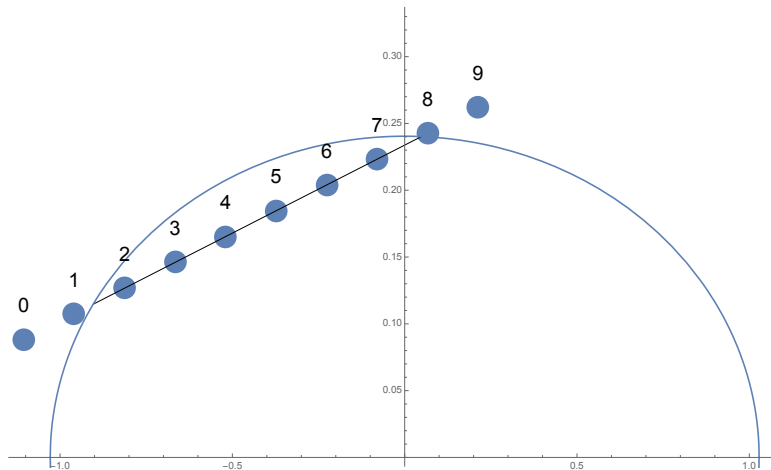
43

Two vantage points, linear point-sets



4352

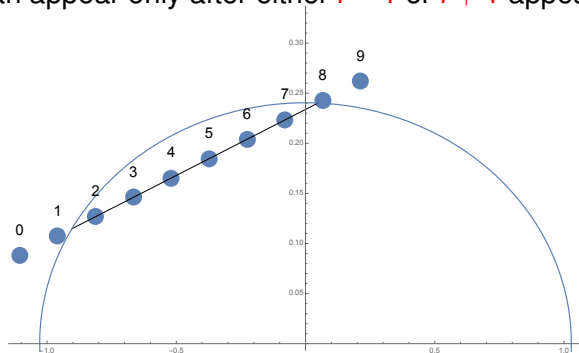
Two vantage points, linear point-sets



4352678910

Two vantage points, linear point-sets

Note: Betweenness property of linear sequences: After the first position, i can appear only after either $i - 1$ or $i + 1$ appears.



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First bound: The minimum number of possible orderings is $\leq 2^{n-1}$.

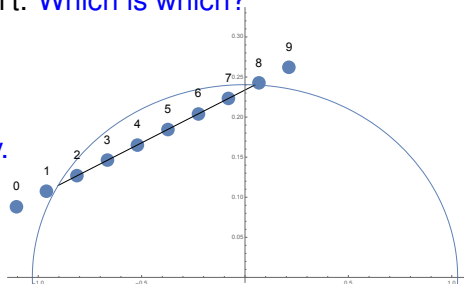
Two vantage points, linear point-sets

Quiz: Suppose we have 8 points equally spaced on a line. One of these is possible, and the other isn't. Which is which?

① 4,3,2,5,6,7,1,8

② 4,3,5,2,6,7,1,8

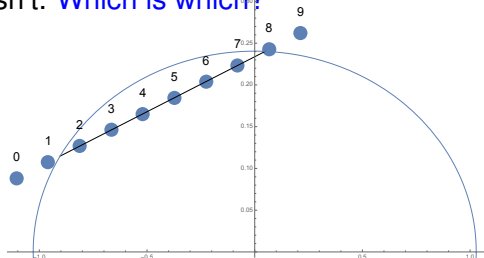
Both satisfy betweenness property.



Two vantage points, linear point-sets

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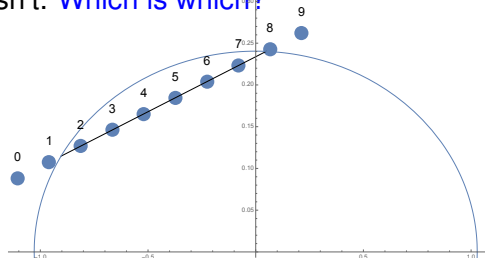
- 1 4,3,2,5,6,7,1,8
- 2 4,3,5,2,6,7,1,8 is possible!



Two vantage points, linear point-sets

Quiz: Suppose we have 8 points equally spaced on a line. One of these is possible, and the other isn't. Which is which?

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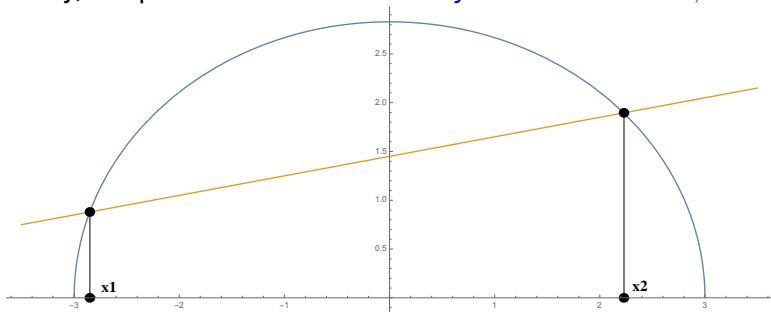


4,3,2,5,6,7,1,8 is impossible because the sequence has two consecutive downs and two consecutive ups:

DDUUUDU

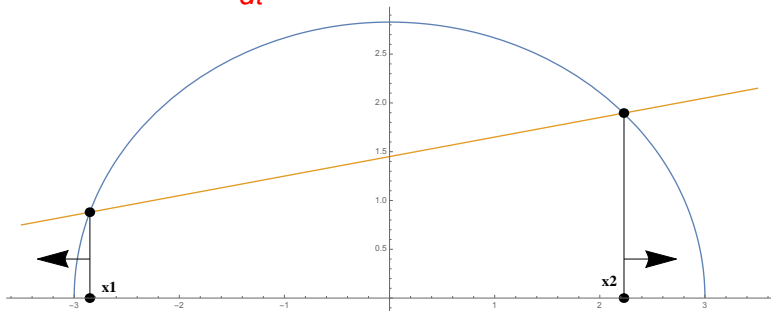
Velocity

- Vantage points are located at $(\pm 1, 0)$.
- Let t represent the (changing) positive x -intercept.
- Then ellipse equation is $\frac{x^2}{t^2} + \frac{y^2}{t^2 - 1} = 1$.
- Finally, the points reside on the line $y = mx + b$ has $m, b > 0$.

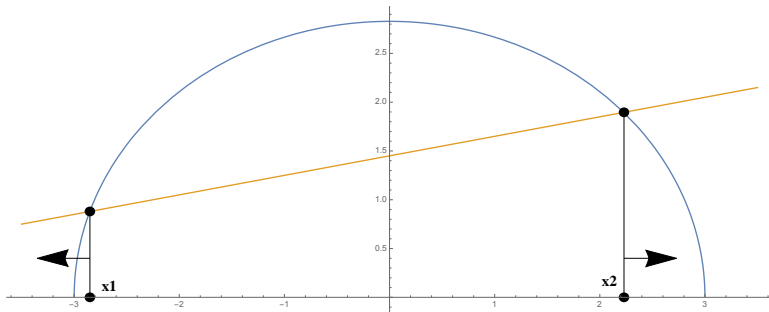


Velocity

- Let $x_1(t)$ and $x_2(t)$ be the x-coordinates of the two intersection points. Let $v_i(t) = \frac{dx}{dt}$ evaluated at $x = x_i$, with $i = 1, 2$.



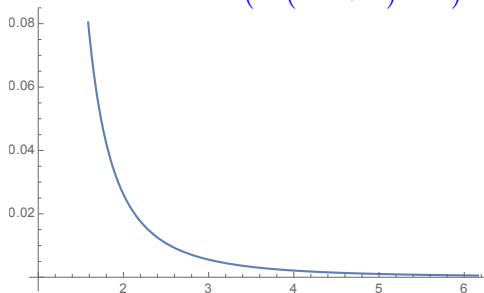
Velocity



$$|v_2(t)| - |v_1(t)| = \frac{4bmt}{(t^2(m^2 + 1) - 1)^2}$$

Velocity

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Difference of speeds.

Note: The point on the right moves faster than the point on the left.

Upper bound on minimum

First, some data:

n	1	2	3	4	5	6	7	8	9	10
min	1	2	4	8	16	30	54	94	160	268

This counts the number of sequences of 0's and 1's of length n that have the property that **consecutive 0's and consecutive 1's cannot both appear except at the beginning or the end of the sequence.**

For instance, the sequence

- 11011101000 is good, but
- but 10001100 is bad.

This is $2a_{n-1}$ where a_n is OEIS sequence A000126.

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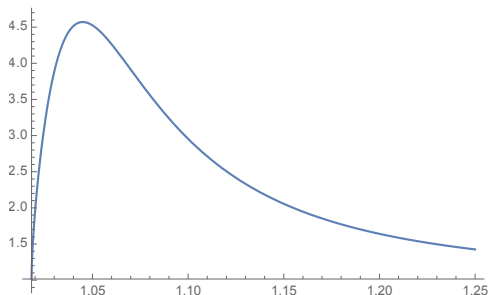
- 11011101000 is good, but
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This is $2a_{n-1}$ where a_n is OEIS sequence A000126.

Very cool fact: This bound equals $2f_{n+4} - 2n - 4$, where f_n is the n^{th} Fibonacci number.

Unfortunately, ...

This bound is too big.



Ratio of speeds. There is a unique max.

Conclusion: Assume $b, m > 0$. Suppose the sequence generated has “up-blocks” a_1, a_2, \dots . Then the up blocks form a **unimodal sequence**, i.e.,

$$a_1 \leq a_2 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_r.$$

Things to do

- For one vantage point:
 - ▶ characterize the permutations that appear;
 - ▶ fill in more of the gaps.
- For two vantage points:
 - ▶ Find any reasonable bounds on the max and the min.
 - ▶ extend to higher dimensions.

