Zero-Locus and Activity-Locus of the Two–Terminal Reliability Polynomial

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Inspiration

This talk is inspired by [1].

[1] Jason Brown and Corey D. C. DeGagné.
Roots of two-terminal reliability polynomials.
Networks, 78(2):153–163, 2021.

Motivation

We have a family of graphs $\mathcal{G}$ with a graph polynomial $G \mapsto Z_G$.

Strategy

Define a *well chosen* set of functions $\mathcal{F}$ and define

- The Zero-Locus ($\overline{\mathcal{Z}}$): The closure of

$$\mathcal{Z} = \{z \in \mathbb{C} : Z_G(z) = 0 \text{ for some } G \in \mathcal{G}\}.$$

- The Activity-Locus ($\mathcal{A}$): Parameters around which $\mathcal{F}$ behaves chaotically.
- The Density-Locus ($\overline{\mathcal{D}}$): Closure of parameters z_0 such that $\{f(z_0) : f \in \mathcal{F}\}$ is *dense*.

Prove that $\overline{\mathcal{Z}} = \mathcal{A} = \overline{\mathcal{D}}$ and use this to prove nice things.

We used this strategy in the case that $\mathcal{G}$ is the set of bounded degree graphs for

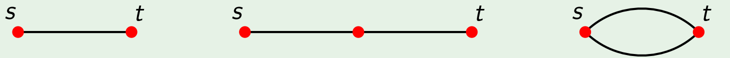
- The ferromagnetic Ising model [B.-Galanis-Patel-Regts '20]
- The hard-core model [de Boer-B.-Guerini-Peters-Regts '21]

Recap on the Two–Terminal Reliability Polynomial

Let G be a multigraph with vertices $s, t \in V(G)$. For $p \in [0,1]$ let every edge of G be independently *operational* with probability p. Denote the probability that the resulting subgraph has an (s,t)-path by

$$\mathrm{Rel}_{s,t}(G;p).$$

Examples



p $\qquad$ p^2 $\qquad$ $1-(1-p)^2$

We let

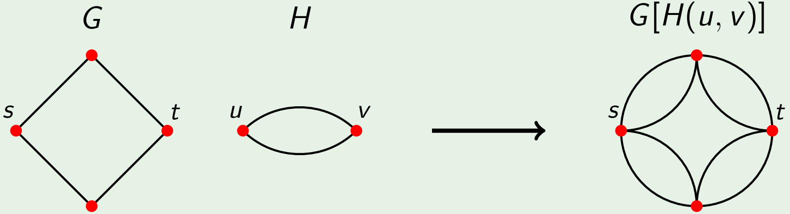
$$\mathcal{F}_{\mathrm{rel}} = \{p \mapsto \mathrm{Rel}_{s,t}(G;p) : \text{for all multigraphs } G \text{ with terminals } s,t\}.$$

They really are polynomials.

Recap on the Two–Terminal Reliability Polynomial

Let (G, s, t) and (H, u, v) be graphs with two terminals. We create a new graph $(G[H(u, v)], s, t)$ by replacing every edge of G with a copy of H.

Example

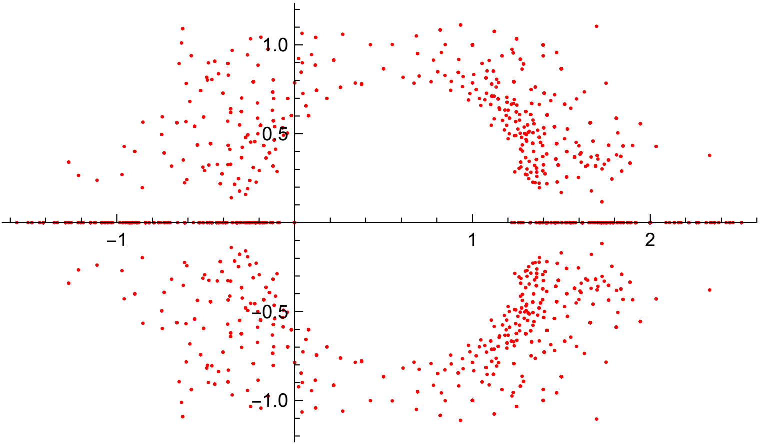


On the level of two–terminal reliability polynomials this has the effect of composition, i.e.

$$\mathrm{Rel}_{s,t}(G[H(u, v)]; p) = \mathrm{Rel}_{s,t}(G; \mathrm{Rel}_{u,v}(H; p)).$$

So $f, g \in \mathcal{F}_{\mathrm{rel}}$ implies $f \circ g \in \mathcal{F}_{\mathrm{rel}}$.

The Zero-Locus



Roots of all two–terminal reliability polynomials of graphs with at most 7 edges

The Zero-Locus

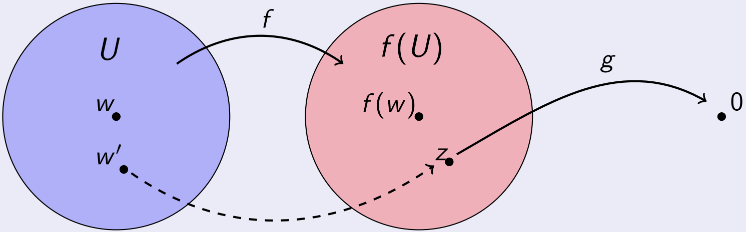
Define *the zero-locus* as the closure of

$$\mathcal{Z} = \{w \in \mathbb{C} : f(w) = 0 \text{ for some non-zero } f \in \mathcal{F}_{\mathrm{rel}}\}.$$

Lemma

Let $w \in \mathbb{C}$, suppose there exists a non-constant $f \in \mathcal{F}_{rel}$ such that $f(w) \in \overline{\mathcal{Z}}$, then $w \in \overline{\mathcal{Z}}$.

Proof.



Normal families

Let $\mathcal{F}$ be a set of rational functions $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Definition

For an open subset $U \subseteq \widehat{\mathbb{C}}$ we say that $\mathcal{F}$ is *normal* on U if every sequence $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}$ has a subsequence that converges to a limit $f : U \to \widehat{\mathbb{C}}$ uniformly on compact subsets of U.

Definition

We say that a parameter $z_0 \in \widehat{\mathbb{C}}$ is active for $\mathcal{F}$ if $\mathcal{F}$ is not normal on any neighborhood of z_0. The *activity-locus* of $\mathcal{F}$ is the set of all active parameters.

Definition

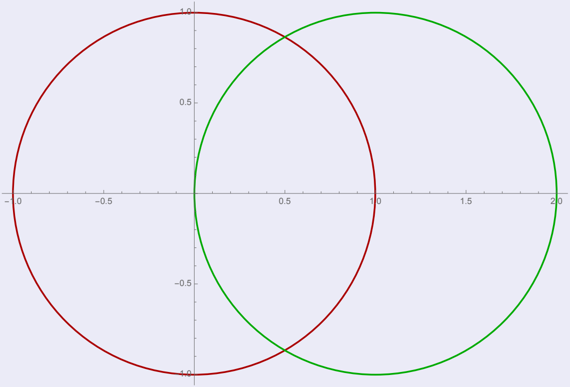
The Julia set of a rational function f is the activity-locus of $\{f^{\circ n}\}_{n\geq 1}$.

Example

Lemma

The Julia sets of f and p given by $f(p) = p^2$ and $g(p) = 1 - (1-p)^2$ are $C(0,1)$ and $C(1,1)$ respectively.

Proof.



The Activity-Locus

Definition

We define $\mathcal{A}$ to be *the activity-locus* of $\mathcal{F}_{\mathrm{Rel}}$.

For any $f \in \mathcal{F}_{\mathrm{Rel}}$ we have $\{f^{\circ n}\}_{n\geq 1} \subseteq \mathcal{F}_{\mathrm{Rel}}$. Therefore, the Julia set of f is contained in $\mathcal{A}$. So

$$C(0,1) \cup C(1,1) \subset \mathcal{A}.$$

Theorem

The activity-locus is equal to the zero-locus, i.e. $\overline{\mathcal{Z}} = \mathcal{A}$.

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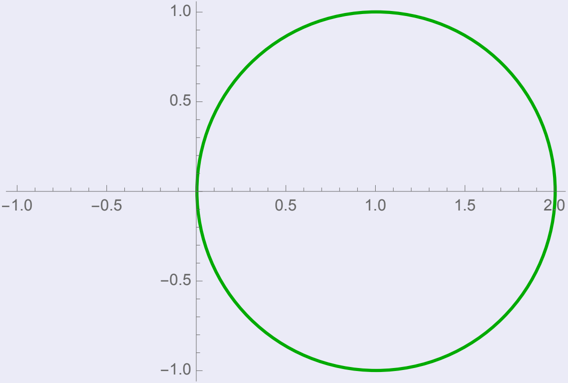
Proof.

Let $z_0 \in \overline{\mathcal{Z}}$ and let U be any neighborhood of z_0.



□

Theorem

The activity-locus is equal to the zero-locus, i.e. $\overline{\mathcal{Z}} = \mathcal{A}$.

Proof.

Let $z_0 \in \mathcal{A}$ and let U be any neighborhood of z_0.

Theorem (Montel's theorem)

Let $\mathcal{F}$ *be a family of polynomials and* $U \subseteq \mathbb{C}$ *an open set. If*

$$\bigcup_{f \in \mathcal{F}} f(U)$$

omits two distinct values in $\mathbb{C}$*, then* $\mathcal{F}$ *is normal on* U*.*

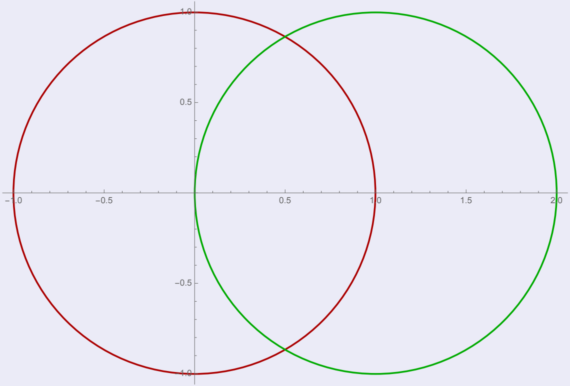
Thus there is a $z \in U$ and $f \in \mathcal{F}_{\mathrm{Rel}}$ such that $f(z) \in \{0, 2\}$, so either $f(z) = 0$ or $1 - (1 - f(z))^2 = 0$. $\square$

Some Results

Lemma

Zeros are dense in $B(0,1)$ *and* $B(1,1)$*, i.e.* $B(0,1) \cup B(1,1) \subset \overline{\mathcal{Z}}$.

Proof.



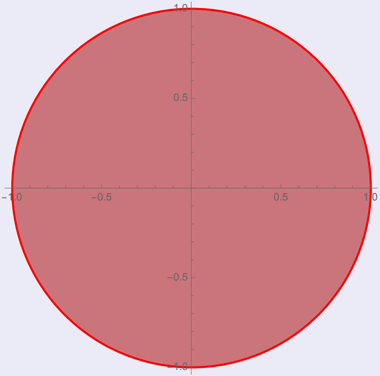
□

Some Results

Corollary

Zeros lie in the interior of the zero-locus, i.e. $\mathcal{Z} \subset \mathrm{int}(\overline{\mathcal{Z}})$.

Proof.



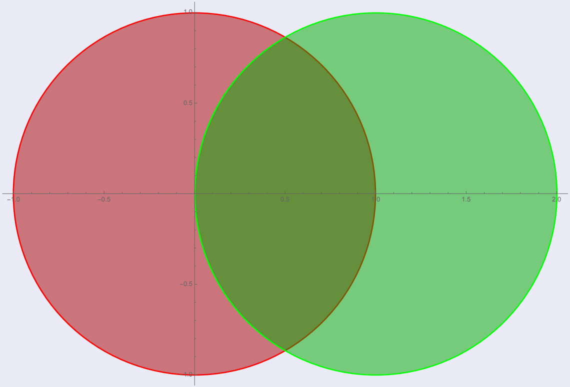
$\square$

Some Results

Lemma (Theorem 3.1. in [Brown, DeGagné])

Zeros are dense in a neighborhood of $\overline{B(0,1) \cup B(1,1)}$.

Proof.

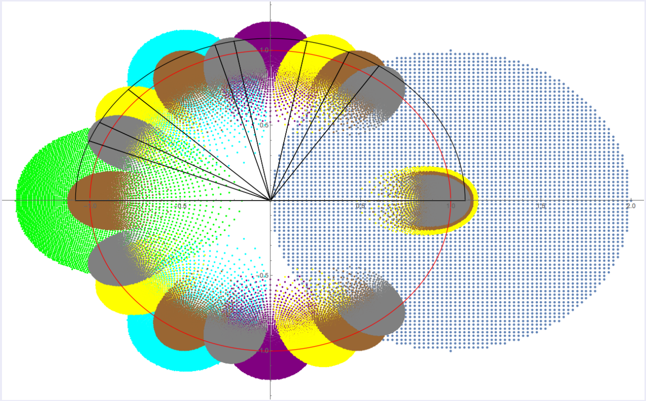


Some Results

Lemma

Zeros are dense in $B(0, 1.08) \cup B(1, 1.08)$.

Proof.



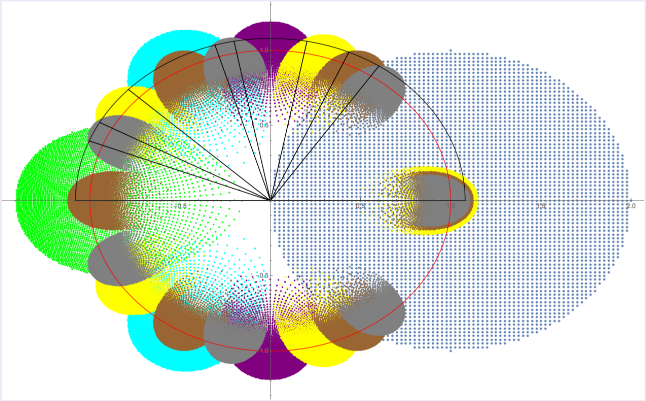
□

Some Results

Lemma

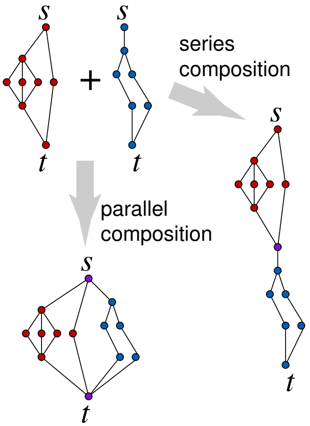
Zeros are dense in $B(0, 1.08) \cup B(1, 1.08)$.

Proof.



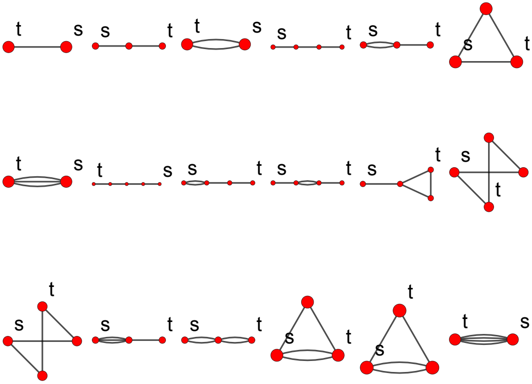
Series–Parallel Graphs

Given two graphs (G, s, t) and (H, u, v) with two terminals we can compose them in the following two ways



Series–Parallel Graphs

The graphs that can be formed by applying these two operations starting with single edges are called *series–parallel* graphs.



Series–parallel graphs with at most 4 edges.

Series–Parallel Graphs

Denote the family of two–terminal reliability polynomials of series–parallel graphs by $\mathcal{F}_{\mathrm{SP}}$. The set $\mathcal{F}_{\mathrm{SP}}$ can be defined as the smallest set satisfying:

- The constant polynomial $p \mapsto p$ is an element of $\mathcal{F}_{\mathrm{SP}}$.
- If $f, g \in \mathcal{F}_{\mathrm{SP}}$ then both $p \mapsto f(p) \cdot g(p)$ and $p \mapsto 1 - (1 - f(p))(1 - g(p))$ are elements of $\mathcal{F}_{\mathrm{SP}}$.

For $\mathcal{F}_{\mathrm{SP}}$ we can also define a zero-locus and an activity-locus. Everything I proved up until now is also true for this zero-locus and activity-locus.

Theorem

The zero-locus of series–parallel graphs is contained in $\overline{B(0,\phi) \cup B(1,\phi)}$*, where* $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.61803$*.*

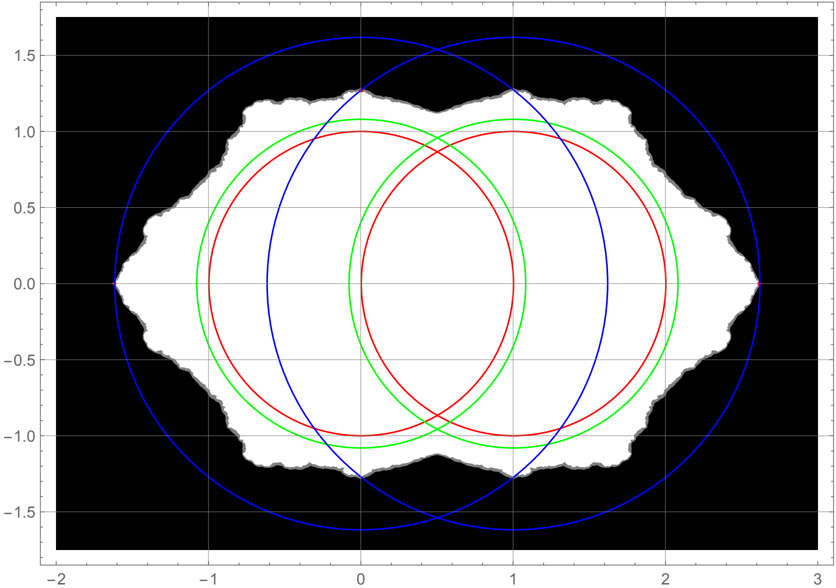


Image of the zero-locus of series–parallel graphs

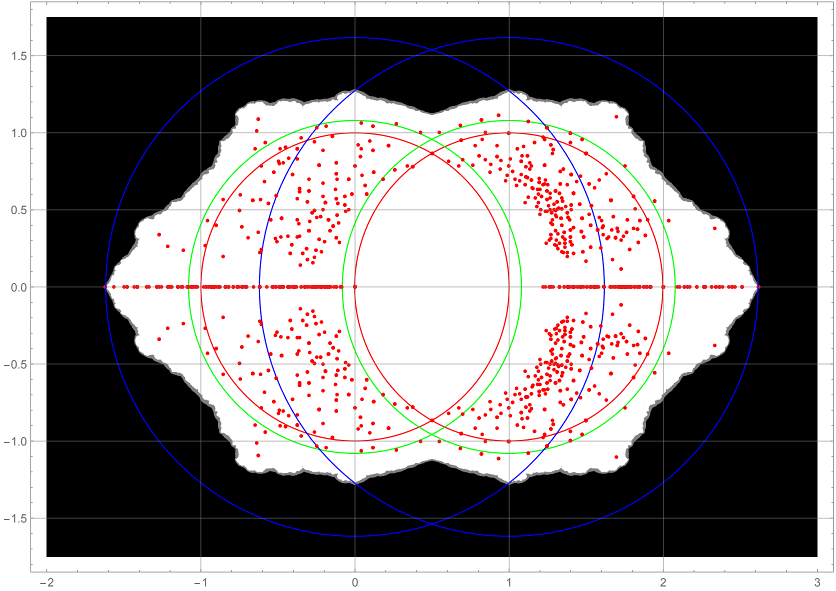


Image of the zero-locus of series–parallel graphs

The Density-Locus

Define *the density-locus* as the closure of

$$\mathcal{D} = \{w \in \mathbb{C} : \{f(w) : f \in \mathcal{F}_{\mathrm{Rel}}\} \text{ is dense in } \mathbb{C}\}.$$

Theorem

The density-locus is equal to the zero-locus, i.e. $\overline{\mathcal{D}} = \overline{\mathcal{Z}}$.

Proof.

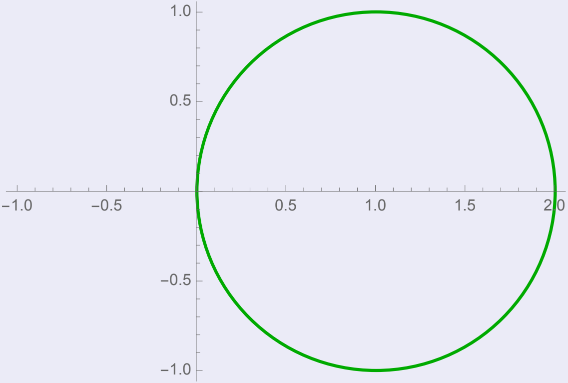
If $w \in \mathcal{D}$ then there is an $f \in \mathcal{F}_{\mathrm{Rel}}$ such that $f(w) \in B(0,1)$ and thus $w \in \overline{\mathcal{Z}}$. □

Theorem

The density-locus is equal to the zero-locus, i.e. $\overline{\mathcal{D}} = \overline{\mathcal{Z}}$.

Proof.

Let $z_0 \in \overline{\mathcal{Z}}$ and let U be any neighborhood of z_0.



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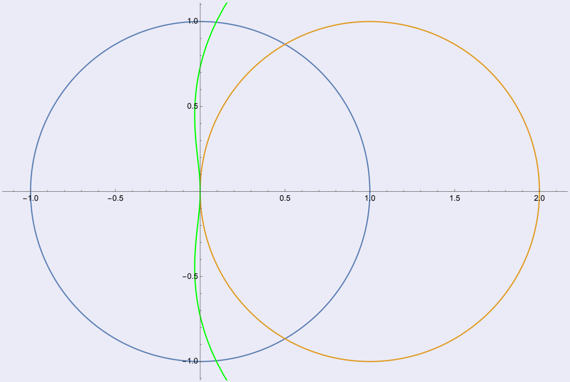

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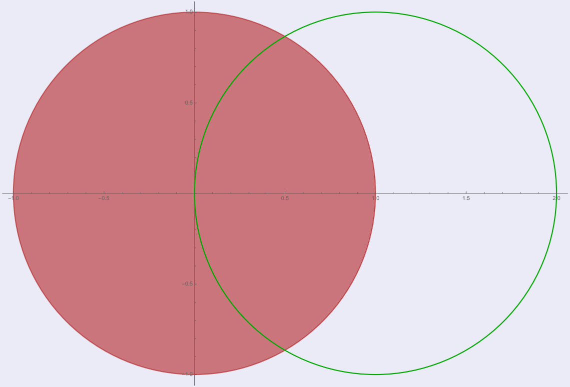
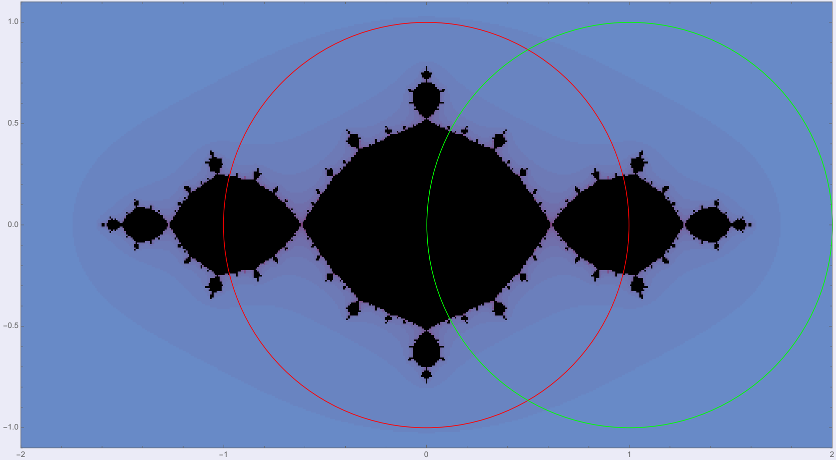


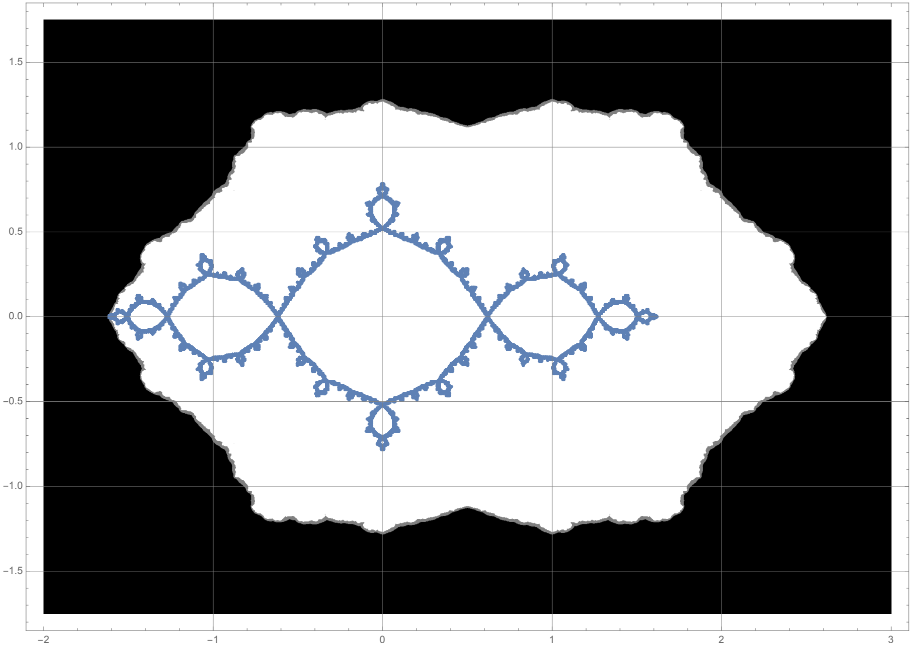
$\square$

Theorem

The zero-locus contains an open neighborhood of the real interval $(-\phi, \phi + 1)$, where $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.61803$.

Proof.




1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2
−1
0
1
2
3

Open questions

- Is the zero-locus of $\mathcal{F}_{\mathrm{rel}}$ bounded?
- Can we prove topological properties of the zero-loci of $\mathcal{F}_{\mathrm{rel}}$ and $\mathcal{F}_{\mathrm{SP}}$, e.g. (simply-)connectedness or (star-)convexity?
- Are there other families of graphs for which the corresponding set of functions $\mathcal{F}$ has interesting properties?
- Can the framework be adapted for the all-terminal reliability polynomial?
- Can the framework be used to tackle problems in computational complexity?

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