CHROMATIC POLYNOMIALS OF 2-EDGE-COLOURED GRAPHS

Joint work with Iain Beaton, Chris Duffy & Nicole Zolkavich

Atlantic Graph Theory Seminar, March 2021
Danielle Cox
Mount Saint Vincent University

ABOUT THE AUTHORS

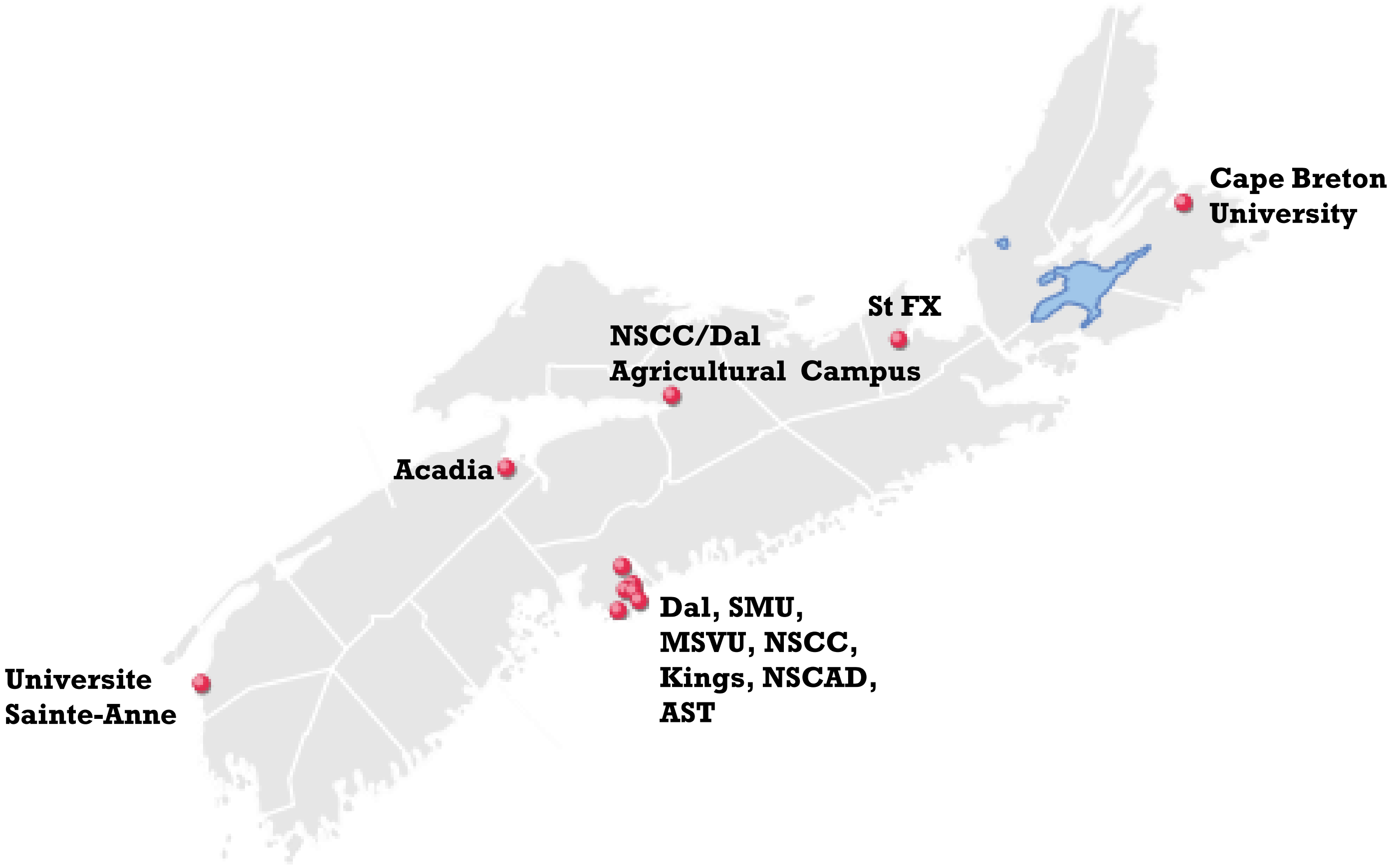
- This work is currently under review, so all results are 2020+
- Iain is a PhD Candidate at Dalhousie University who expertise is graph polynomials.
- Chris is at the University of Saskatchewan and he studies graph homomorphisms.
- Nicole was a student of Chris' who first started looking at a problem similar to this in a summer project. She is now pursuing a graduate degree at McGill with a focus on algebraic geometry.
- I am at Mount Saint Vincent University and come from the world of graph polynomials.

MOUNT SAINT VINCENT UNIVERSITY

Mount Saint Vincent University is located in Mi'kma'ki the ancestral and unceded territory of the Mi'kmaq

THE UNIVERSITIES OF HALIFAX

THE UNIVERSITIES OF NOVA SCOTIA

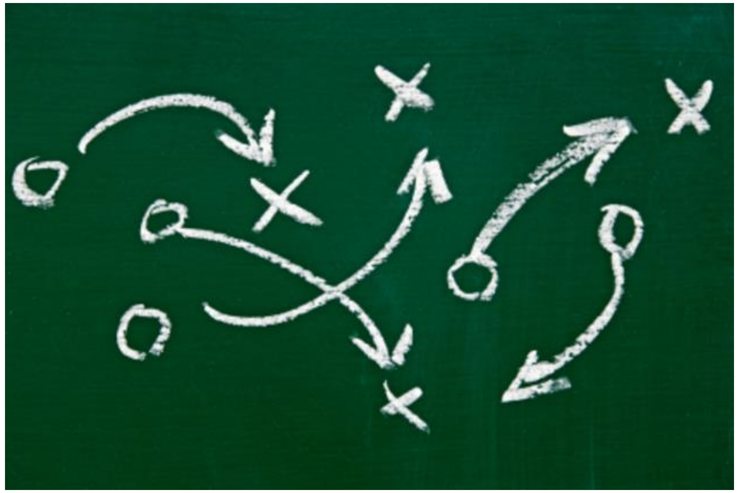


A QUOTE FROM LAST WEEK

- “No one ever complained that a talk ended early” – a great mathematician

THE GAME PLAN

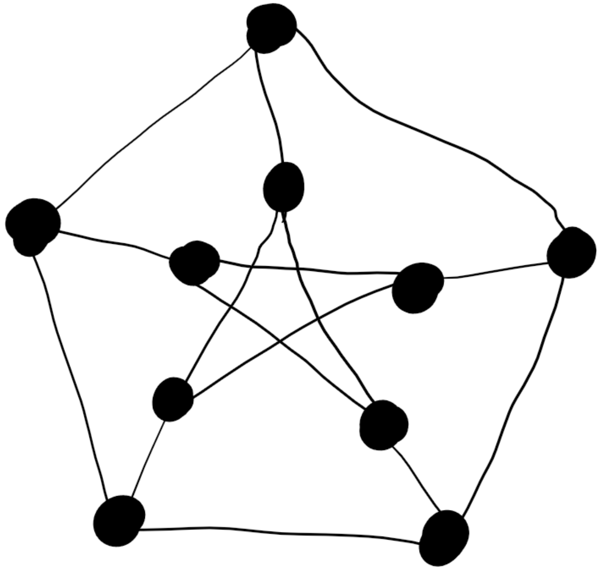
- Define the terms in the title
- Go over some examples
- Provide some recent results
- Open problems & questions



K –COLOURING OF A GRAPH

- A proper **colouring** of a graph *G* is labelling of the vertices of *G* so that adjacent vertices obtain different labels.

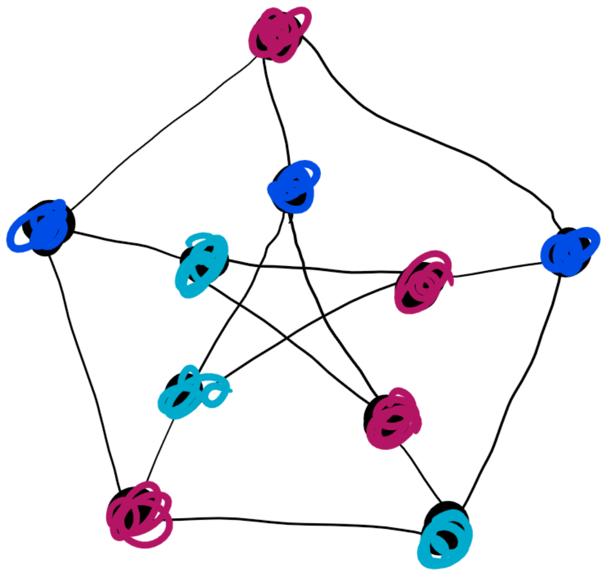
- If *k* labels are used, we call this a ***k*-colouring of *G***.



K –COLOURING OF A GRAPH

- A proper **colouring** of a graph G is labelling of the vertices of G so that adjacent vertices obtain different labels.

- If k labels are used, we call this a ***k*-colouring of *G***.



THE CHROMATIC POLYNOMIAL

- In 1912 George Birkhoff wanted to prove the Four Colour Conjecture.

- Plan of attack: POLYNOMIALS!

- He defined the chromatic polynomial and it counts the number of colourings of a graph as a function of the number of colours.

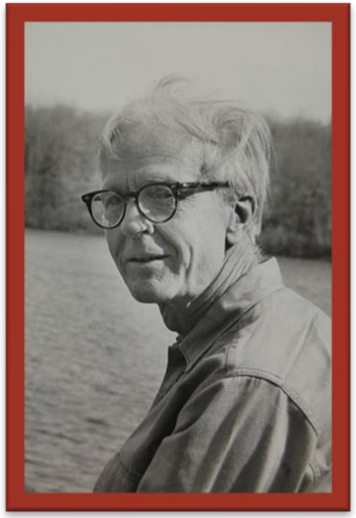
- He was focused only on planar graphs (because he cared only about making the conjecture a theorem).

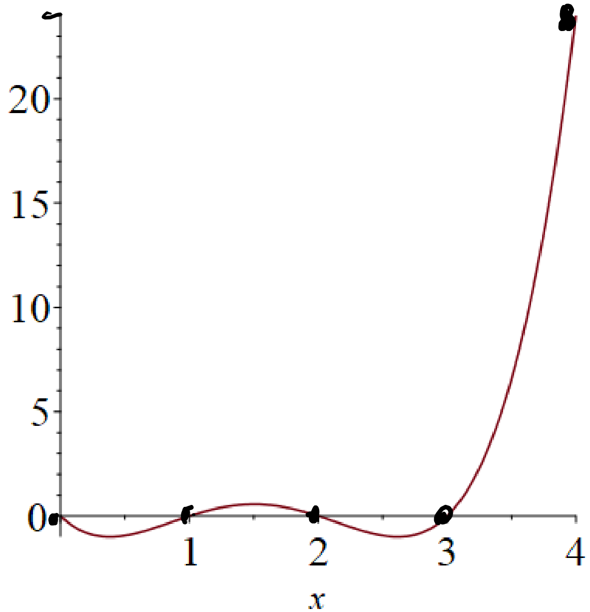
- We know how his plan turned out…

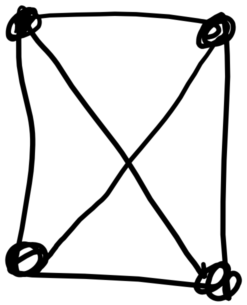
THE CHROMATIC POLYNOMIAL

- In 1932 Hassler Whitney generalized Birkoff's polynomial to all graphs.

- For a graph G of order n, the **chromatic polynomial** is the unique interpolating polynomial, $P(G,x)$ of degree at most n that passes through the points $(c, P(G,c))$, $c=0,\ldots,n$ where c is the number of colours and $P(G,c)$ the number of colourings.

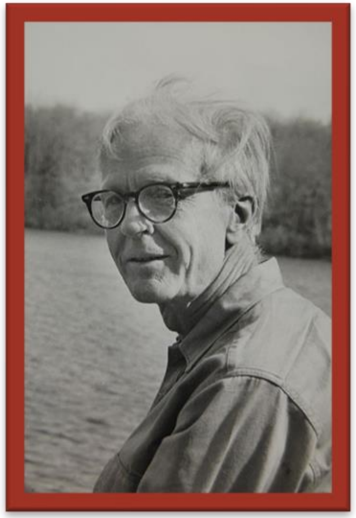






$P(K_4, x) = x(x-1)(x-2)(x-3)$

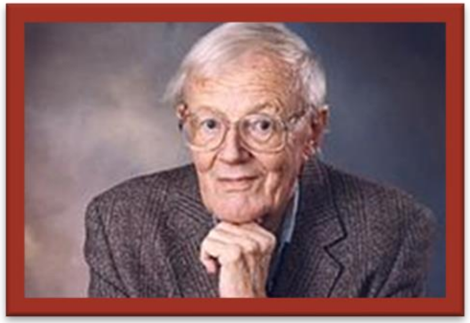
THE CHROMATIC POLYNOMIAL



- In 1932 Hassler Whitney generalized Birkoff's polynomial to all graphs.

- For a graph G of order n, the **chromatic polynomial** is the unique interpolating polynomial, $P(G,x)$ of degree at most n that passes through the points $(c, P(G,c))$, $c=0,\ldots,n$ where c is the number of colours and $P(G,c)$ the number of colourings.

- There are nice deletion-contraction formulas for computing the chromatic polynomial of a graph.

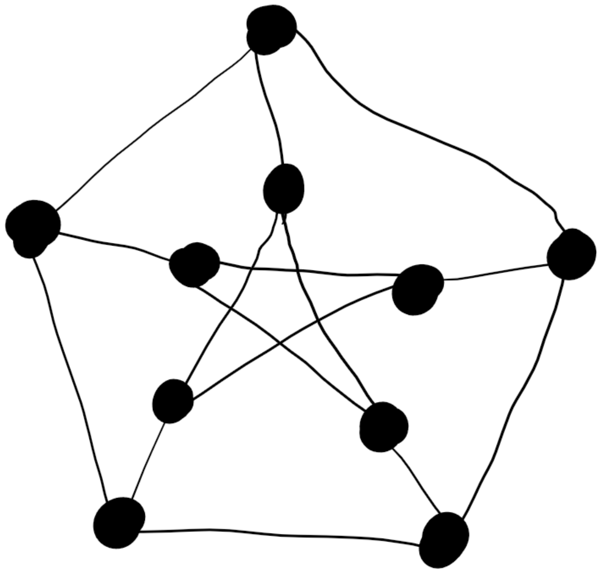


- W.T. Tutte looked at the bivariate generalization of this polynomial, which became the well known Tutte Polynomial.

- The chromatic polynomial is really just an evaluation of the Tutte Polynomial.

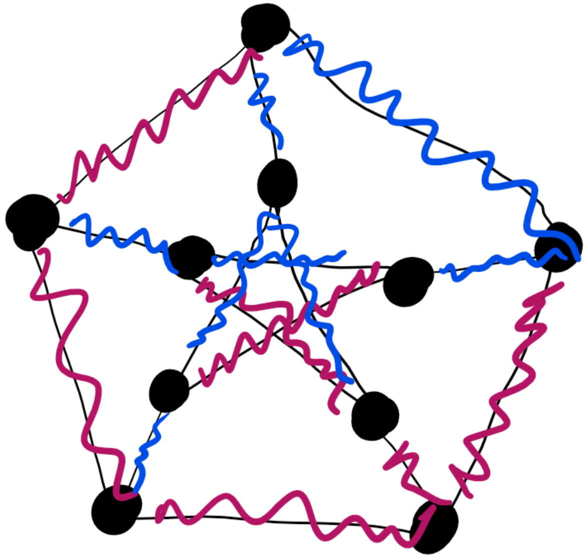
2-EDGE-COLOURED GRAPHS

- A **2-edge-coloured graph** G is a triple (Γ, R, B) where Γ is a simple graph, R and B disjoint subsets that partition $E(\Gamma)$.
- We call G a **2-edge-colouring** of Γ.



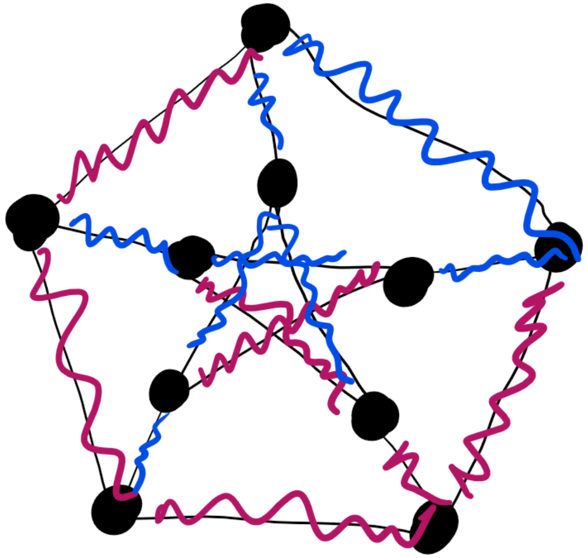
2-EDGE-COLOURED GRAPHS

- A **2-edge-coloured graph** G is a triple (Γ, R, B) where Γ is a simple graph, R and B disjoint subsets that partition $E(\Gamma)$.
- We call G a **2-edge-colouring** of Γ.



2-EDGE-COLOURED GRAPHS

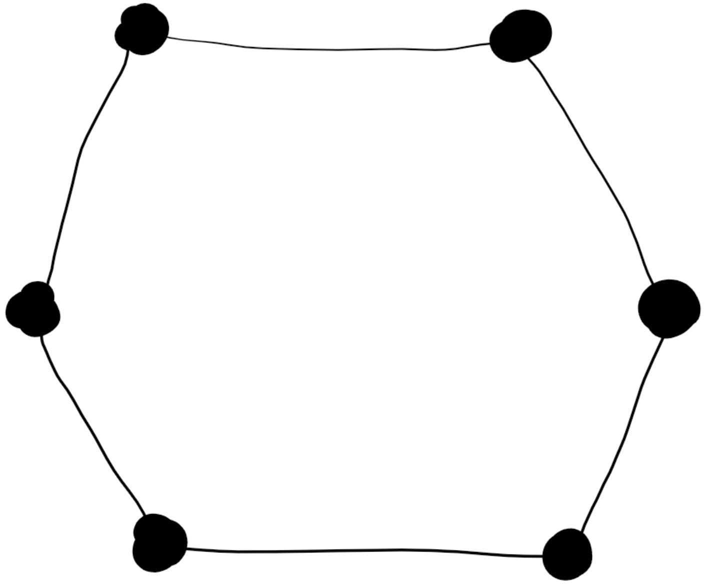
- A **2-edge-coloured graph** G is a triple (Γ, R, B) where Γ is a simple graph, R and B disjoint subsets that partition $E(\Gamma)$.
- We call G a **2-edge-colouring** of Γ.



- Can we also colour the vertices of G ?
- How would we define a k-colouring of a 2-edge-coloured graph?

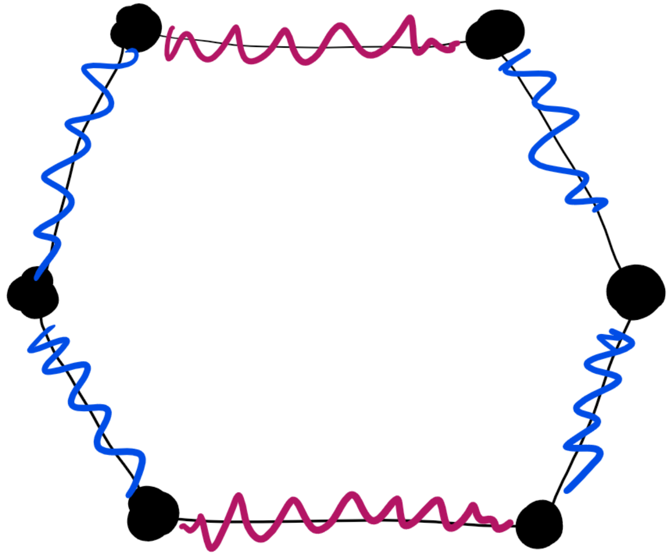
K-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of homomorphims:*
  For 2-edge-coloured graphs *G* and *H* a homomorphism from mapping the vertices of *G* to those of *H* that preserves the existence of edges and their colours is called a ***k*-colouring of G** when *H* has k vertices.



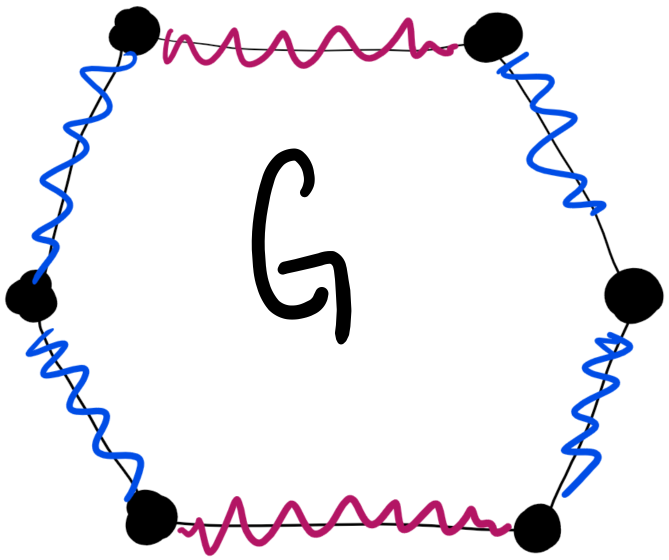
K-COLOURING OF A 2-EDGE-COLOURED GRAPH

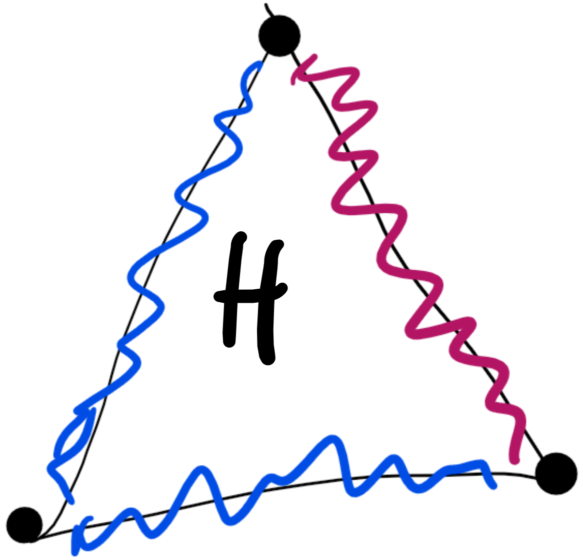
- *In terms of homomorphims:*
  For 2-edge-coloured graphs *G* and *H* a homomorphism from mapping the vertices of *G* to those of *H* that preserves the existence of edges and their colours is called a ***k*-colouring of G** when *H* has k vertices.



K-COLOURING OF A 2-EDGE-COLOURED GRAPH

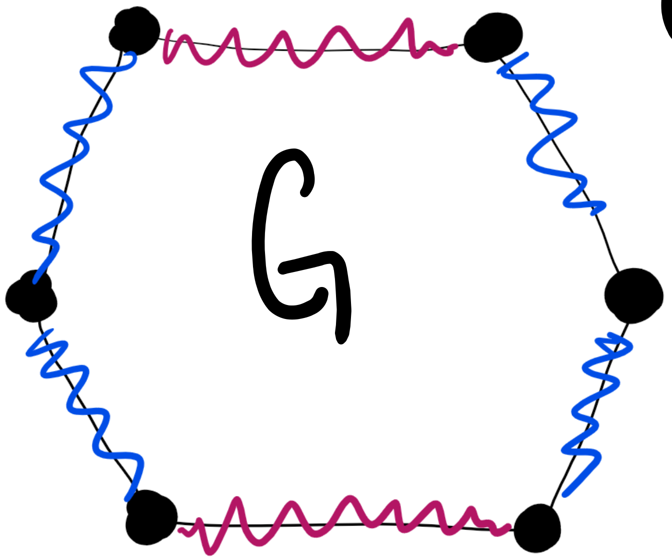
- *In terms of homomorphims:*
  For 2-edge-coloured graphs *G* and *H* a homomorphism from mapping the vertices of *G* to those of *H* that preserves the existence of edges and their colours is called a ***k*-colouring of G** when *H* has k vertices.



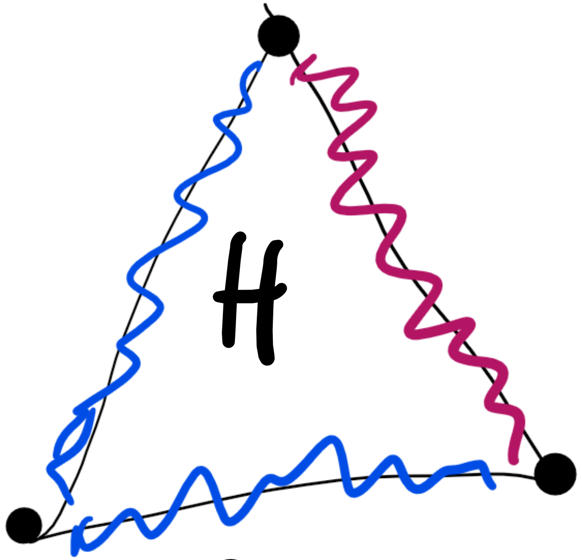


k-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of homomorphims:*
  For 2-edge-coloured graphs *G* and *H* a homomorphism from mapping the vertices of *G* to those of *H* that preserves the existence of edges and their colours is called a ***k*-colouring of G** when *H* has k vertices.

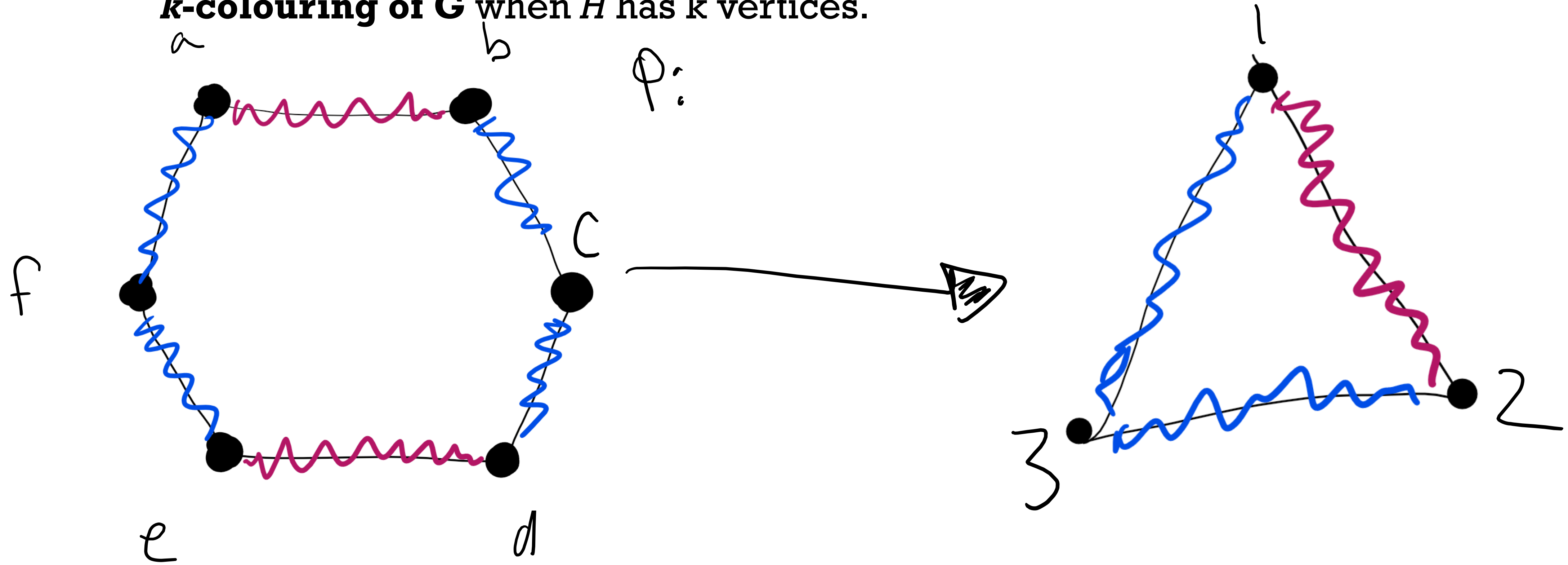


Can we find a 3-coloring of G using this target graph?



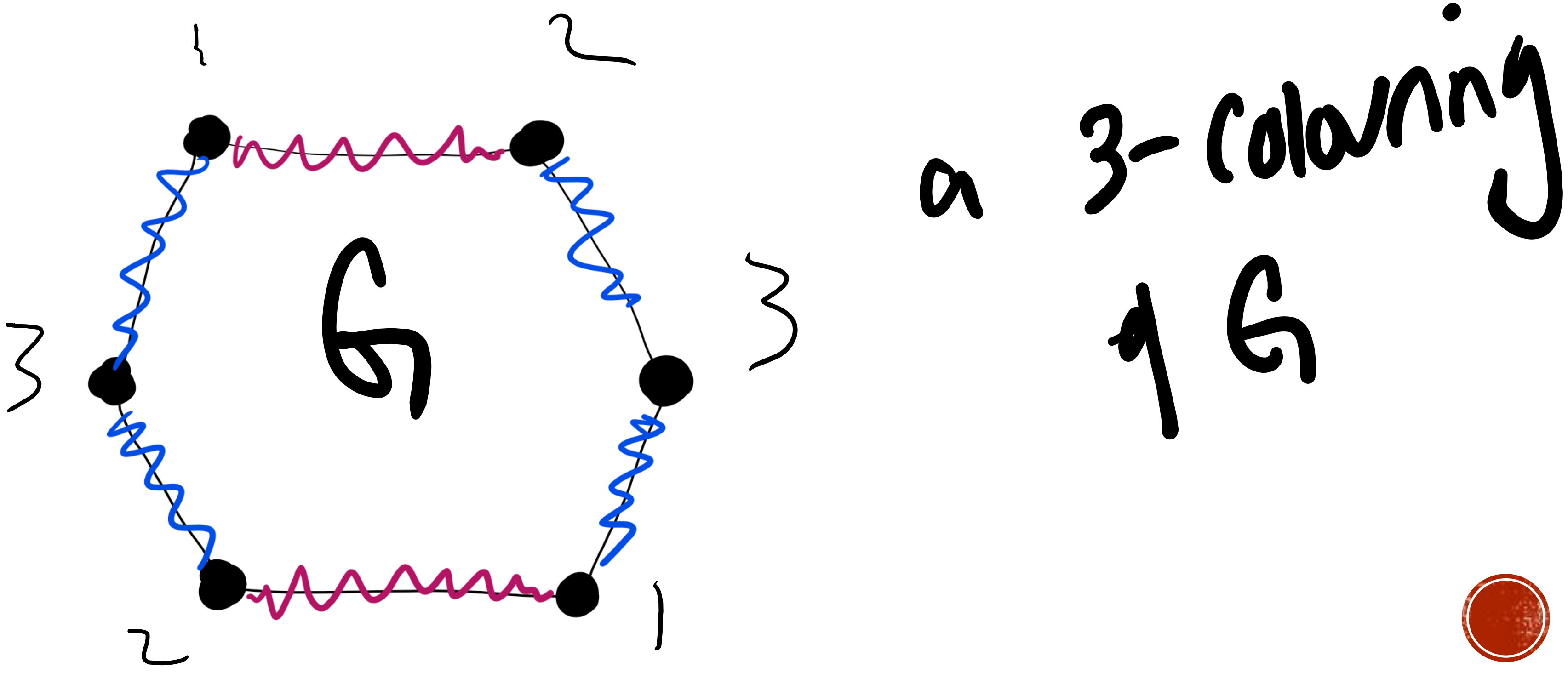
k-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of homomorphims:*
  For 2-edge-coloured graphs *G* and *H* a homomorphism from mapping the vertices of *G* to those of *H* that preserves the existence of edges and their colours is called a ***k*-colouring of G** when *H* has k vertices.



k-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of homomorphims:*
  For 2-edge-coloured graphs *G* and *H* a homomorphism from mapping the vertices of *G* to those of *H* that preserves the existence of edges and their colours is called a ***k*-colouring of G** when *H* has k vertices.



k-COLOURING OF A 2-EDGE-COLOURED GRAPH

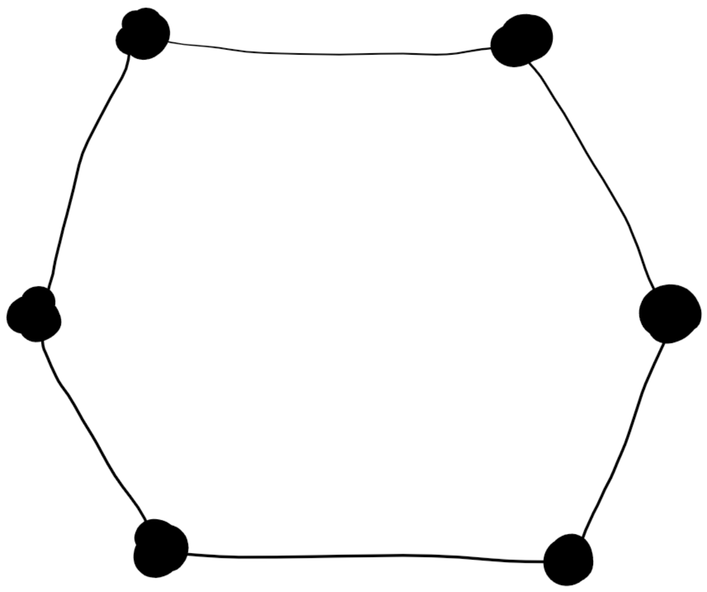
- *In terms of how I normally think of colourings:*

  Let $c: V(G) \to \{1,2,\ldots,k\}$
  - *for all edges* yz, $c(y) \neq c(z)$

K-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of how I normally think of colourings:*

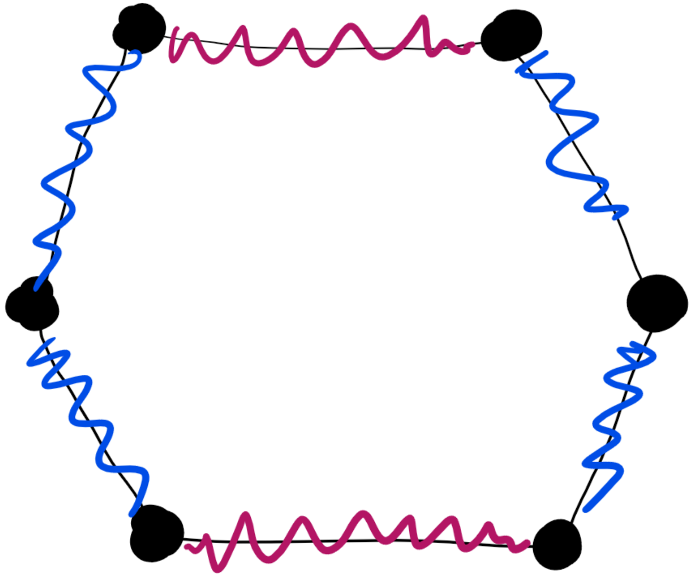
  Let $c: V(G) \rightarrow \{1, 2, \ldots, k\}$
  - *for all edges yz,* $c(y) \neq c(z)$



K-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of how I normally think of colourings:*

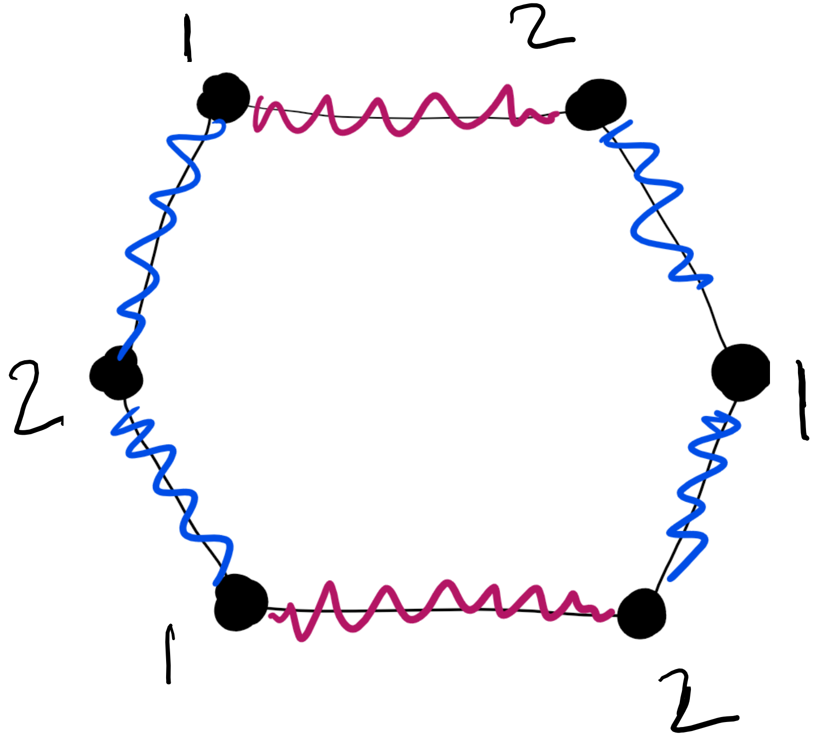
  Let $c{:}V(G) \to \{1,2,\ldots,k\}$
  - *for all edges yz,* $c(y) \neq c(z)$



k-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of how I normally think of colourings:*

  Let $c: V(G) \to \{1,2,\ldots,k\}$
  - *for all edges yz,* $c(y) \neq c(z)$



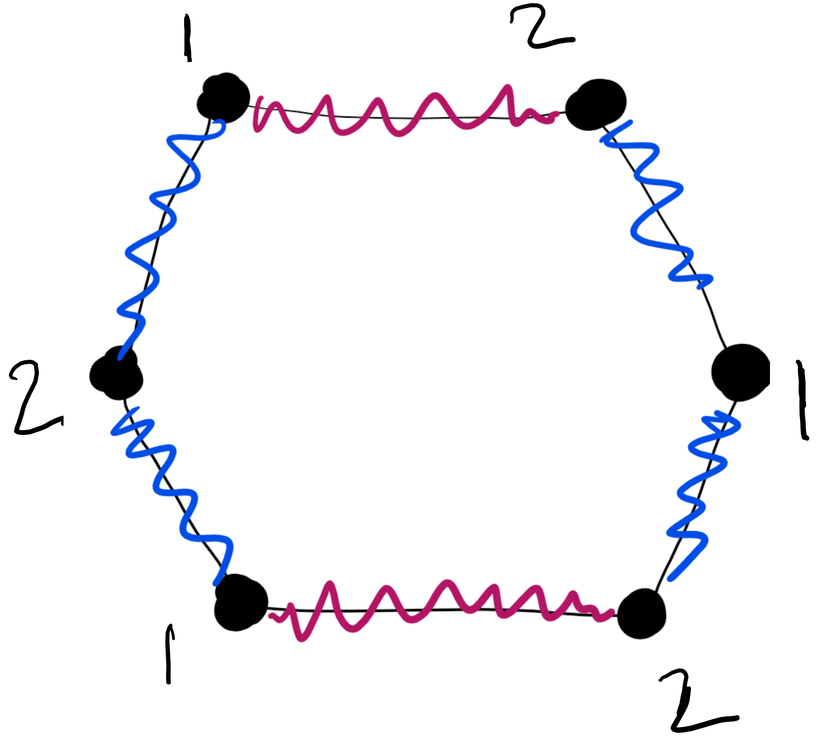
k-COLOURING OF A 2-EDGE-COLOURED GRAPH

- *In terms of how I normally think of colourings:*

  Let $c: V(G) \rightarrow \{1,2,\ldots,k\}$

  - *for all edges* yz, $c(y) \neq c(z)$

Do we see any problems with this colouring?

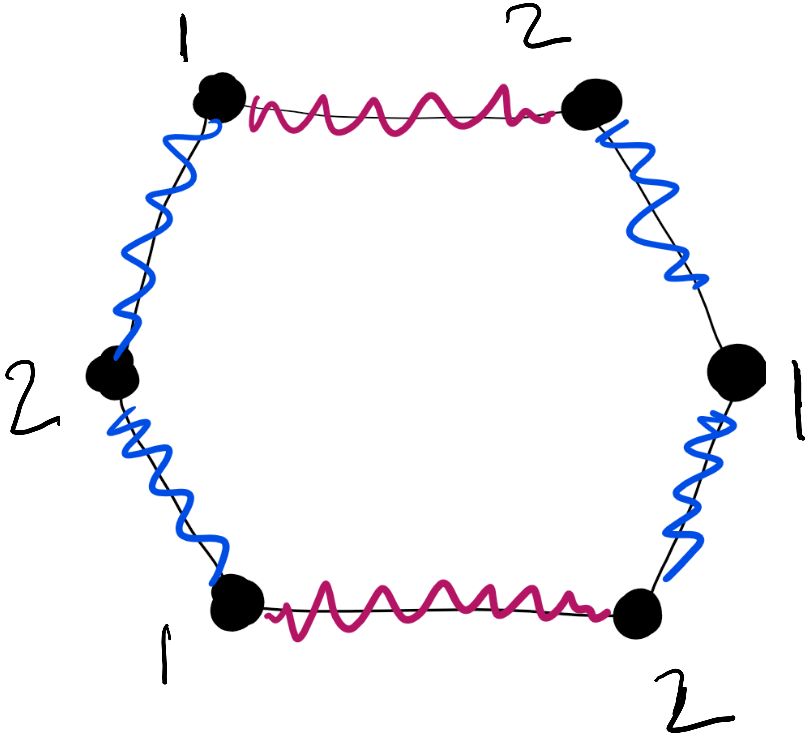


K-COLOURING OF A 2-EDGE-COLOURED GRAPH

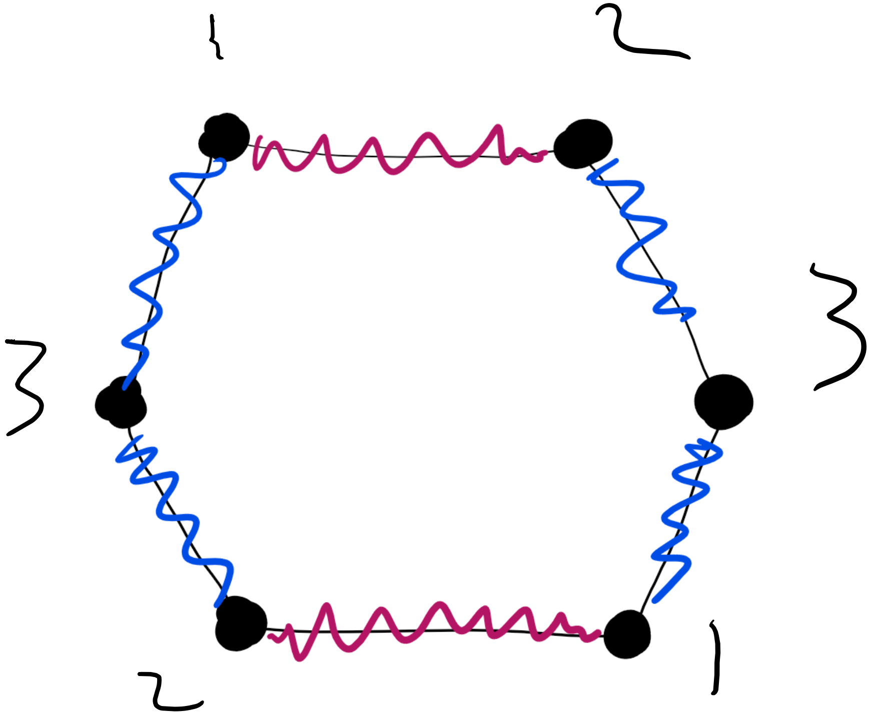
- *In terms of how I normally think of colourings:*

  Let $c: V(G) \to \{1, 2, \ldots, k\}$

  - *for all edges yz,* $c(y) \neq c(z)$
  - *for all ux in R and vy in B if* $c(u) = c(v)$ *then* $c(x) \neq c(y)$



K-COLOURING OF A 2-EDGE-COLOURED GRAPH



CHROMATIC NUMBER

- Just as with graphs, the chromatic number, $\chi(G)$ of a 2-edge-coloured graph, G is just the least t that admits a t-colouring.

- There has been work done regarding this parameter.
  - N. Alon and T. Marshall. Homomorphisms of Edge-Colored Graphs and Coxeter Groups. Journal of Algebraic Combinatorics, 8(1):5-13, 1998.
  - R. C. Brewster, F. Foucaud, P. Hell, and R. Naserasr. The complexity of signed graph and edge-coloured graph homomorphisms. Discrete Mathematics, 340(2):223{235, 2017.
  - R. C. Brewster and P. Hell. On homomorphisms to edge-coloured cycles. Electronic Notes in Discrete Mathematics, 5:46-49, 2000.
  - A. Montejano, P. Ochem, A. Pinlou, A. Raspaud, and E. Sopena. Homomorphisms of 2-edge-coloured Graphs. Discrete Applied Mathematics, 158(12):1365-1379, 2010.
  - P. Ochem, A. Pinlou, and S. Sen. Homomorphisms of 2-edge-colored triangle-free planar graphs. Journal of Graph Theory, 85(1):258-277, 2017.

CHROMATIC POLYNOMIAL FOR 2-EDGE-COLOURED GRAPHS

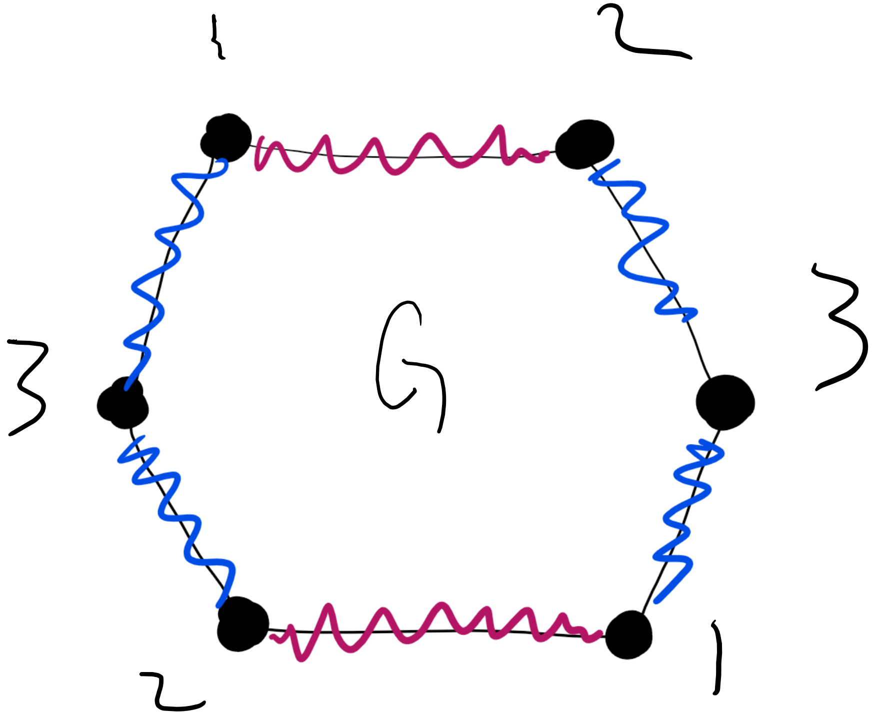
- Since the proper colouring of a 2-edge-coloured graph is really just a proper colouring of the underlying graph, that satisfies some extra constraints (to take edge colours into consideration) we can define the chromatic polynomial for 2-edge-coloured graphs in the same way as for graphs.
-

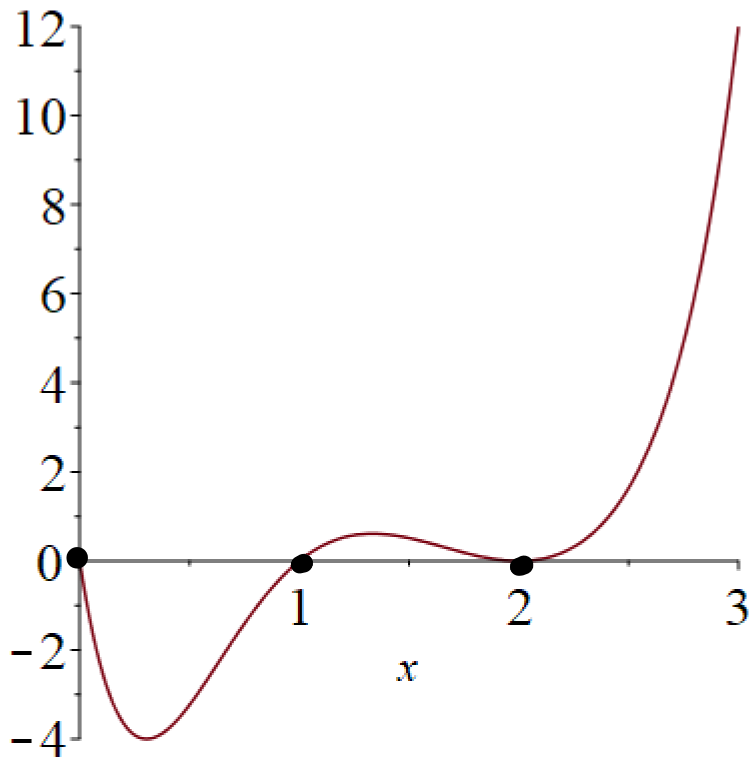
CHROMATIC POLYNOMIAL FOR 2-EDGE-COLOURED GRAPHS

- Since the proper colouring of a 2-edge-coloured graph is really just a proper colouring of the underlying graph, that satisfies some extra constraints (to take edge colours into consideration) we can define the chromatic polynomial for 2-edge-coloured graphs in the same way as for graphs.

- For a 2-edge-coloured graph G of order n, the **chromatic polynomial** is the unique interpolating polynomial, $P(G,x)$ of degree at most n that passes through the points $(c, P(G,c))$, $c=0,\ldots,n$ where c is the number of colours and $P(G,c)$ the number of colourings

CHROMATIC POLYNOMIAL FOR 2-EDGE-COLOURED GRAPHS





$$P(G, x) = x^6 - 10x^5 + 41x^4 - 84x^3 + 84x^2 - 32x$$

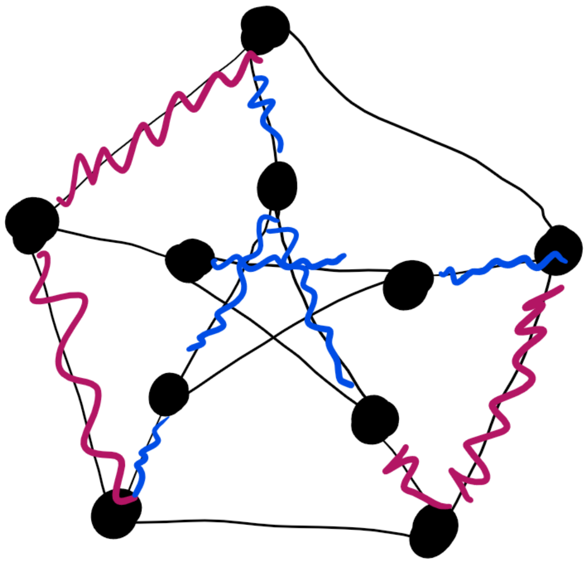
I LIED A BIT…

- So I said this talk was about the chromatic polynomial of 2-edge-coloured graphs … and it is!

- But to help us out we are going to look at a more generalized graph object – a ***mixed 2-edge-coloured graph***

MIXED 2-EDGE-COLOURED GRAPHS

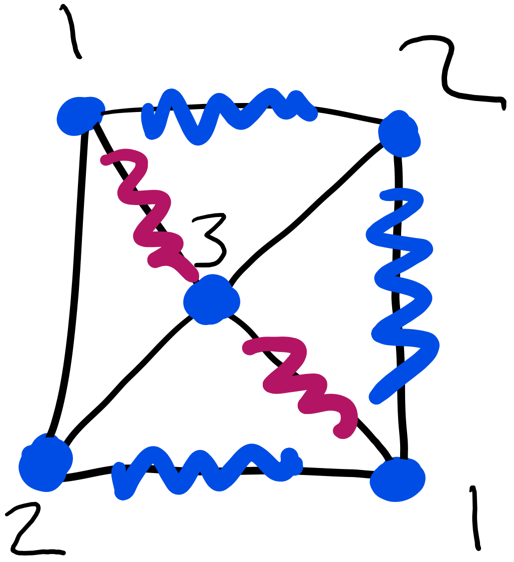
- A mixed 2-edge-coloured graph is a pair $M=(G,F)$ where G is a 2-edge-coloured graph and M is a subset of edges that are not in R or B



K-COLOURING OF A MIXED 2-EDGE-COLOURED GRAPH

- Let $M=(G,F)$ where G is a 2-edge-coloured graph and M is a subset of edges that are not in R or B

- *Let* $c: V(G) \rightarrow \{1,2,\ldots,k\}$
  - *for all edges yz,* $c(y) \neq c(z)$
  - *for all ux in R and vy in B if* $c(u)=c(v)$ *then* $c(x) \neq c(y)$

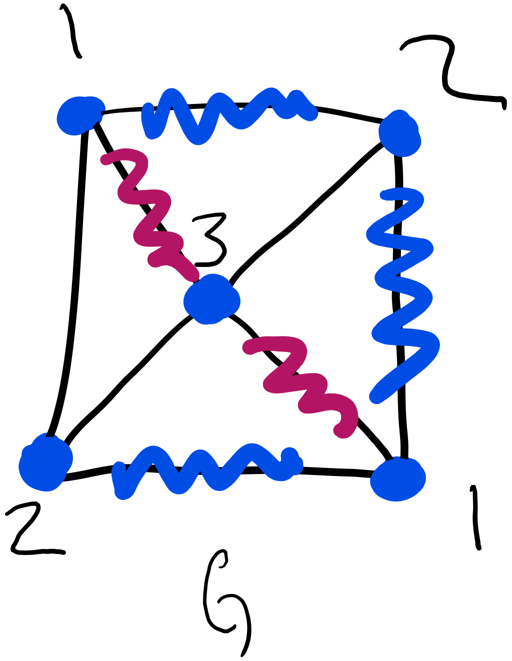


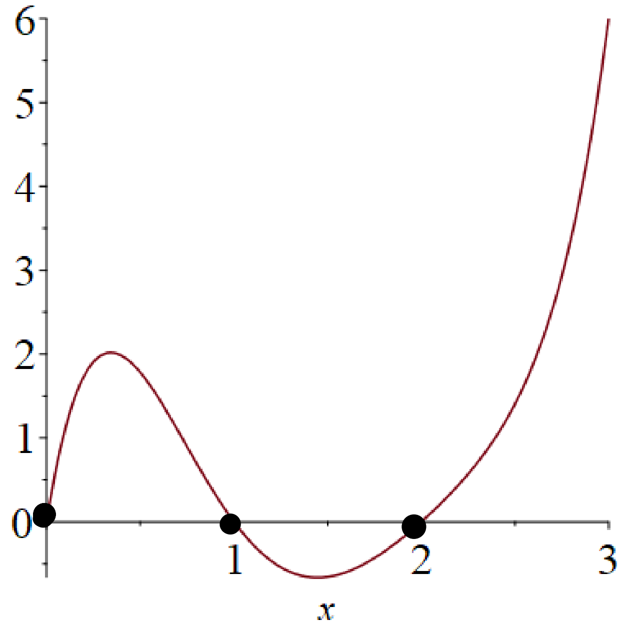
- *Note:* it is just a 2-edge-colouring of G ensuring that the vertices of a non-coloured edge have different labels.

CHROMATIC POLYNOMIAL FOR MIXED 2-EDGE-COLOURED GRAPHS

- Again – this chromatic polynomial is defined as one would expect
- For a mixed 2-edge-coloured graph M of order n, the **chromatic polynomial** is the unique interpolating polynomial, $P(M,x)$ of degree at most n that passes through the points $(c, P(M,c))$, $c=0,\ldots,n$ where c is the number of colours and $P(M,c)$ the number of colourings

CHROMATIC POLYNOMIAL FOR MIXED 2-EDGE-COLOURED GRAPHS



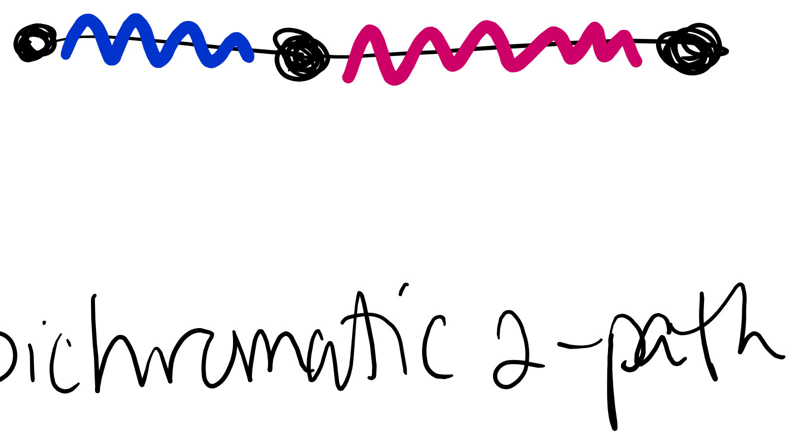


$$P(G,x) = x^5 - 8x^4 + 24x^3 - 31x^2 + 14x$$

COMPUTING CHROMATIC POLYNOMIAL OF MIXED 2-EDGE-COLOURED GRAPHS

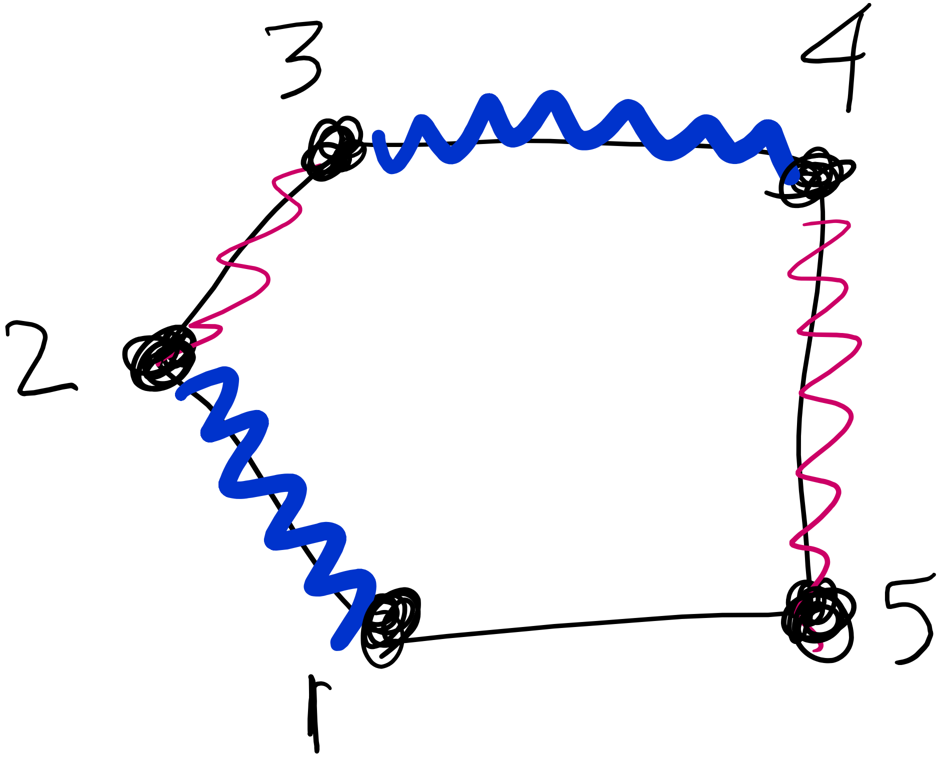
- We don't have the deletion-contraction formulas that the chromatic polynomial of a graph has – but we do have a recursive way to compute this polynomial.

- First an observation:



COMPUTING CHROMATIC POLYNOMIAL OF MIXED 2-EDGE-COLOURED GRAPHS

- This means that if you have a mixed 2-edge-coloured graph M on n vertices where every pair is either adjacent or at the ends of a bichromatic 2-path, then all vertices receive different colours.



COMPUTING CHROMATIC POLYNOMIAL OF MIXED 2-EDGE-COLOURED GRAPHS

- Pairs of vertices that **are adjacent or at the end of a bichromatic 2-path** receive different colours
-
-

COMPUTING CHROMATIC POLYNOMIAL OF MIXED 2-EDGE-COLOURED GRAPHS

- Pairs of vertices that **are adjacent or at the end of a bichromatic 2-path** receive different colours

- Pairs of vertices, *u* and *v* **not adjacent or at the end of a bichromatic 2-path** either receive different colours or the same colours.

-

COMPUTING CHROMATIC POLYNOMIAL OF MIXED 2-EDGE-COLOURED GRAPHS

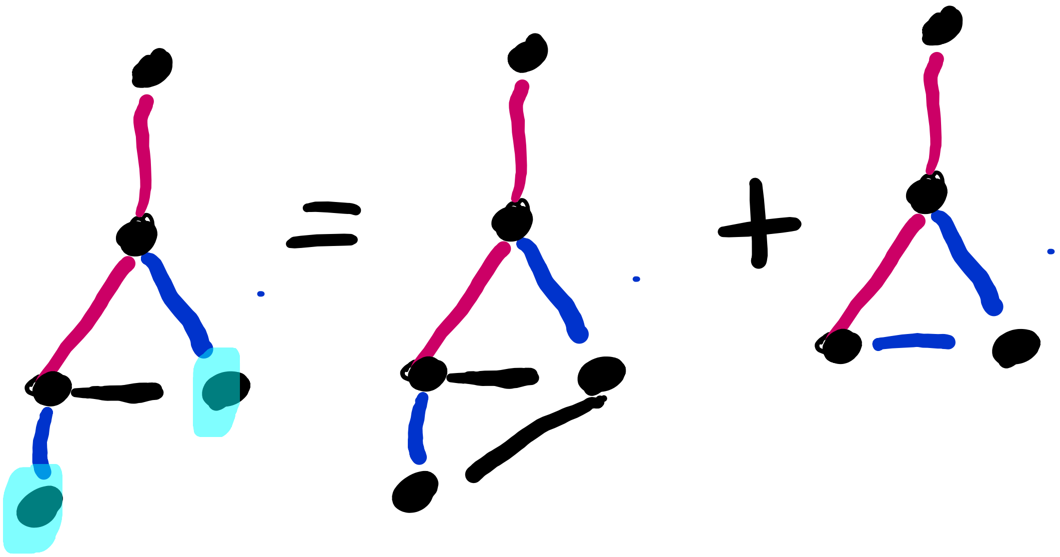
- Pairs of vertices that **are adjacent or at the end of a bichromatic 2-path** receive different colours

- Pairs of vertices, *u* and *v* **not adjacent or at the end of a bichromatic 2-path** either receive different colours or the same colours.

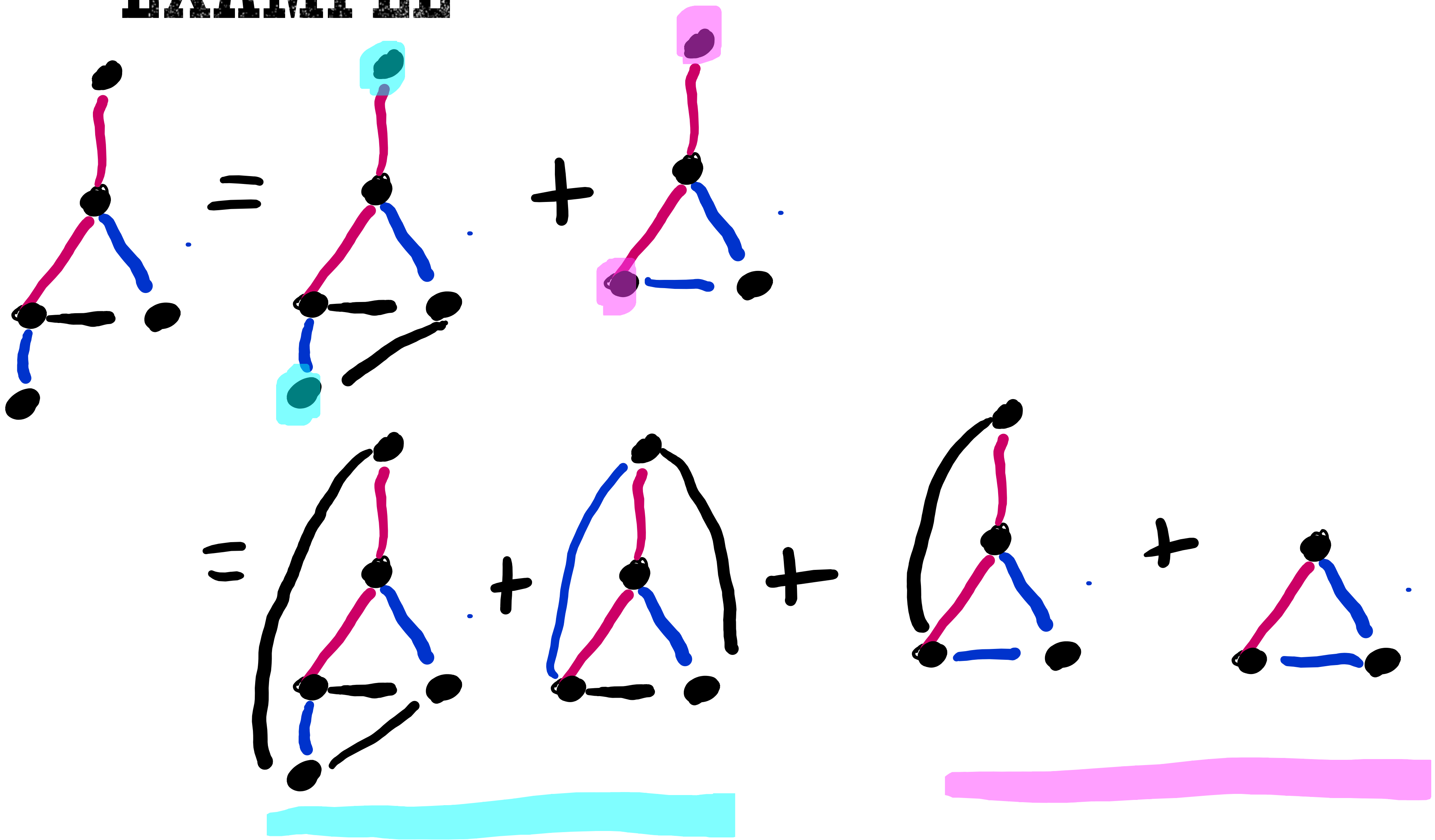
- For *u*,*v* not adjacent or ends of a bichromatic 2-path, we can compute the chromatic polynomial as:

$$P(M, x) = P(M + uv, x) + P(M_{uv}, x)$$

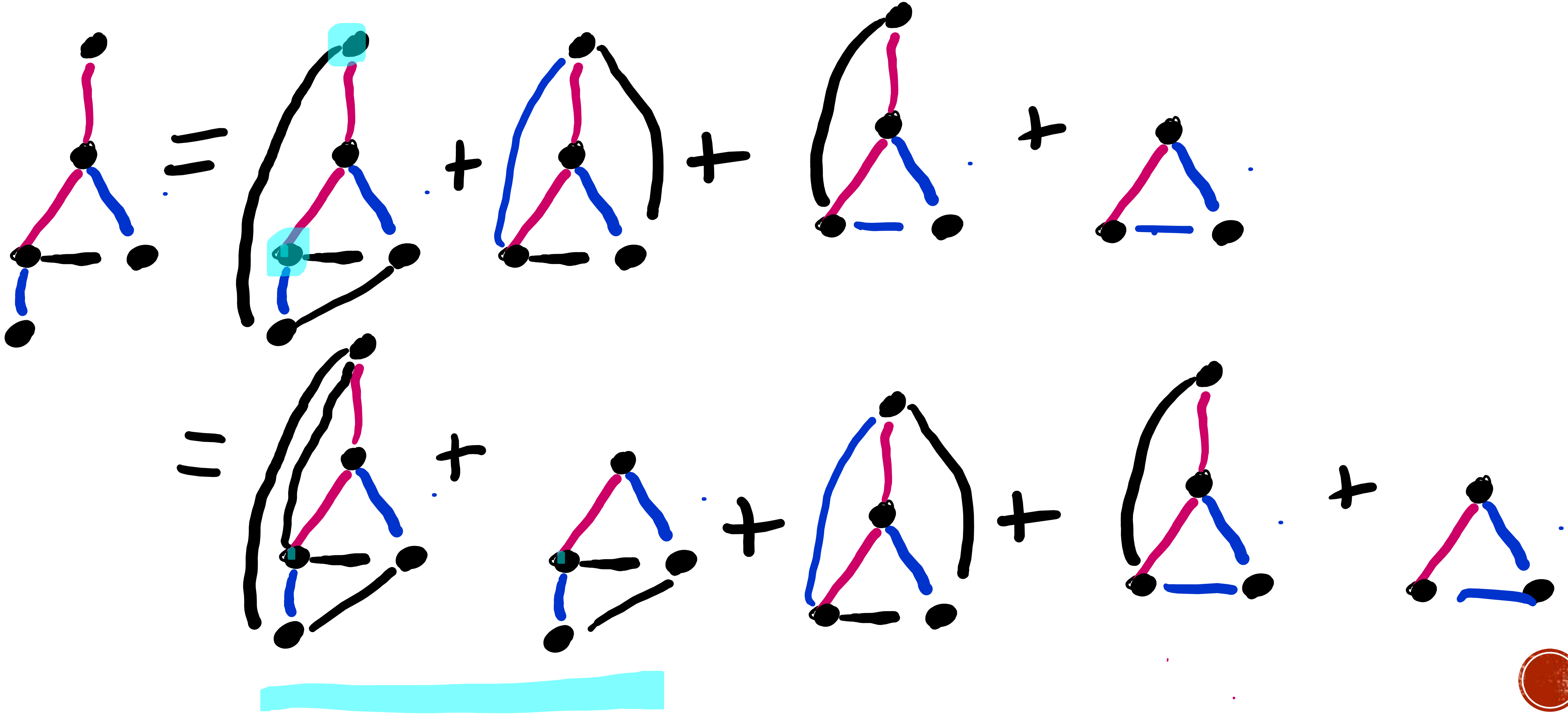
EXAMPLE



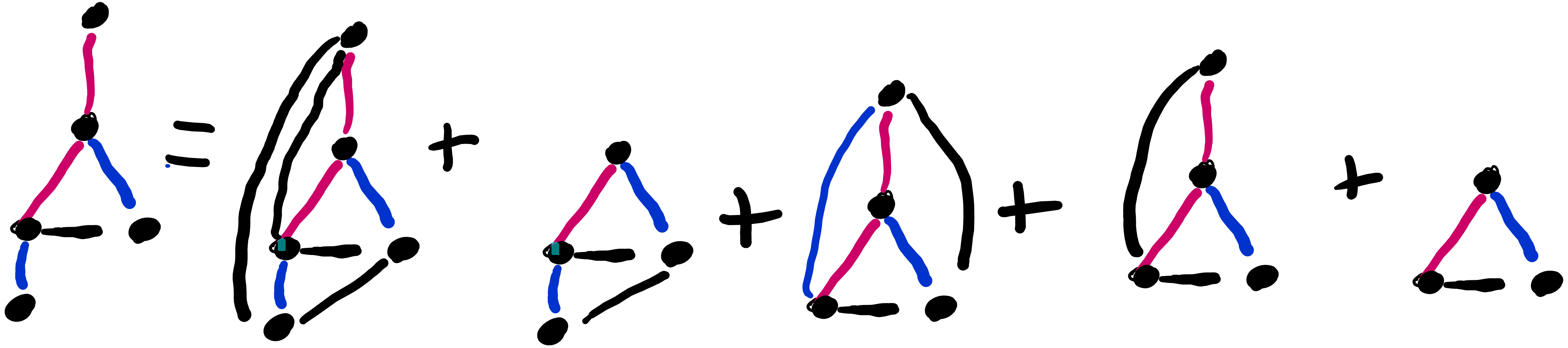
EXAMPLE



EXAMPLE



EXAMPLE



CHROMATICALLY INVARIANT 2-EDGE-COLOURED GRAPHS

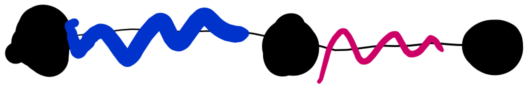
- For every 2-edge-coloured graph has an underlying graph, so it is natural to ask the following:

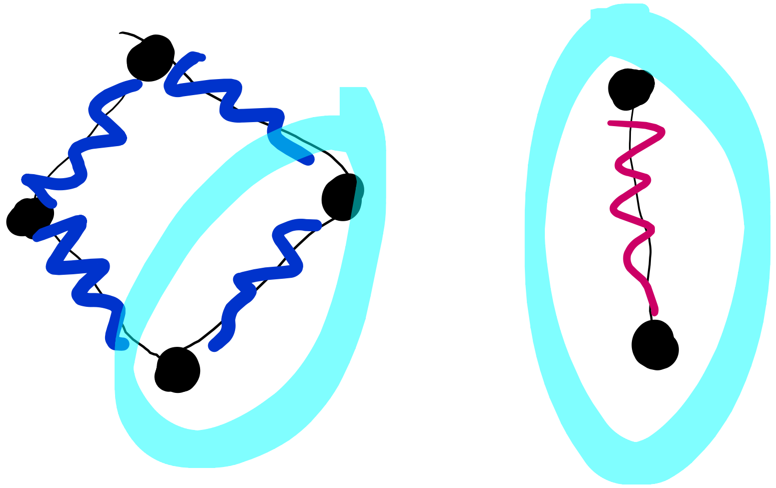
What graphs, Γ admit a 2-edge-colouring, G, so that G and Γ have the same chromatic polynomials?

CHROMATICALLY INVARIANT 2-EDGE-COLOURED GRAPHS

- Theorem [IB,**DC**, CD, NZ 2020+]

  A 2-edge coloured graph *G* is chromatically invariant if and only if *G* has no induced bichromatic 2-path and no induced bichromatic copy of $2K_2$





CHROMATICALLY INVARIANT 2-EDGE-COLOURED GRAPHS

- Similar to the previous result, we can characterize these chromatically invariant 2-edge-coloured graphs in terms of pairs of independent sets.

CHROMATICALLY INVARIANT 2-EDGE-COLOURED GRAPHS

- Theorem [IB,**DC**, CD, NZ 2020+]

  A 2-edge coloured graph $G=(\Gamma, R, B)$ is chromatically invariant if and only for *every disjoint pair of non-empty independent sets* I and I' in Γ the *2-edge-coloured subgraph induced by I and I' is monochromatic.*

CHROMATICALLY INVARIANT 2-EDGE-COLOURED GRAPHS

- Theorem [IB,**DC**, CD, NZ 2020+]

  A graph Γ *admits a non-trivial* chromatically invariant 2-edge-colouring in *which every vertex is incident with both a red edge and blue edge* if and only if Γ is the *join of two graphs, each which has no isolated vertices*.

CHROMATICALLY INVARIANT 2-EDGE-COLOURED GRAPHS

- Open problem:

  A graph Γ admits a non-trivial chromatically invariant 2-edge-colouring in which every vertex is **not** incident with both a red edge and blue edge if and only if ….??

AND NOW FOR SOMETHING COMPLETELY DIFFERENT

BICHROMATIC ROOTS

- A **bichromatic root** is a real or complex number that is the root of the chromatic polynomial of some 2-edge coloured graph
- A **chromatic root** is a real or complex number that is the root of the chromatic polynomial of some graph

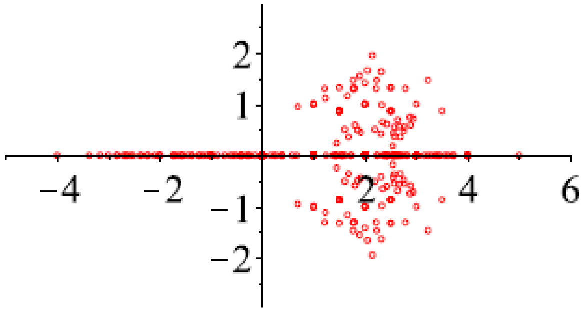


FIGURE 3. Bichromatic roots of all connected 2-edge-coloured graphs on 6 vertices

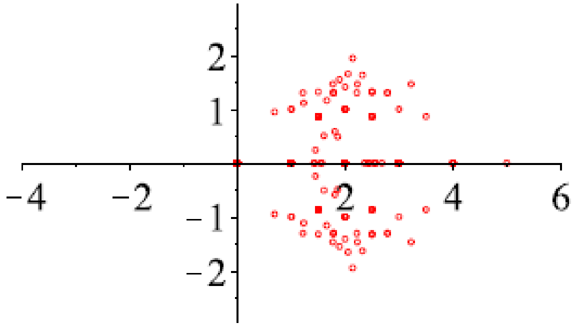


FIGURE 4. Chromatic roots of all connected graphs on 6 vertices

BICHROMATIC ROOTS VS CHROMATIC ROOTS

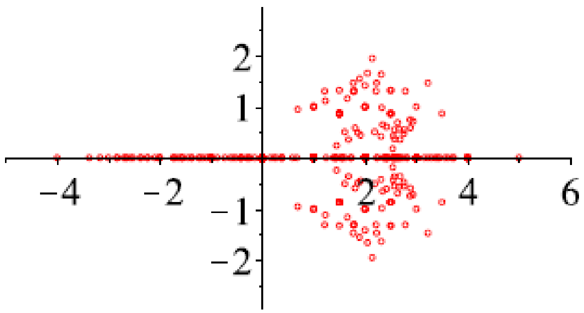


FIGURE 3. Bichromatic roots of all connected 2-edge-coloured graphs on 6 vertices

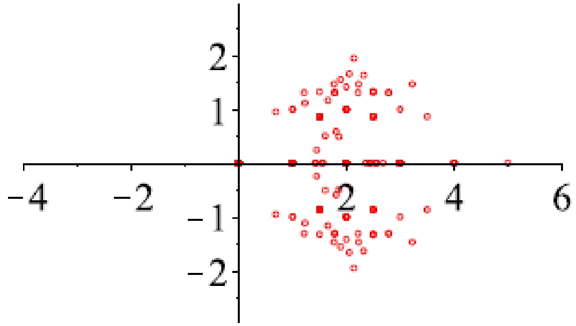


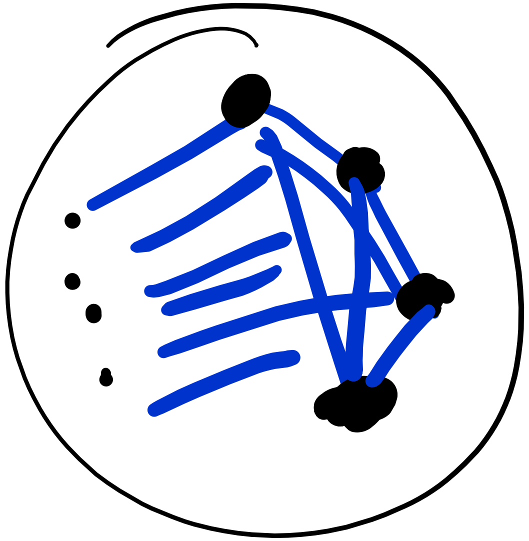
FIGURE 4. Chromatic roots of all connected graphs on 6 vertices

BICHROMATIC ROOTS

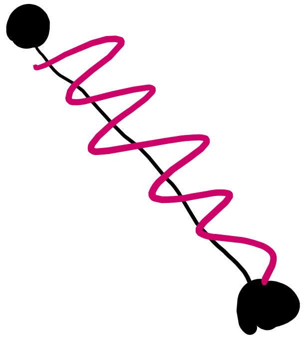
- The roots of the chromatic polynomial of graphs is well studied.

- Some of the well-known results are:
  - Real roots are always positive
  - No real roots in (0,1) and (1,32/27] [B. Jackson, 2003]
  - Closure is the entire complex plane [A. Sokal, 2004]

BICHROMATIC ROOTS



K_n^b



K_2^r

BICHROMATIC ROOTS

- Theorem [IB,**DC**, CD, NZ 2020+]

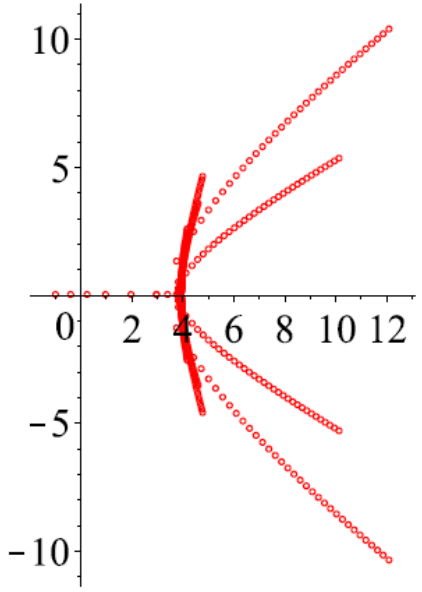
  The closure of the rational roots of 2-edge-coloured graph is the integers.

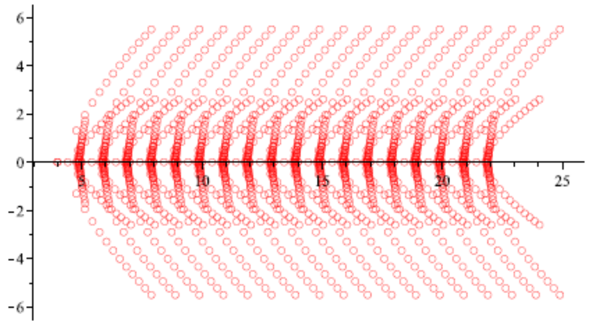
BICHROMATIC ROOTS

- Theorem [IB,**DC**, CD, NZ 2020+]

  The closure of the real roots of 2-edge-coloured graph is the reals.

BICHROMATIC ROOTS





OPEN PROBLEMS

- Further study chromatic invariance
- Explore the analytic properties of the chromatic polynomial of 2-edge-coloured graphs, for example, what is the closure of the complex roots?
- Further compare chromatic polynomials of graphs and 2-edge-coloured graphs

- Thanks to NSERC & MSVU for funding my work and AARMS for the Zoom license to give these talks.