The Hamilton-Waterloo Problem

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Joint work with

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Outline

- Triple Systems
- Oberwolfach Problem
- Generalised Oberwolfach Problem
- Hamilton Waterloo Problem
- Hamilton Waterloo Problem for Uniform Cycle Lengths
  - Even cycles
  - Odd Cycles
  - Opposite Parities

Kirkman's Schoolgirl Problem

In (1847) Rev. T.P. Kirkman posed the following riddle:

Fifteen young ladies in a school walk out three abreast for seven days in succession:
it is required to arrange them daily so that no two shall walk twice abreast.

Girls are numbered from 0 to 14, the following is a solution:

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
0, 5, 10	0, 1, 4	1, 2, 5	4, 5, 8	2, 4, 10	4, 6, 12	10, 12, 3
1, 6, 11	2, 3, 6	3, 4, 7	6, 7, 10	3, 5, 11	5, 7, 13	11, 13, 4
2, 7, 12	7, 8, 11	8, 9, 12	11, 12, 0	6, 8, 14	8, 10, 1	14, 1, 7
3, 8, 13	9, 10, 13	10, 11, 14	13, 14, 2	7, 9, 0	9, 11, 2	0, 2, 8
4, 9, 14	12, 14, 5	13, 0, 6	1, 3, 9	12, 13, 1	14, 0, 3	5, 6, 9

A solution to this problem is an example of a Kirkman triple system.

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1, 6, 11	2, 3, 6	3, 4, 7	6, 7, 10	3, 5, 11	5, 7, 13	11, 13, 4
2, 7, 12	7, 8, 11	8, 9, 12	11, 12, 0	6, 8, 14	8, 10, 1	14, 1, 7
3, 8, 13	9, 10, 13	10, 11, 14	13, 14, 2	7, 9, 0	9, 11, 2	0, 2, 8
4, 9, 14	12, 14, 5	13, 0, 6	1, 3, 9	12, 13, 1	14, 0, 3	5, 6, 9

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Kirkman Triple Systems

Can be easily greneralised to arbitrary v.

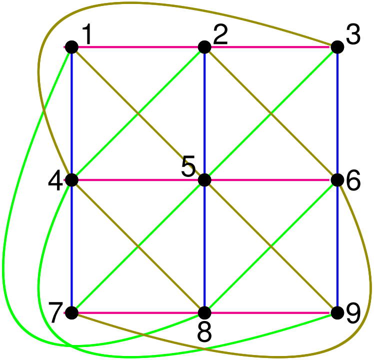
A Triangle Factor of a graph G is a spanning subgraph of G, every component of which is a triangle.

A Triangle Factorisation is a partition of the edges G into triangle factors.

A Kirkman Triple System, KTS(v), is a asks for a Triangle Factorisation of the complete graph on v points K_v.

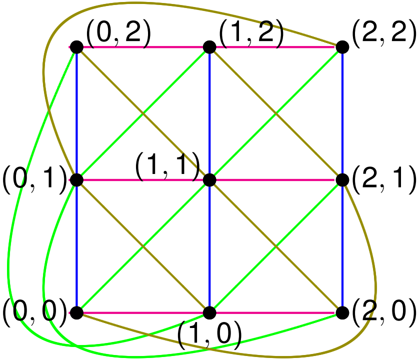
Example: KTS(9)

{1,2,3}	{4,5,6}	{7,8,9}
{1,6,8}	{2,4,9}	{3,5,7}
{1,5,9}	{2,6,7}	{3,4,8}
{1,4,7}	{2,5,8}	{3,6,9}



Example: Affine Plane

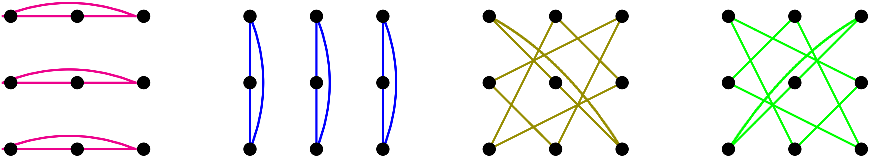
$$
\begin{array}{lll}
y = c & \pmod 3 & c = 0, 1, 2 \\
x + 2y = c & \pmod 3 & c = 0, 1, 2 \\
x + y = c & \pmod 3 & c = 0, 1, 2 \\
x = c & \pmod 3 & c = 0, 1, 2
\end{array}
$$



4 edge disjoint Triangle Factors

Example

KTS(9), $r = 4$



Theorem (Ray-Chaudhuri and Wilson (1971))

A KTS(v) exists if and only if $v \equiv 0 \bmod 3$

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The Oberwolfach problem can be thought of as a generalisation of Kirkman Triple Systems

The Oberwolfach problem

The Oberwolfach problem was posed by Ringel in the 1960s.
At the Conference center in Oberwolfach, Germany

The Oberwolfach problem was originally motivated as a seating problem:

Oberwolfach Problem

In the 1960s, Ringel posed the following problem:

- There are v mathematicians attending a conference.
- The dining venue has t round tables, which seat $m_1, m_2, \ldots, m_t$ people (where $\sum_{i=1}^{t} m_i = v$).
- Can the attendees be seated over r successive days of the conference in such a way that every person is seated next to every other person exactly once?

2-Factorizations

- A 2-factor of a graph G is a spanning 2-regular subgraph.
- A 2-factor F containing α_i cycles of length m_i, $1 \leq i \leq t$, will be denoted $F = [{m_1}^{\alpha_1}, {m_2}^{\alpha_2}, \ldots, {m_t}^{\alpha_t}]$.
- If $F = [m^t]$, we will call it uniform and refer to a C_m-factor.
- A 2-factorization is a decomposition of a graph G into 2-factors.
- If $\mathcal{H}$ is a collection of 2-factors of G, an $\mathcal{H}$-factorization is a 2-factorization in which every 2-factor is isomorphic to an element of $\mathcal{H}$.
- If $\mathcal{H} = \{F\}$, we will write F-factorization.

Oberwolfach Problem

- So the Oberwolfach problem asks:

  Given a 2-factor $F = [m_1, m_2, \ldots, m_t]$, of order v
  is there an F-factorization of K_v?

- Since each 2-factor "uses" 2 edges incident with a given vertex, v must be odd. $\left(r = \frac{v-1}{2}\right)$

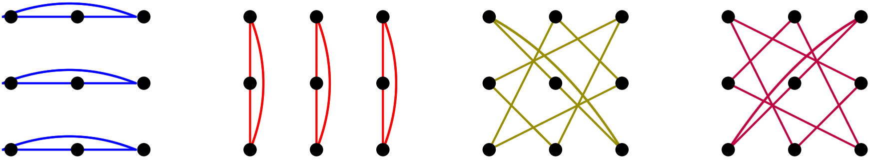
  For even v, we consider instead an F-factorization of $K_v - I$ and $\left(r = \frac{v-2}{2}\right)$.

- More generally, given a graph G and a 2-factor $F = [m_1, m_2, \ldots, m_t]$, is there an F-factorization of G?

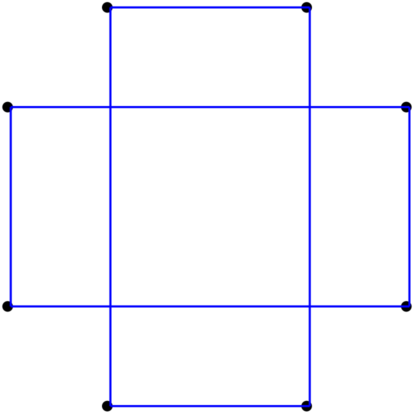
A $[3^3]$-factorization of K_9

Example

$F = [3^3], r = 4$



Example $n = 8, \quad F = [4, 4]$



A $[4, 4]-$Factor

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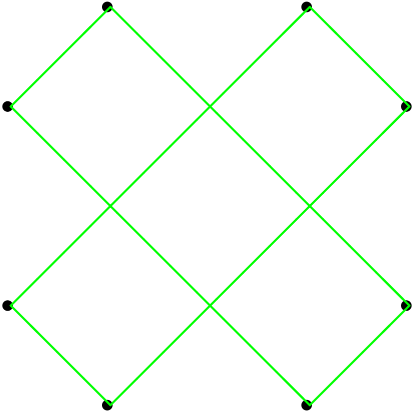

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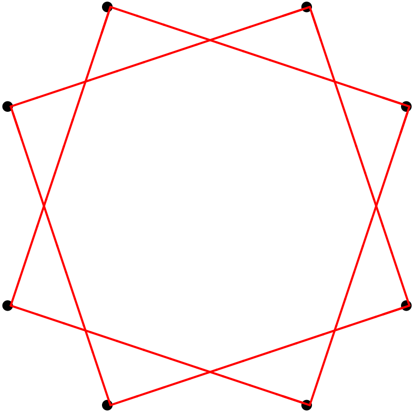

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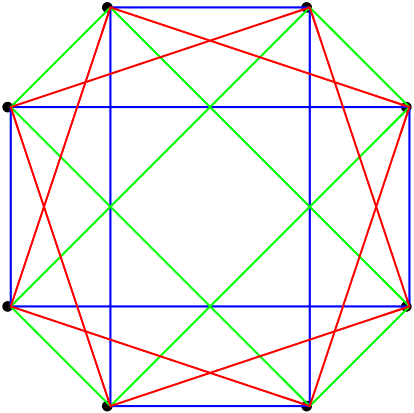

A $[4, 4]-$Factorisation of K_8

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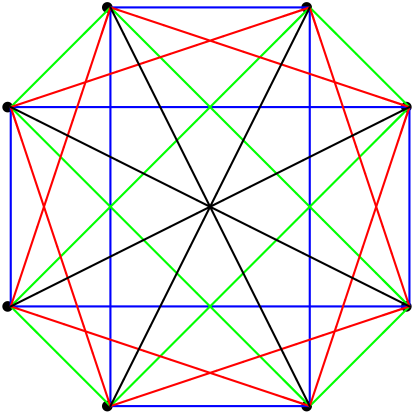


A $[4, 4]-$Factorisation of K_8 with a $1-$factor remaining

Oberwolfach problem – major known results

- $\mathrm{OP}([3^5])$ was solved by Kirkman in 1850. (15 schoolgirls problem)
- $\mathrm{OP}([v])$ was solved by Walecki in 1892.(Hamiltonian Factorization)
- There is no solution to $\mathrm{OP}([3^2])$, $\mathrm{OP}([3^4])$, $\mathrm{OP}([4,5])$, $\mathrm{OP}([3^2,5])$. These are the only known exceptions.
- Every other instance has a solution when $v \leq 60$ (Deza, Franek, Hua, Meszka, Rosa, 2010; Salassa, Dragotto, Traetta, Buratti, Della Croce, 2021+)
- $\mathrm{OP}([m^t])$ is solved (Alspach, Stinson, Schellenberg and Wagner, 1989; Hoffman and Schellenberg, 1991)
- $\mathrm{OP}([m_1, m_2])$ is solved (Traetta, 2013)
- $\mathrm{OP}(F)$ is solved when F has only even cycles (Häggkvist, 1985; Bryant and Danziger, 2011)
- The general problem is still open.

Notational Interlude: Lexicographic Products

For a graph G, $G[n]$ denotes the lexicographic product of G with $\overline{K_n}$, the independent graph on n vertices and no edges.

$V(G[n]) = V(G) \times \mathbb{Z}_n$,
$E(G[n]) = \{\{(x,a),(y,b)\} : \{x,y\} \in E(G),\ a,b \in \mathbb{Z}_n\}$.

We are particularly interested in $K_m[n]$, the multipartite graph with m parts of size n.

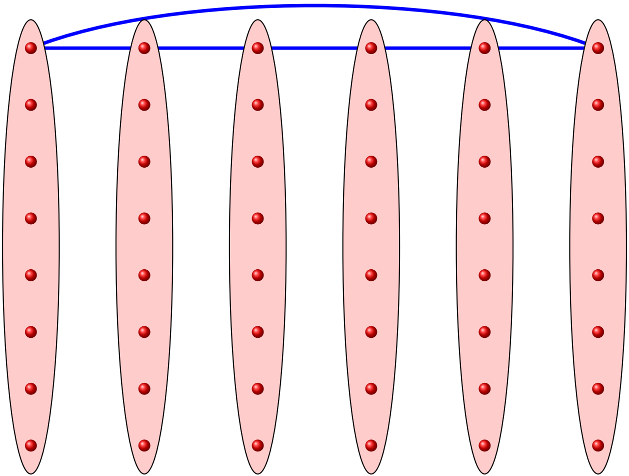
And are also interested in $C_m[n]$, where consecutive parts are joined in a cycle.

Example $C_5[7]$

We start with C_5,

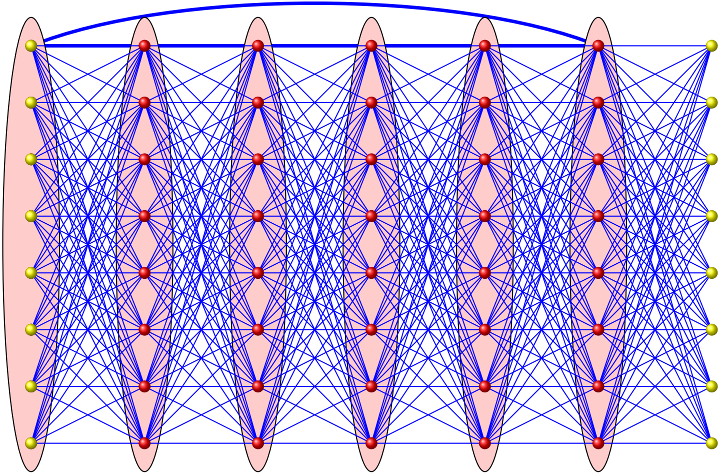
Example $C_5[7]$

We start with C_5, and "blow up" each point by 7



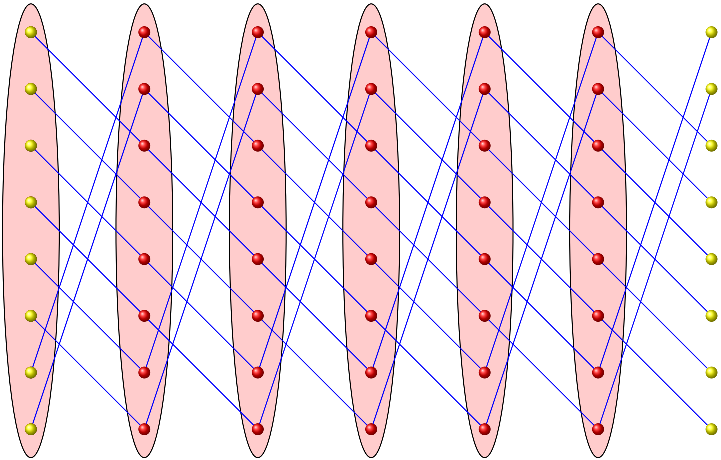
Example $C_5[7]$

Wherever G has an edge, join all blown up points



Example $C_5[7]$

We can talk about edges with difference d, here $d = 2$



Generalised Oberwolfach Problem $\mathrm{OP}(F_1, \ldots, F_t)$

Given t 2−factors $F_1, F_2, \ldots, F_t$ order v and non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_t$ such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_t = \begin{cases} \frac{v-1}{2} & v \text{ odd} \\ \frac{v-2}{2} & v \text{ even} \end{cases}$$

Find a 2-factorisation of K_v, or $K_v - I$ if v is even, in which there are exactly α_i 2-factors isomorphic to F_i for $i = 1, 2, \ldots, t$.

Asymptotic Results

Theorem (Glock, Joos, Kim, Kühn, Osthus, 2019)

For every $\eta > 0$, there exists an $v_0 \in \mathbb{N}$ such that for all odd $v \geq v_0$, given 2-regular graphs of order v, $F_1, \ldots, F_t$, and $\alpha_1, \ldots, \alpha_t \in \mathbb{N}$, with $\alpha_1 + \ldots + \alpha_t = (v-1)/2$ and $\alpha_1 \geq \eta v$ then $\mathrm{OP}(v; F_1, \ldots, F_t)$ has a solution.

Pros:

Shows eventual existence (Yay!)

Cons:

v_0 is hard to pin down - Bounds are hard to find - it is very large

Methods are probabilistic and not constructive.

Dukes, Ling 2007 Showed asymptotic existence in the Uniform Case - Wilson type Constructive methods with (very large) Explicit bounds.

Generalise to other Graphs G, $\mathrm{OP}(G; F_1, \ldots, F_t)$

We can also consider Factorisations of other graphs G.

Of particular interest are the cases when:

- $G = K_m[n]$, the multipartite complete graph with m parts of size n and;

Theorem (Liu 2003)

The complete multipartite graph $K_m[n]$, $m \geq 2$, has a 2−factorisation into k−cycles if and only if $k \mid mn$, $(n-1)m$ is even, further k is even when $n = 2$, and $(k, n, m) \notin \{(3,3,2), (3,6,2), (3,3,6), (6,2,6)\}$.

- $G = C_m[n]$ a cycle of length m "blown up" by n.

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Hamilton - Waterloo: Include the Pub ($t = 2$)

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In the Hamilton-Waterloo variant of the problem the conference has two venues ($t = 2$)

The first venue (Hamilton) has circular tables corresponding to a $2-$factor F_1 of order v.

The second venue (Waterloo) circular tables each corresponding to another $2-$factor F_2 also of order v.

Hamilton-Waterloo

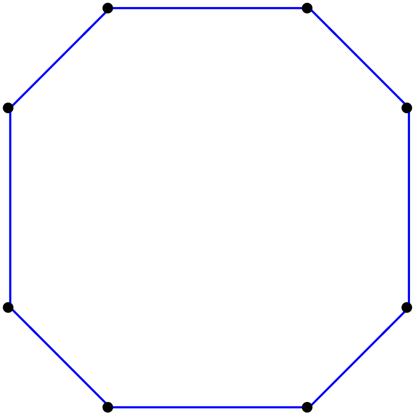
- The Hamilton-Waterloo problem thus requires a factorization of K_v (or $K_v - I$ if v is even) into two 2-factors, F_1 and F_2, with α factors of the form F_1 and β factors of the form F_2.
- Again, the number of days (times a factor appears) is
$$r = \alpha + \beta = \begin{cases} \frac{v-1}{2} & v \text{ odd} \\ \frac{v-2}{2} & v \text{ even} \end{cases} = \left\lfloor \frac{v-1}{2} \right\rfloor.$$
- We will generally assume that $\alpha, \beta > 0$, so there is at least one factor of each type.
- We denote a solution to this problem by HWP$(v; F_1, F_2; \alpha, \beta)$
- If $F_1 = [m^{t_1}]$ and $F_2 = [n^{t_2}]$, we write HWP$(v; m, n; \alpha, \beta)$. Such factors are called uniform.

Generalized Hamilton-Waterloo

More generally:

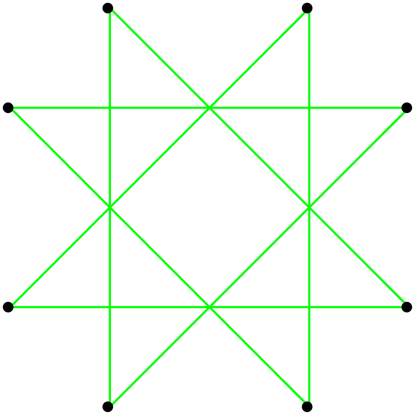
- Given a graph G, two 2-factors F_1 and F_2, and integers α and β, with $\alpha + \beta = |E(G)|/2$, is there a $\{F_1, F_2\}$-factorization of G, with α F_1 factors and β F_2 factors?
- We write $\mathrm{HW}(G; F_1, F_2, \alpha, \beta)$.
- If $F_1 = [m^{t_1}]$ and $F_2 = [n^{t_2}]$, ie they are uniform, we write $\mathrm{HWP}(G; m, n; \alpha, \beta)$.
- Of particular interest is the case when $G = C_m[n]$ a cycle of length m "blown up" by n.

Example $v = 8, \; F_1 = [8], \alpha = 2, \; F_2 = [4,4], \; \beta = 1$



An F_1−Factor of K_8

Example $v = 8$, $F_1 = [8], \alpha = 2$, $F_2 = [4, 4]$, $\beta = 1$



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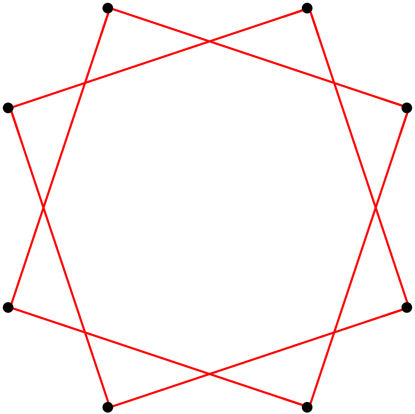
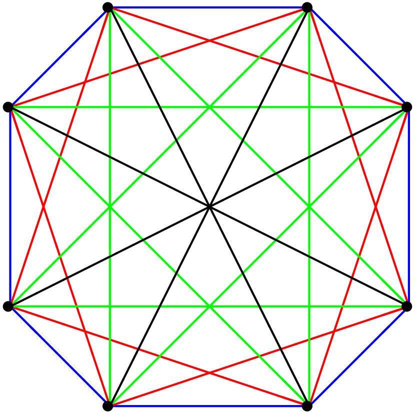


An F_2−Factor of K_8

Example $v = 8$, $F_1 = [8], \alpha = 2$, $F_2 = [4, 4]$, $\beta = 1$



HWP$(8; F_1, F_2; 2, 1)$

Hamilton? - Waterloo?

Hamilton?

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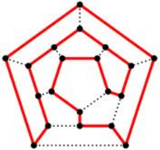
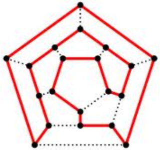


Hamiltonian?

Hamilton? - Waterloo?

Hamilton?



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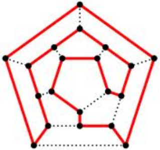

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3rd Ontario Combinatorics Workshop
McMaster University, **Hamilton**, Ontario, Feb. 1988.
University of Waterloo, **Waterloo**, Ontario, Oct. 1987;

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Organizers

Alex Rosa
McMaster University
Hamilton, Ontario

Charlie Colbourn
University of Waterloo
Waterloo, Ontario

Hamilton - Waterloo

Uniform cycle lengths

Theorem (Necessary conditions)

If $\mathrm{HWP}(v; m, n; \alpha, \beta)$ *has a solution, then* $m, n \geq 3$, $m \mid v$, $n \mid v$ *and* $\alpha + \beta = \lfloor (v-1)/2 \rfloor$.

The case $m = n$ is the Oberwolfach problem with uniform cycle lengths. So we'll assume $m \neq n$.

From now on, we'll also assume (WLOG) that $n > m \geq 3$.

Hamilton-Waterloo Problem Small Cases

Theorem (Franek and Rosa, 2000; Franek, Holub and Rosa, 2004; Adams and Bryant, 2006; (D 2021+))

If v is odd and $v \leq 17$ or v is even and $v \leq 10$ (16), there is a solution to every instance of the Hamilton-Waterloo problem, except that there is no solution to:

- $\mathrm{HWP}(7; [3, 4], [7]; 2, 1)$,
- $\mathrm{HWP}(8; [3, 5], [4^2]; 2, 1)$,
- $\mathrm{HWP}(8; [3, 5], [4^2]; 1, 2)$,
- $\mathrm{HWP}(9; [3^3], [4, 5]; 2, 2)$,
- $\mathrm{HWP}(9; [3^3], F; 3, 1)$, $F \in \{[4, 5], [3, 6], [9]\}$,
- $\mathrm{HWP}(15, [3^5], F; 6, 1)$, $F \in \{[3^2, 4, 5], [3, 5, 7], [5^3], [4^2, 7], [7, 8]\}$

Solutions to $\mathrm{HW}(v; m, n; \alpha, \beta)$ Early Results

- $(m, n) = (3, 15)$ or $(5, 15)$, v odd
(Adams, Billington, Bryant and El-Zanati, 2002);

- $(m, n) = (3, 5)$, v odd, except when $(v, \alpha, \beta) = (15, 6, 1)$ and possibly when $\beta = 1$ or $v \equiv 0 \pmod{15}$
(Adams, Billington, Bryant and El-Zanati, 2002);

- $(m, n) = (3, v)$, v odd, except possibly for 14 values of v (Dinitz and Ling, 2009). Partial solutions for v even (Lei and Shen, 2012).

- $(m, n) = (3, 4)$, (Danziger, Quattrocchi and Stevens, 2009);

Known solutions to $\mathrm{HW}(v; m, n; \alpha, \beta)$ 2016

- Sparse families of cyclic solutions have been found. (Buratti and Danziger, 2015)
- $(m, n) = (4, 2k + 1)$ (Obadasi and Ozkan, 2016)
- $(m, n) = (3, 7)$, v odd (Lei and Fu, 2016);
- $(m, n) = (3, 3x)$ v odd, Except a finite number of x values, also considered v even, (Asplund, Kamin, Keranen, Pastine and Özkan, 2016);
- Many Families (Kerenan and Pastine 2016) - Considered $K_t[w]$.

Hamilton Waterloo Even Cycle sizes

We first consider the case when both m and n are both Even.

Generalised Oberwolfach Problem and Häggkvist

Theorem (Häggkvist (1985))

Let $n \equiv 2 \mod 4$, and $F_1, \ldots, F_t$ be bipartite 2−factors of order n then $\mathrm{OP}(F_1, \ldots, F_t)$ has solution, with an even number of factors isomorphic to each F_i.

Corollary ($t = 1$)

Let $n \equiv 2 \mod 4$, and F be a bipartite 2−factor of order n then $\mathrm{OP}(F)$ has solution.

Corollary ($t = 2$)

Let $n \equiv 2 \mod 4$, and F_1, F_2 be bipartite 2−factors of order n then $\mathrm{OP}(F_1, F_2)$ (Hamilton-Waterloo) has solution where there are an even number of each of the factors. (Both α and β are even)

Häggkvist Doubling: Factoring $C_m[2]$

Lemma (Häggkvist (1985))

For any $m > 1$ and for each bipartite 2-regular graph F of order $2m$, there exists a 2-factorisation of $C_m[2]$ in which each 2-factor is isomorphic to F.

Given a cycle C

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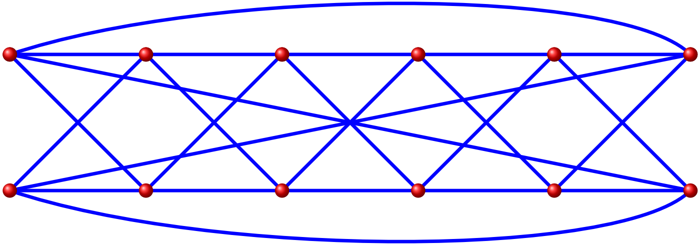
C[2]



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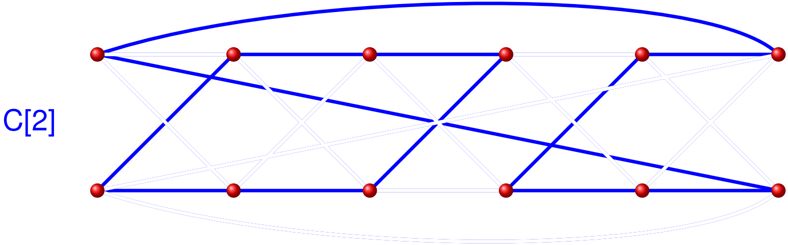
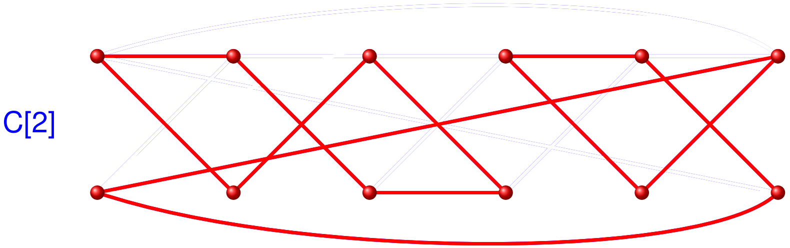


Häggkvist Doubling: Factoring $C_m[2]$

Lemma (Häggkvist (1985))

For any $m > 1$ and for each bipartite 2-regular graph F of order $2m$, there exists a 2-factorisation of $C_m[2]$ in which each 2-factor is isomorphic to F.

Given a cycle C, consider $C[2]$; Choosing a cycle in this way, leaves a factor of the same type



Häggkvist Solution to $v = 2s \equiv 2 \bmod 4$

Let s be odd, and $v = 2s \equiv 2 \bmod 4$.

Given bipartite 2−factors $F_1, \ldots, F_{\frac{m-1}{4}}$, each of order $2s$ (not necessarily distinct).

Since s is odd, K_s has a factorisation into Hamiltonian cycles H_i, $1 \leq i \leq \frac{s-1}{2}$.

Now doubling, we have a $H[2]$ factorisation of $K_s[2]$

We can factor $H_i[2]$ into 2 copies of F_i by Häggkvist doubling as above.

Result is a factorisation of $K_s[2] \cong K_{2s} \setminus I$ into pairs of factors each isomorphic to F_i, $i = 1, \ldots \frac{m-1}{4}$ plus a 1-factor I.

Even Cycle Sizes

Theorem (Bryant, Danziger 2011)

If $n \equiv 0 \bmod 4$ and $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order n and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are non-negative integers such that

- $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$,
- α_i *is even for* $i = 2, 3, \ldots, t$,
- $\alpha_1 \geq 3$ *is odd*,

then $\mathrm{OP}(F_1, \ldots, F_t)$ *has a solution with* α_i *2-factors isomorphic to* F_i *for* $i = 1, 2, \ldots, t$.

...and so...

Corollary ($t = 1$)

Let n be even and F be a bipartite $2-$factor of order n then $\mathrm{OP}(F)$ has solution.

Corollary ($t = 2$)

Let n be even and F_1, F_2 be bipartite $2-$factors of order n then $HWP(v, F_1, F_2, \alpha, \beta)$ if and only if $\alpha + \beta = \frac{v-2}{2}$ except possibly when

- $v \equiv 0 \pmod 4$ *and either* $\alpha = 1$ *or* $\beta = 1$
- $v \equiv 2 \pmod 4$ *and* α *and* β *are both odd.*

Hamilton Waterloo - Uniform Factors - Even Cycles

Let m and n both be even.

Theorem (Bryant, Danziger and Dean, 2013)

If $m \mid n$ there is a solution to HWP(v, m, n, α, β) if and only if $\alpha + \beta = \frac{v-2}{2}$ and $n \mid v$.

Theorem (Burgess, Danziger, Traetta 2019)

If $m \nmid n$, then there is a solution to HWP$(v; m, n; \alpha, \beta)$ if and only if m and n are both divisors of v and $\alpha + \beta = \frac{v-2}{2}$, except possibly when at least one of the following holds:

- $\beta = 1$;
- $\beta = 3$ and $v \equiv 2 \pmod 4$;
- $\alpha = 1$ and $m, n \equiv 0 \pmod 4$;
- $\alpha = 1$, $v \equiv 2 \pmod 4$ and $mn \nmid v$;
- $v = 2mn/\gcd(m, n) \equiv 2 \pmod 4$ α and β both odd.

Hamilton Waterloo Odd Cycle sizes

We now consider the case when both m and n are both odd.

Uniform Odd Cycles

Theorem (Burgess, Danziger, Traetta, 2018)

Let m and n be odd integers with $n > m \geq 3$ and $\alpha, \beta \geq 0$
Then $\mathrm{HWP}(v; m, n, \alpha, \beta)$ has a solution if and only if m and n are divisors of v and $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$,
except when $(v, m, \alpha, \beta) \in \{(6, 3, 2, 0), (12, 3, 5, 0)\}$ and possibly when at least one of the following holds:

1. $\beta = 1$;
2. $\beta = 3$ and $m \nmid n$;
3. $\alpha = 1$, $m \nmid n$ and $mn \nmid v$;
4. $v = s \cdot \frac{mn}{\gcd(m,m)}$ where $s \in \{1, 2, 4\}$;
5. $(v, m) \in \{(6n, 3), (18n, 3\gcd(m, n))\}$;
6. v is even and $(m, n, \beta) = (5, 7, 5)$.

Uniform Odd Cycles - v Odd

In particular, if v is odd, we have the following near-complete solution to $\mathrm{HWP}(v; m, n; \alpha, \beta)$.

Corollary (Burgess, Danziger, Traetta, 2018)

Let m, n and v be odd integers with $n \geq m \geq 3$, and let α and β be non-negative integers. Then $\mathrm{HWP}(v; m, n, \alpha, \beta)$ has a solution if and only if $m, n \mid v$ and $\alpha + \beta = \frac{v-1}{2}$, except possibly when one of the following holds:

1. $\beta = 1$;
2. $\beta = 3$ and $m \nmid n$;
3. $\alpha = 1$, $m \nmid n$ and $mn \nmid v$;
4. $v = \frac{mn}{\gcd(m,n)}$.

Hamilton Waterloo Problem for the complete equipartite graph

Theorem (Burgess, Danziger, Traetta, 2018)

Let t and w be positive integers with $t \geq 3$. Also, let m and n be odd divisors of w with $n > m \geq 3$, and let $\alpha, \beta > 0$. Then $HWP(K_t[w]; m, n, \alpha, \beta)$ has a solution if and only if $2(\alpha + \beta) = (t-1)w$, except possibly when at least one of the following conditions holds:

1. $\beta = 1$;
2. $\beta = 3$ and $m \nmid m$;
3. $\alpha = 1$, $m \nmid n$, and $mn \nmid v$;
4. $(m, n, \beta) \in \{(3, 7, 5), (5, 7, 5)\}$;
5. $(m, t, w) \in \{(3\gcd(M, N), 3, 6N), (3, 3, 2N)\}$.

Sketch: Solving Case $m \nmid n$, $v > \frac{mn}{\gcd(m,n)}$

Let $g = \gcd(m, n)$, $m = m'g$, $n = n'g$ and $v = m'n'gt = wt$.

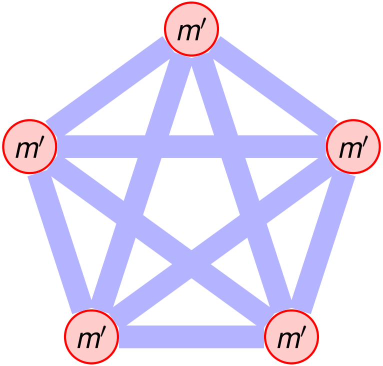
$$n > m \text{ and } m \nmid n \quad \Longrightarrow \quad n' > m' > 1,$$

and

$$v > \frac{mn}{\gcd(m, n)} \quad \Longrightarrow \quad t > 1.$$

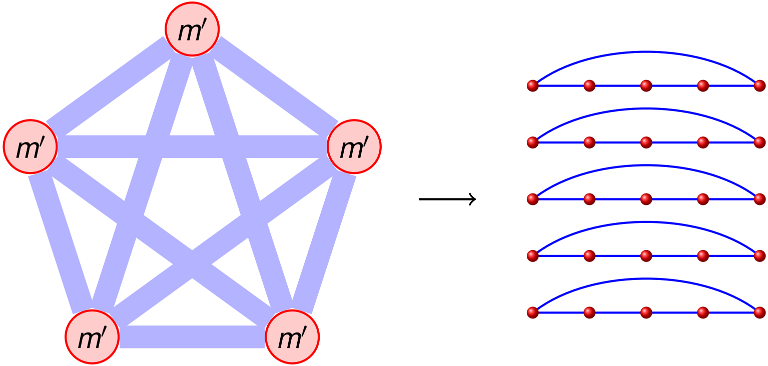
Sketch: Solving Case $m \nmid n$, $v > \frac{mn}{\gcd(m,n)}$

Start with $K_t[m']$ (the complete multipartite graph with t parts of size m').



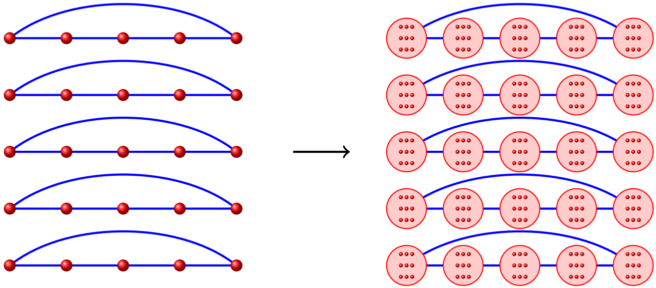
Sketch: Solving Case $m \nmid n$, $v > \frac{mn}{\gcd(m,n)}$

By the result of Liu (2003), there exists a $C_{m'}$-factorization of $K_t[m']$.



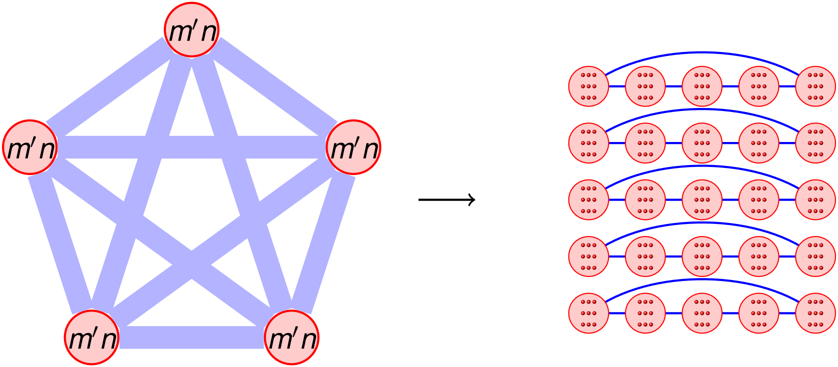
Sketch: Solving Case $m \nmid n$, $v > \frac{mn}{\gcd(m,n)}$

Blow up vertices by $n = n'g$, turning each $C_{m'}$ into a $C_{m'}[n'g]$.



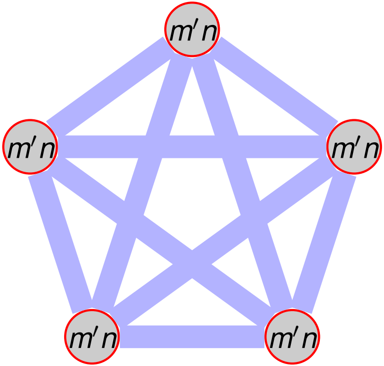
Sketch: Solving Case $m \nmid n$, $v > \frac{mn}{\gcd(m,n)}$

This gives us a $C_{m'}[n]$-factorization of $K_t[m'][n] \cong K_t[m'n'g] = K_t[w]$.

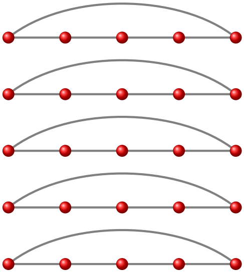


Sketch: Solving Case $m \nmid n$, $v > \frac{mn}{\gcd(m,n)}$

Fill in the parts of size $m'n = mn'$ by C_m or C_n factors as desired.

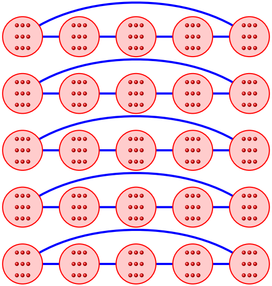


$\longrightarrow$



Sketch: Solving Case $m \nmid n$, $v > \frac{mn}{\gcd(m,n)}$

Need to be able to factor $C_{m'}[n]$ into C_m and C_n factors.



The Hamilton-Waterloo problem for $C'_m[n'g]$, $m = m'g$, $n = n'g$

Theorem (Burgess, Danziger, Traetta, 2018)

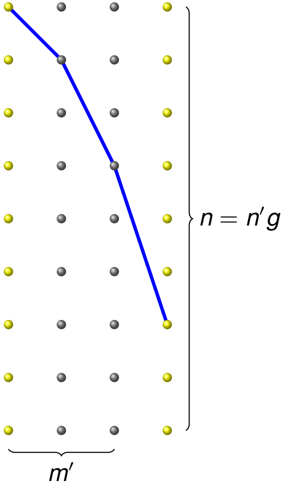
Let m and n be odd integers with $n > m \geq 3$, and let $g \neq m$ be a common divisor of m and n. Then there is a solution to $\mathrm{HW}(C_{m/g}[n]; m, n; \alpha, \beta)$ whenever $\alpha, \beta \geq 0$, $\alpha + \beta = n$ except possibly when one of the following conditions hold:

$\beta \in \{1, 3\}$ *or*

- $g = 1$, *and*
  - $\alpha = 2$ *and the smallest prime divisor of* n *is greater than* m, *or*
  - $(\alpha, M) = (4, 3)$;
- $g > 1$ *and* $\alpha = 1$.

C_m-factors in $C_{m'}[n]$

A C_m-factor is formed from m' differences with sum of order g in $\mathbb{Z}_n$.



1	2	3
2	3	1
−1	−2	−3
−3	−1	−2
−2	−3	−1

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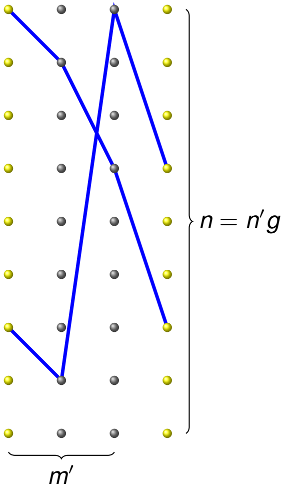

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & -2 & -3 \\ -3 & -1 & -2 \\ -2 & -3 & -1 \end{bmatrix}$$

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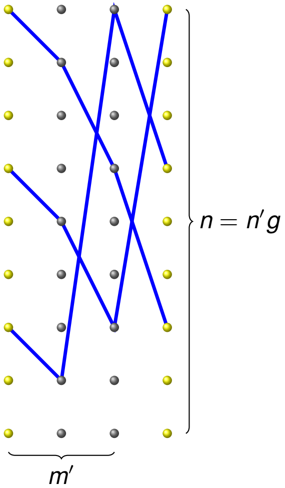

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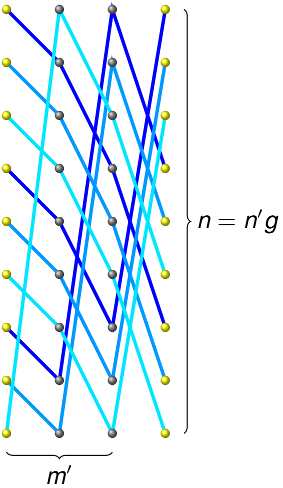

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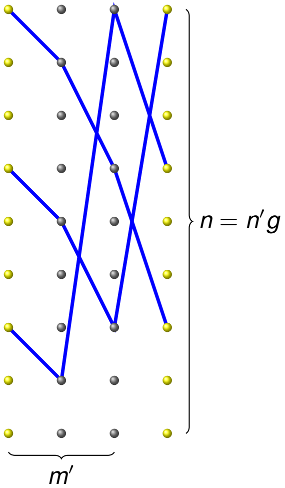

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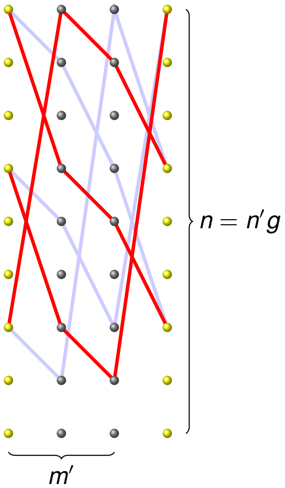

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \\ -1 & -2 & -3 \\ -3 & -1 & -2 \\ -2 & -3 & -1 \end{bmatrix}$$

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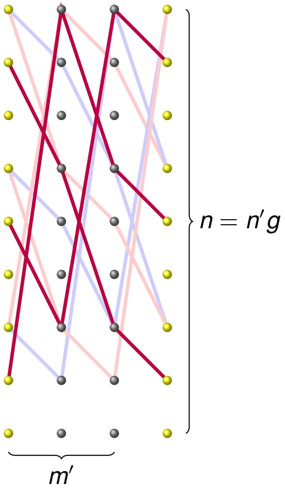

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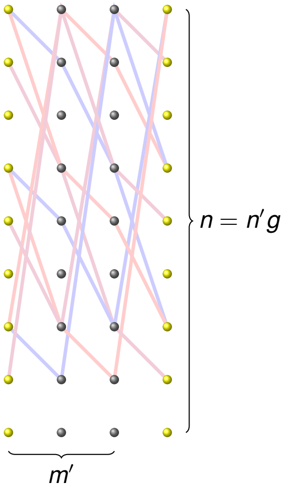

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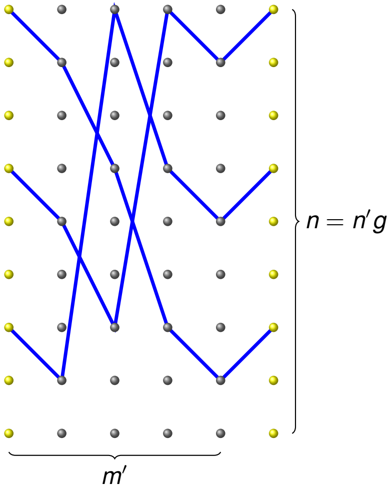
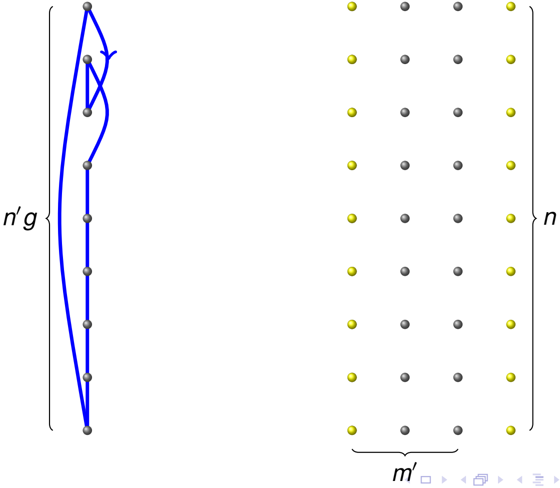


$$\begin{bmatrix} 1 & 2 & 3 & 1 & -1 \\ 3 & 1 & 2 & 2 & -2 \\ 2 & 3 & 1 & 3 & -3 \\ -1 & -2 & -3 & -1 & 1 \\ -3 & -1 & -2 & -2 & 2 \\ -2 & -3 & -1 & -3 & 3 \end{bmatrix}$$

C_n-factors in $C_{m'}[n]$

We use a projection technique from Alspach, Stinson, Schellenberg and Wagner 1989. Uses all edges $\pm d$.



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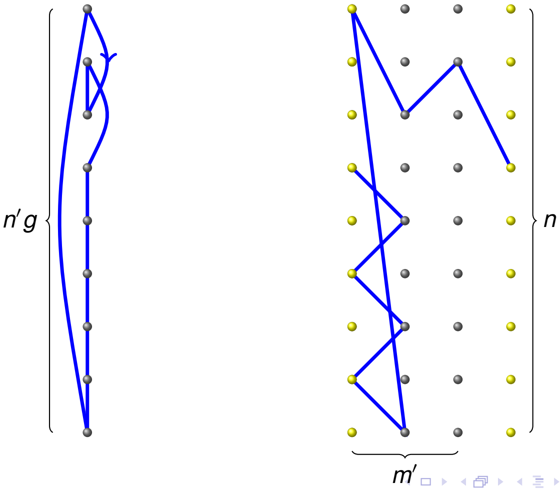

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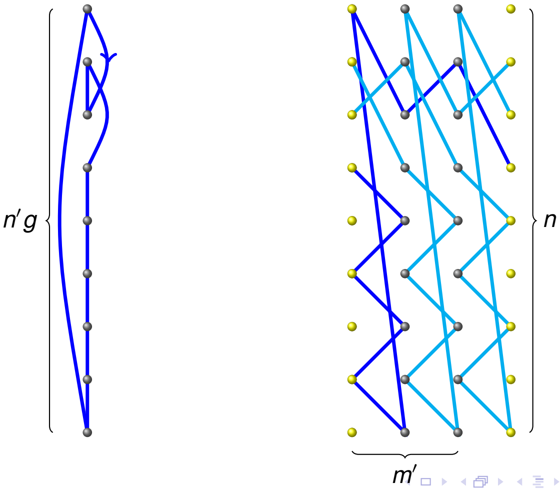

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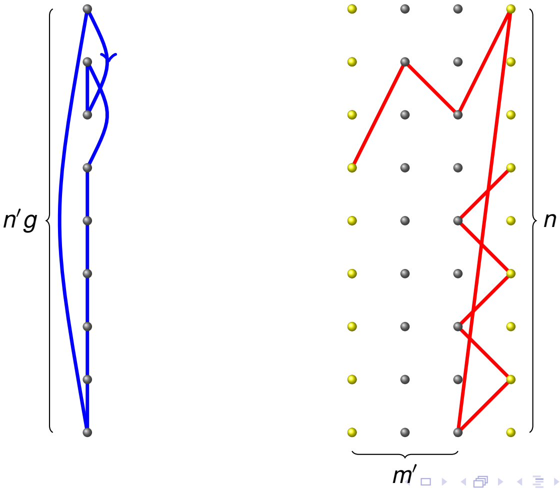

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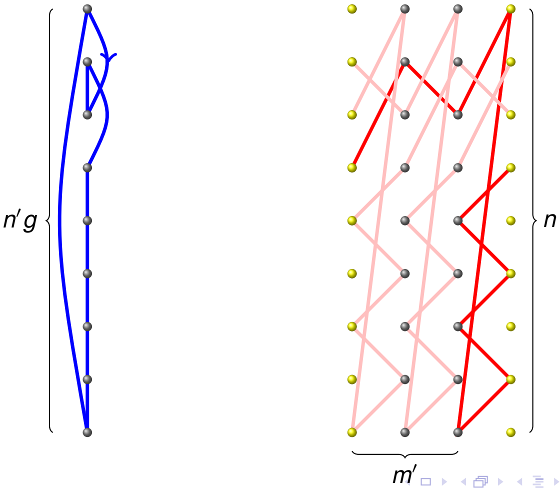

The new frontier: Cycles with opposite parities

Theorem (Kerenan and Pastine 2018)

Let x and y be odd with $\gcd(x, y) \geq 3$, and both x and y divide v then $\mathrm{HW}(tv; 2^k x, y; \alpha, \beta)$ has solution for $t \geq 3$, except possibly when $\alpha, \beta = 1$.

The new frontier: Cycles with opposite parities

Theorem (Burgess, Danziger, Traetta 2018)

Let m, n, v, α and β be positive integers such that $n > m \geq 3$ and m is an odd divisor of n. Then, $\mathrm{HWP}(v; m, n; \alpha, \beta)$ has solution if and only if $n \mid v$ and $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$, except possibly when at least one of the following conditions holds:

- $\beta = 1$;
- $\beta = 2$, $n \equiv 2m \bmod 4m$;
- $n \in \{2m, 6m\}$;
- $v \in \{n, 2n, 4n\}$;
- $(m, v) = (3, 6n)$.

The new frontier: Cycles with opposite parities

Corollary

Let $m \geq 3$ be an odd divisor of n. The necessary conditions for the solvability of $\mathrm{HWP}(v; m, n; \alpha, \beta)$ are sufficient whenever $v > 6n > 36m$ and $\beta \neq 1$.

Conclusions and Future Work

To do . . .

- Solve the annoying exceptions
- Deal with $\beta = 1$
- Deal with small v, small β
- Case n and m have different parities, particularly when m, n co-prime
- Generalise to more than two 2-factors when v is odd.

Generalised Oberwolfach Problem

GOP$(v; F_1, \dots, F_t)$

When the factors are uniform, ie $F_i = [m_i]$, we write
OP$(v; m_1, \dots, m_t; \alpha_1, \dots, \alpha_t)$.
Clearly we require that $m_i \mid v$ for each i.

Let $3 \le m_1 < \dots < m_t$ and $\ell = lcm(m_1, \dots, m_t)$.

Theorem (Burgess, Danziger, Traetta, 2019+)

For v odd, $OP(v; m_1, \dots, m_t; \alpha_1, \dots, \alpha_t)$ has a solution whenever $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{v-1}{2}$ and $m_i \mid v$ for each i, and

- $\alpha_i \neq 1$ *for every* i*;*
- $gcd(m_1, \dots, m_t) \ge 3$*;*

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The End

Thank You