Enumerating Digitally Convex Sets in Graphs

MacKenzie Carr
Simon Fraser University

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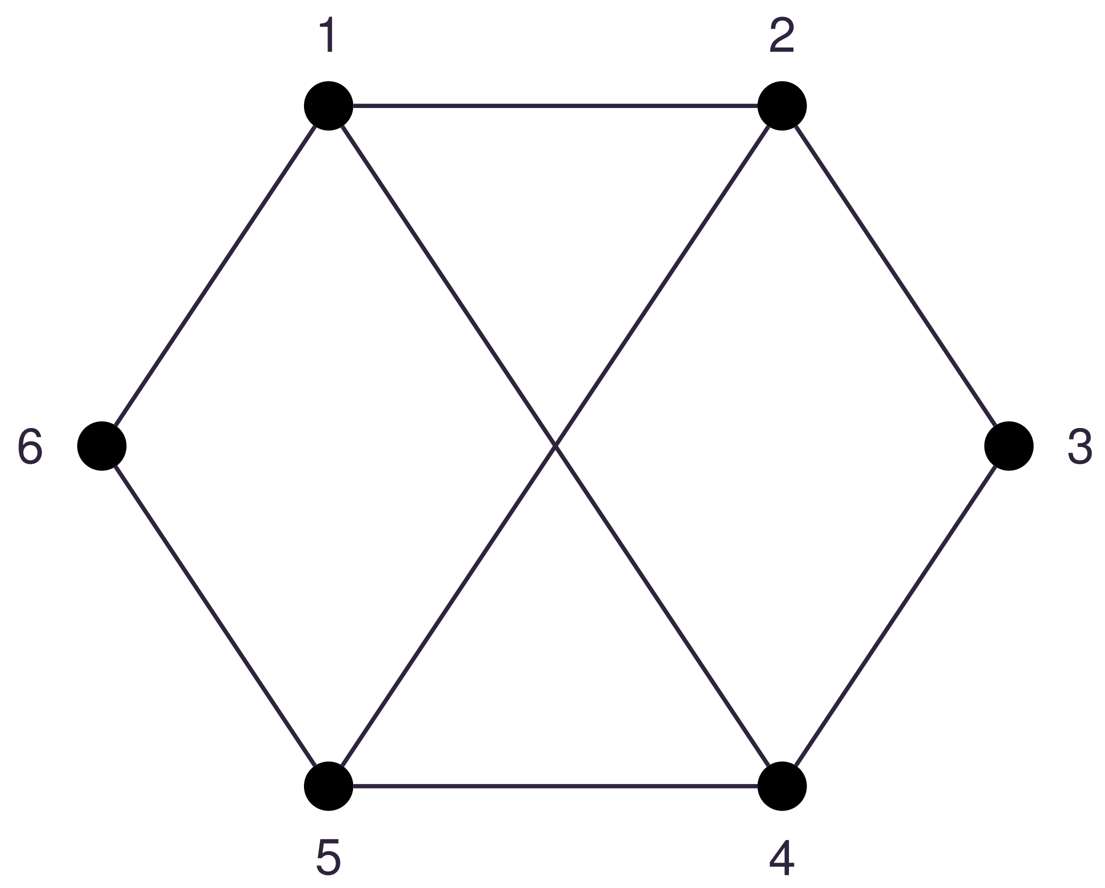
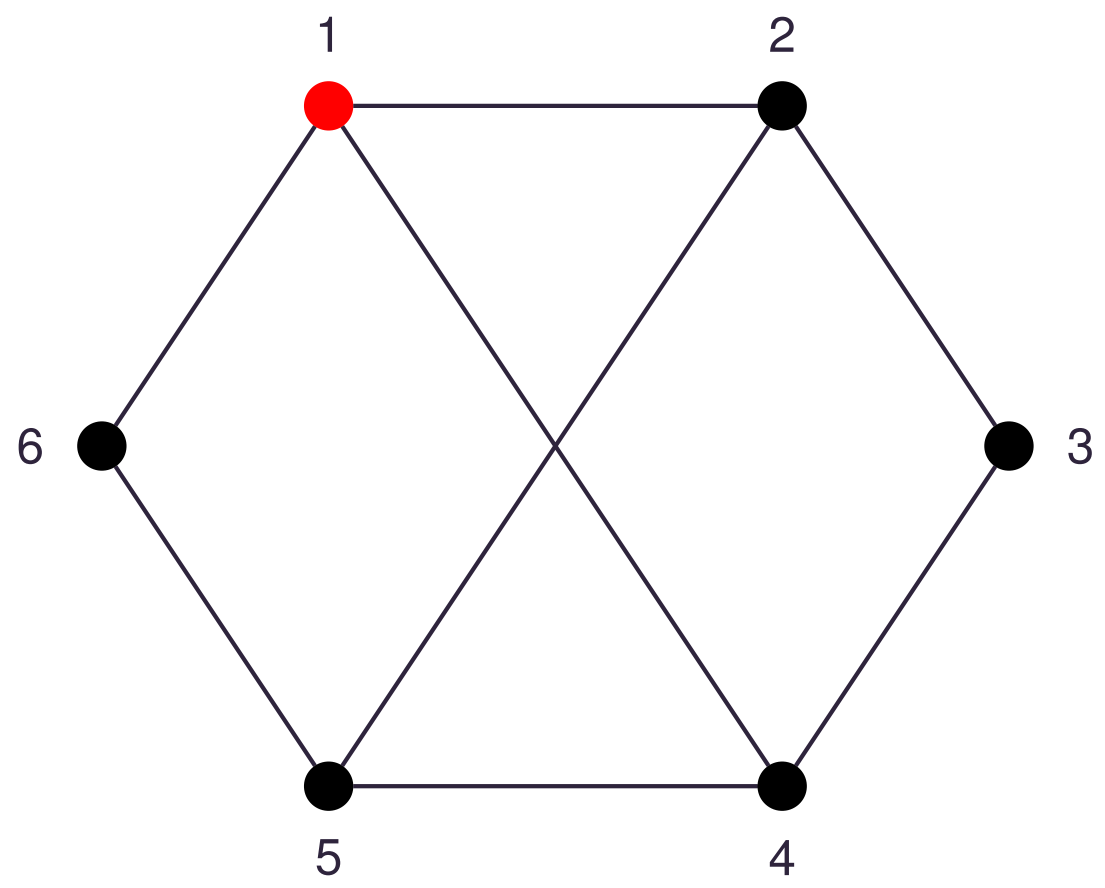
Definition: Let V be a finite set. A *convexity*, $\mathscr{C}$, is a collection of subsets of V that includes $\emptyset$ and V and is closed under taking intersections.

Definition: A set $S \subseteq V(G)$ is *digitally convex* if, for every $v \in V(G)$, we have $N[v] \subseteq N[S] \Rightarrow v \in S$.

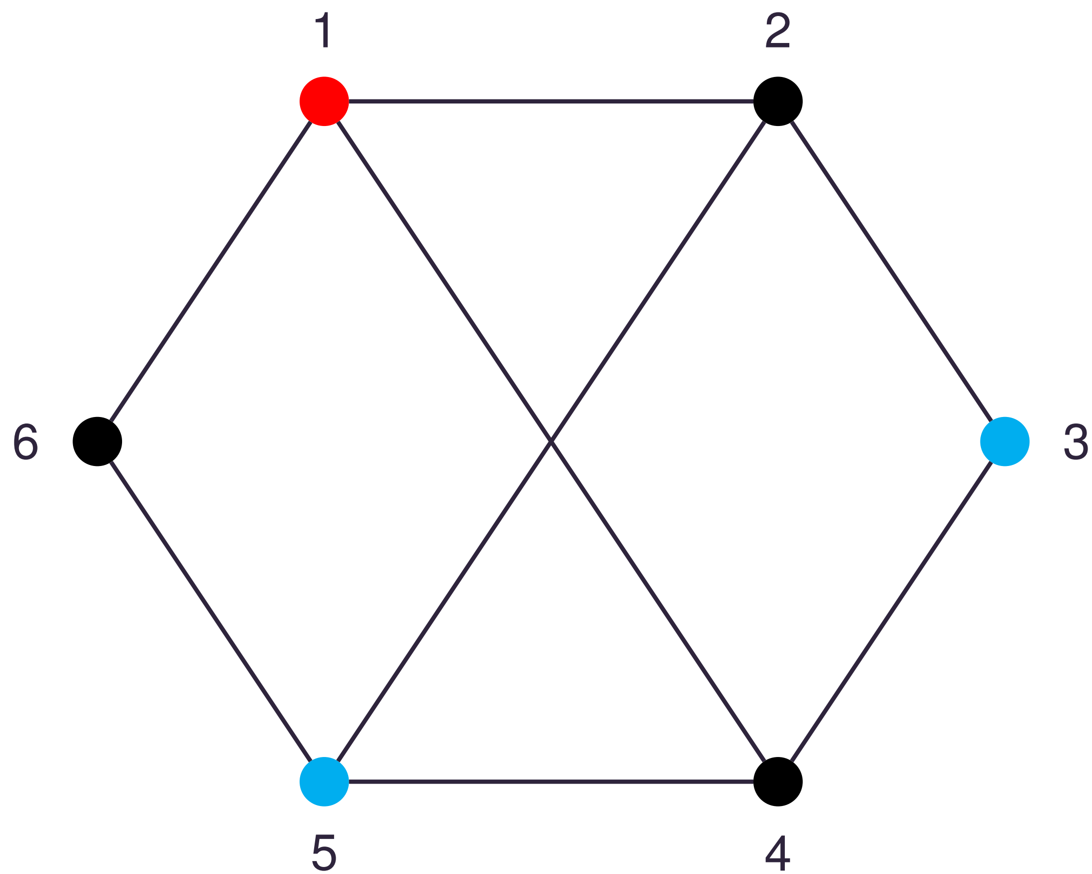
- $\rightarrow$ Every vertex $v \notin S$ must have private neighbour with respect to S.
- $\rightarrow$ Collection of digitally convex sets in a graph G is the *digital convexity* of G: $\mathscr{D}(G)$
- $\rightarrow$ Number of digitally convex sets in G: $n_{\mathscr{D}}(G)$

Example




1
2
6
3
5
4

Example



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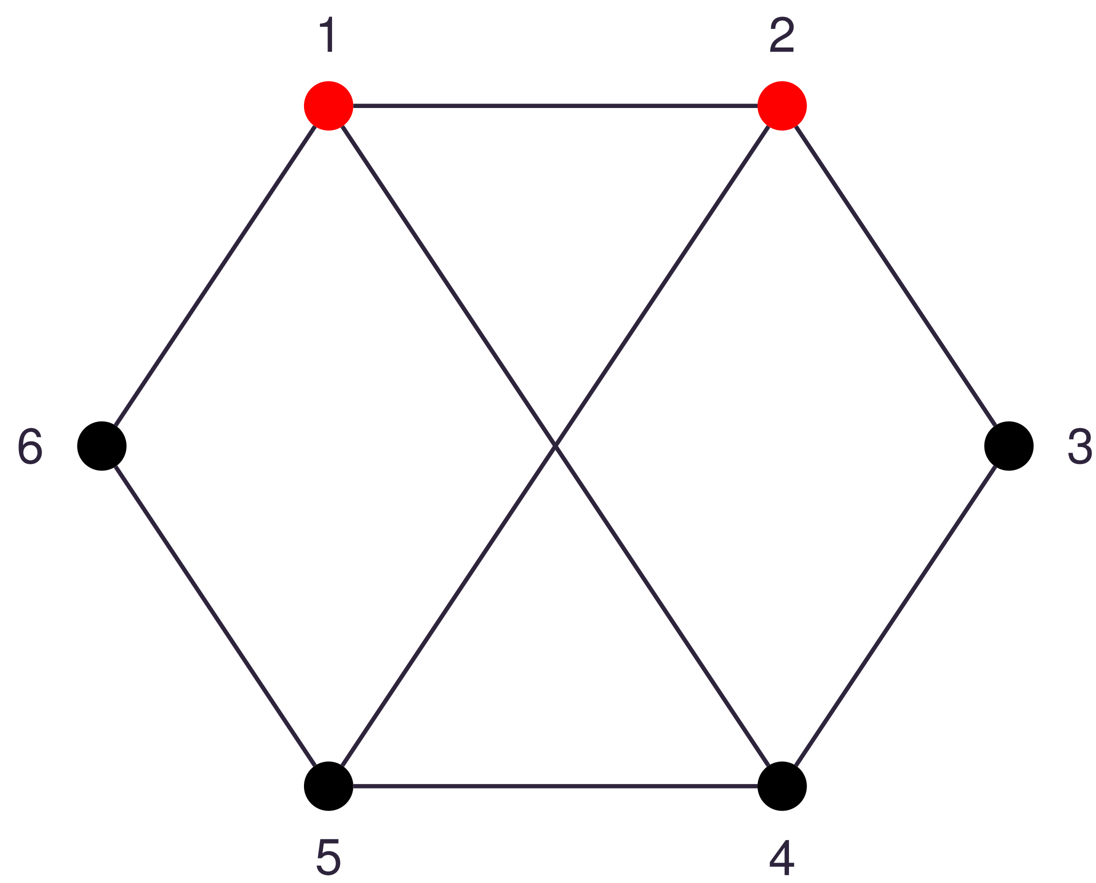
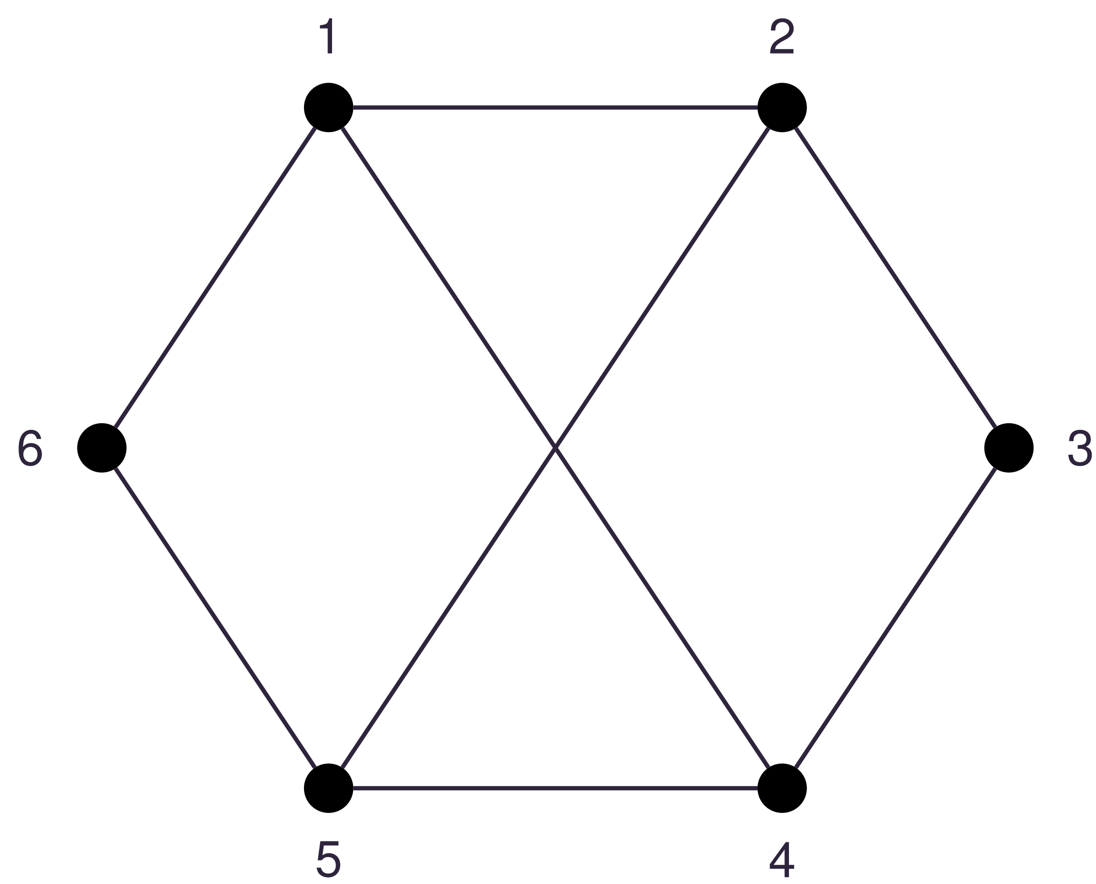
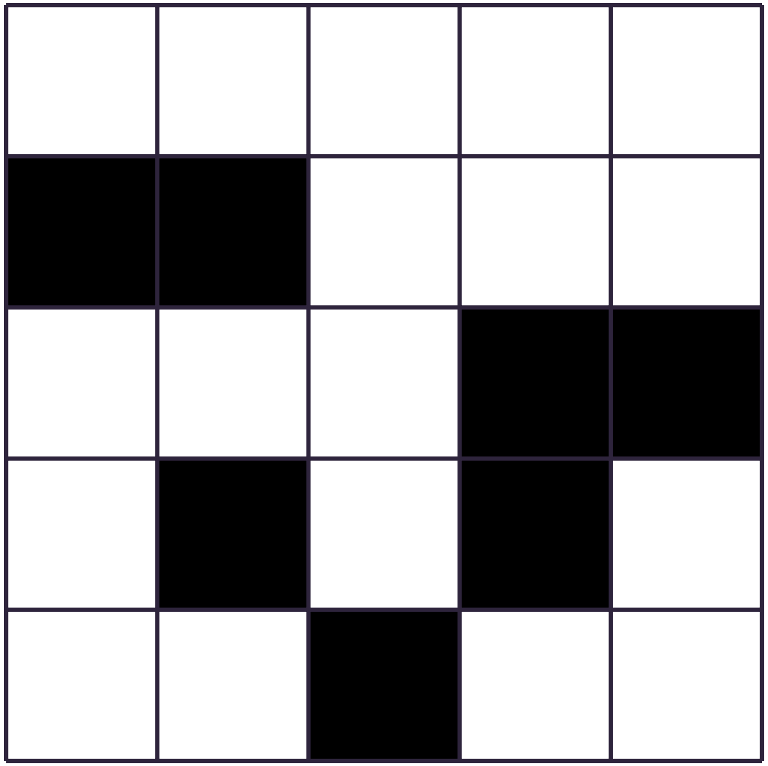
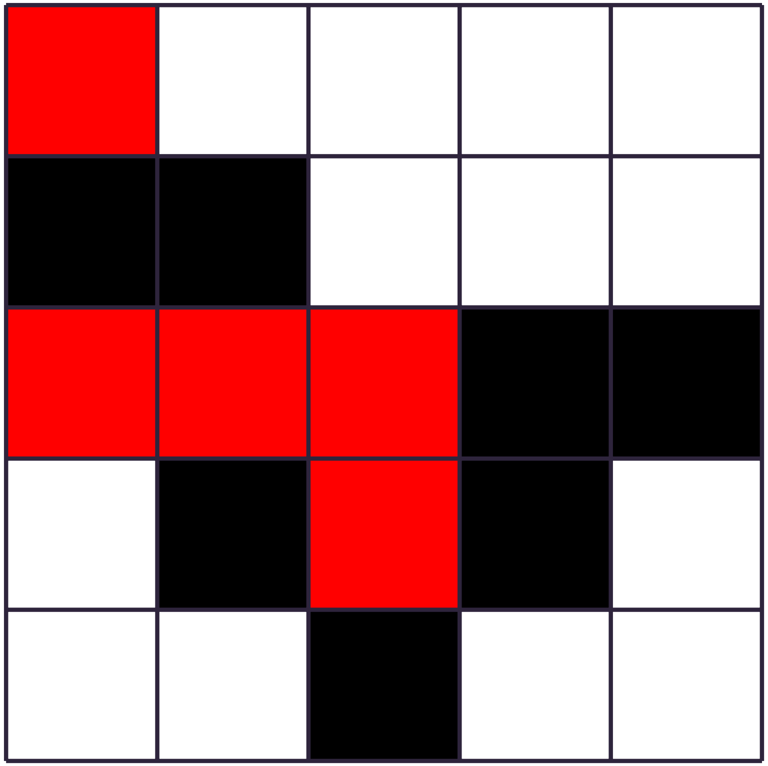


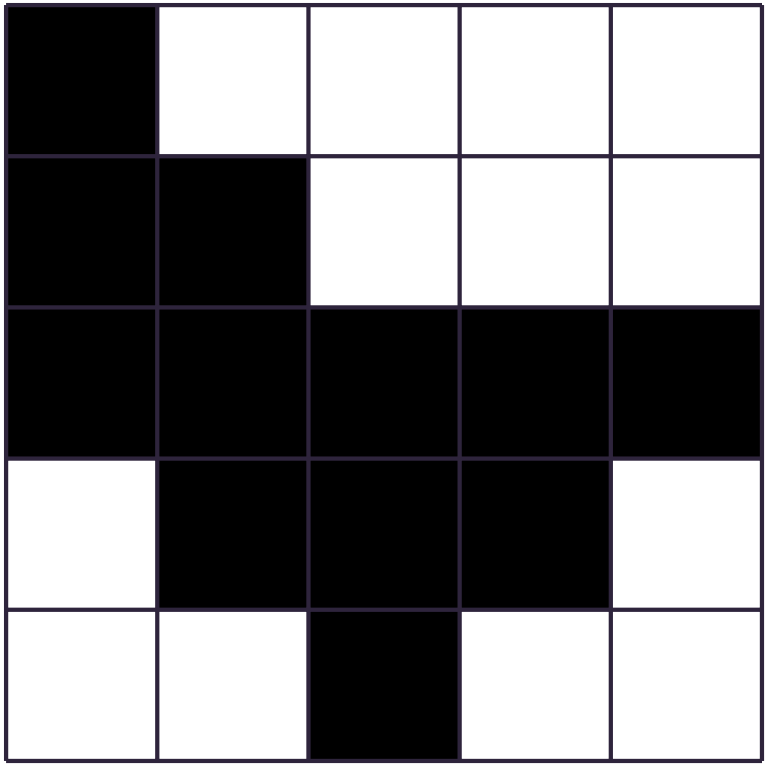


$$n_{\mathscr{D}}(G) = 14$$



Motivation: Image Smoothing





Theorem. (Lafrance, Oellermann, Pressey, 2016) *Let G be a graph. If S is a digitally convex set, then $\varphi(S) = V(G) - N[S]$ is also a digitally convex set.*

- $\to$ This function φ is a bijection from $\mathscr{D}(G)$ to itself.
- $\to$ Every graph must have an even number of digitally convex sets.

Properties of Digitally Convex Sets

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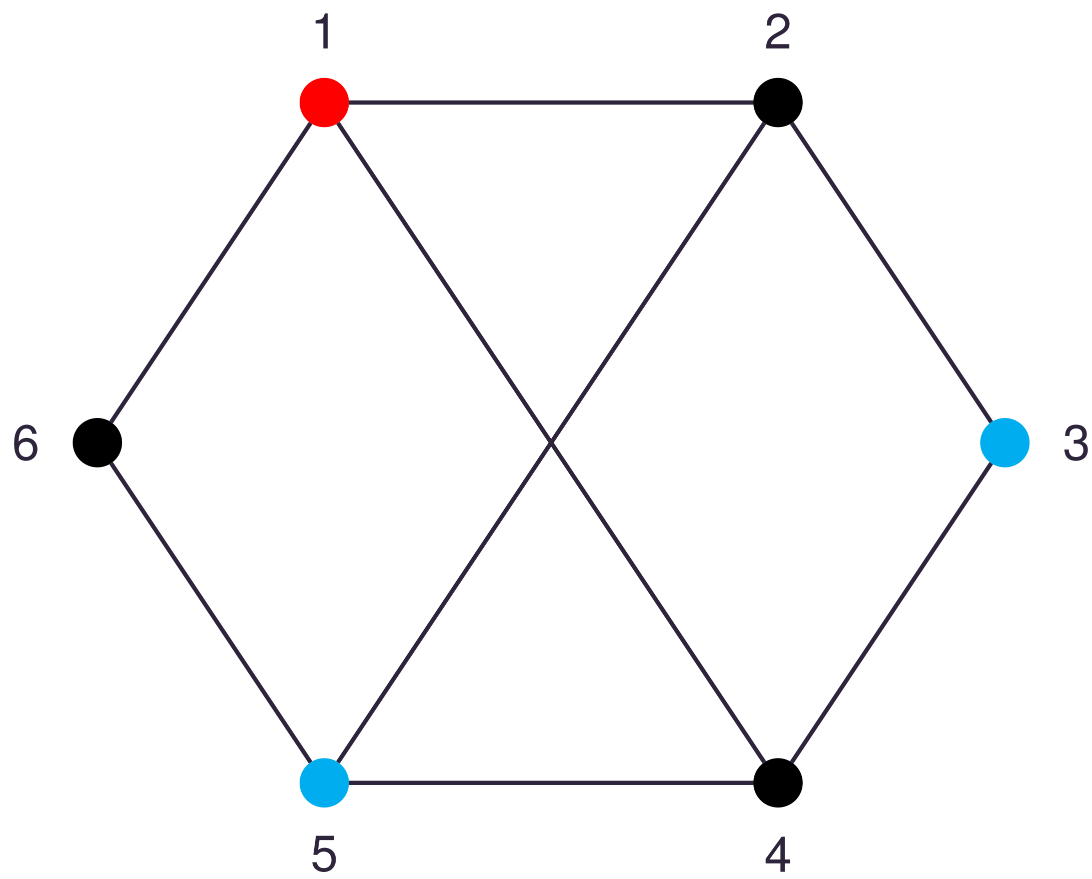
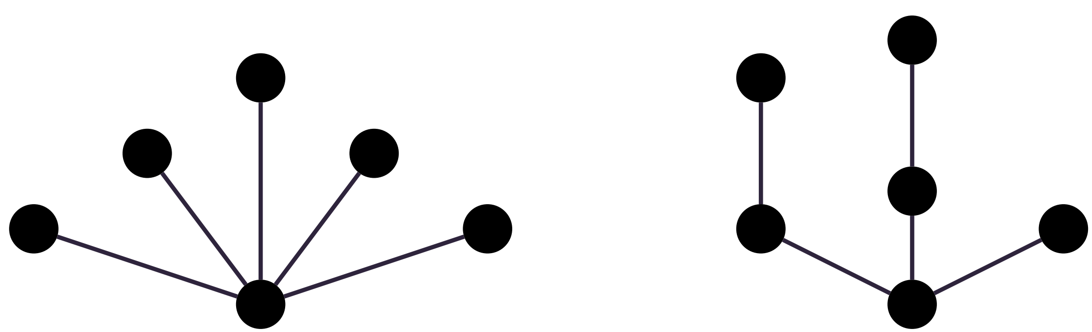


Theorem: (*Lafrance, Oellermann, Pressey, 2016*) Let T be a tree of order n. Then

$$\left.\begin{array}{lr} for\ n\ even & 2\cdot 2^{n/2}-2 \\ for\ n\ odd & 3\cdot 2^{(n-1)/2}-2 \end{array}\right\} \leq n_{\mathscr{D}}(T) \leq 2^{n-1}$$

The lower bound is attained by the spiderstar S_n and the upper bound by the star $K_{1,n-1}$.



$K_{1,5}$ S_6

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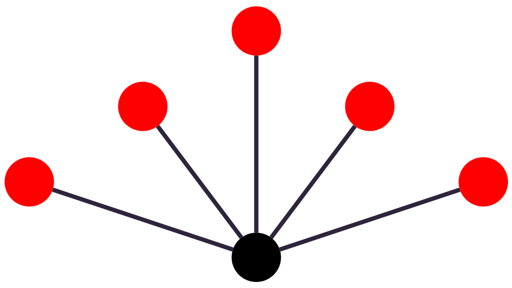
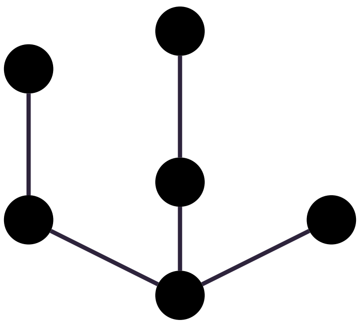


$K_{1,5}$



S_6

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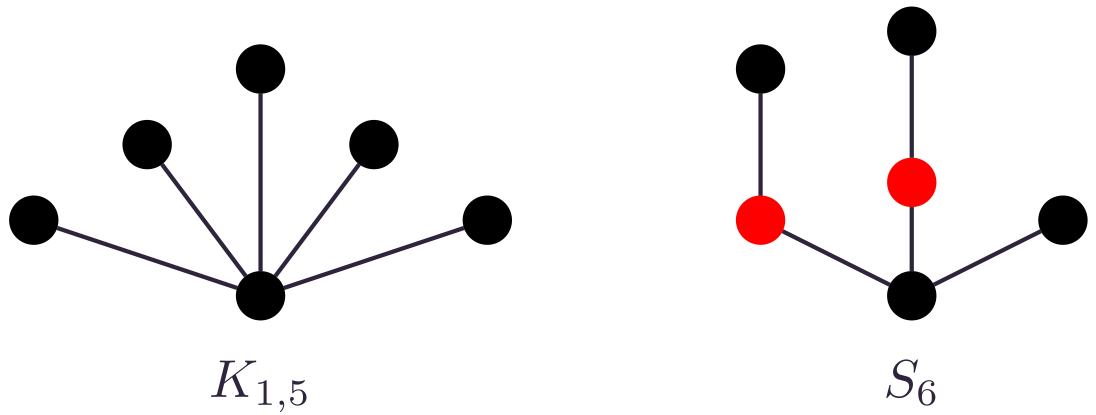
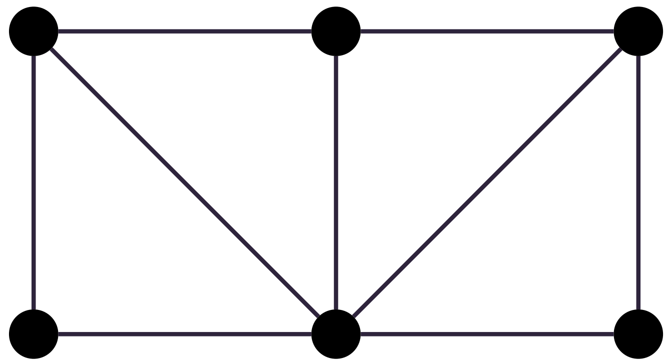
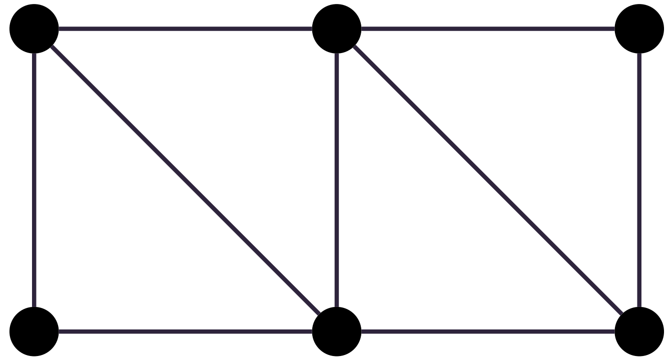


Definition: A 2-tree is a graph defined as follows: a 3-clique is a 2-tree and a 2-tree of order $n > 3$ is constructed by adding a vertex v adjacent to 2 pairwise adjacent vertices in a 2-tree of order $n - 1$.

Definition: A k-tree is a graph defined as follows: a $k + 1$-clique is a k-tree of order $n > k + 1$ is constructed by adding a vertex v adjacent to k pairwise adjacent vertices in a k-tree of order $n - 1$.

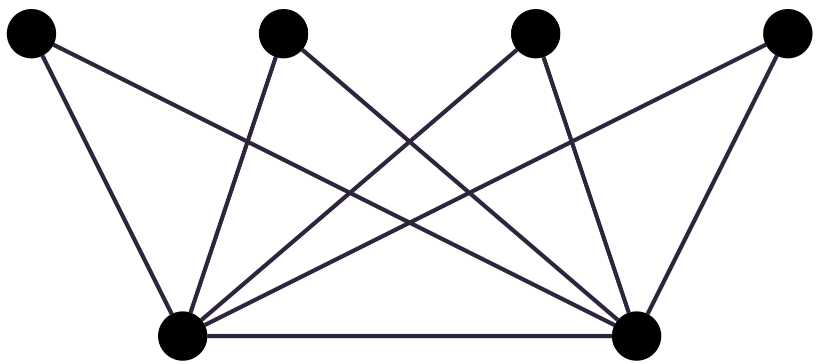


$$n_{\mathscr{D}}(G_1) = 10$$



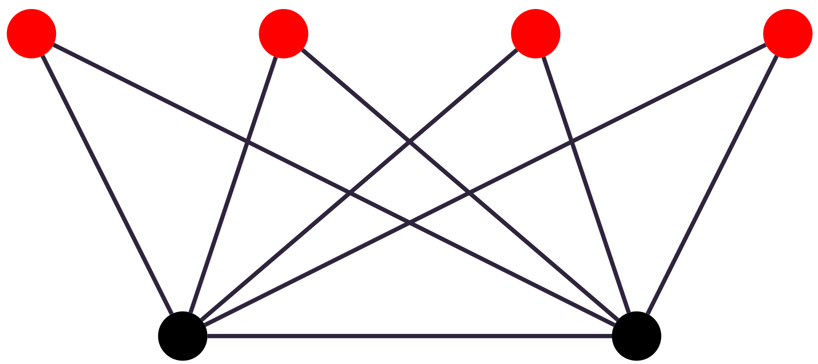
$$n_{\mathscr{D}}(G_2) = 8$$

Theorem: Let G be a 2-tree of order n. Then $n_{\mathscr{D}}(G) \leq 2^{n-2}$. This bound is attained by the 2-trees $K_2 + \overline{K}_{n-2}$.



$$n_{\mathscr{D}}(K_2 + \overline{K}_4) = 16$$

Theorem: Let G be a 2-tree of order n. Then $n_{\mathscr{D}}(G) \leq 2^{n-2}$. This bound is attained by the 2-trees $K_2 + \overline{K}_{n-2}$.



$$n_{\mathscr{D}}(K_2 + \overline{K}_4) = 16$$

Theorem: Let G be a k-tree of order n. Then $n_{\mathscr{D}}(G) \leq 2^{n-k}$.

$\rightarrow$ This bound is attained by the k-trees $K_k + \overline{K}_{n-k}$

Conjecture: Let G be a 2-tree of order n. Then

$$n_{\mathscr{D}}(G) \geq \begin{cases} 3 \cdot 2^{n/3} - 4, & for\; n \equiv 0 \pmod 3 \\ 4 \cdot 2^{(n-1)/3} - 4, & for\; n \equiv 1 \pmod 3 \\ 5 \cdot 2^{(n-2)/3} - 4, & for\; n \equiv 2 \pmod 3 \end{cases}$$

$\rightarrow$ Conjecture might be proven by induction on n and by dividing all 2-trees into subclasses based on the structure of the neighbourhoods of vertices of degree 2.

The conjectured lower bound is attained by the 2-*spiderstars*, which are constructed as follows:

- $\rightarrow$ Begin with a K_2 with vertices x and y
- $\rightarrow$ Repeat to add the remaining $n-2$ vertices:
  - Add vertex w adjacent to x and y
  - Add vertex u adjacent to x and w
  - Add vertex v adjacent to w and u

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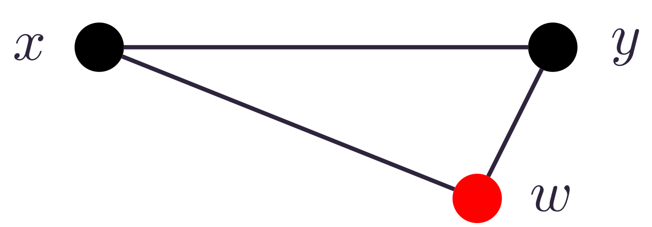

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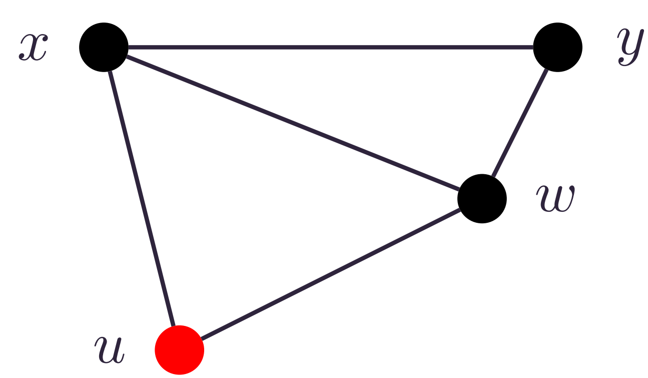

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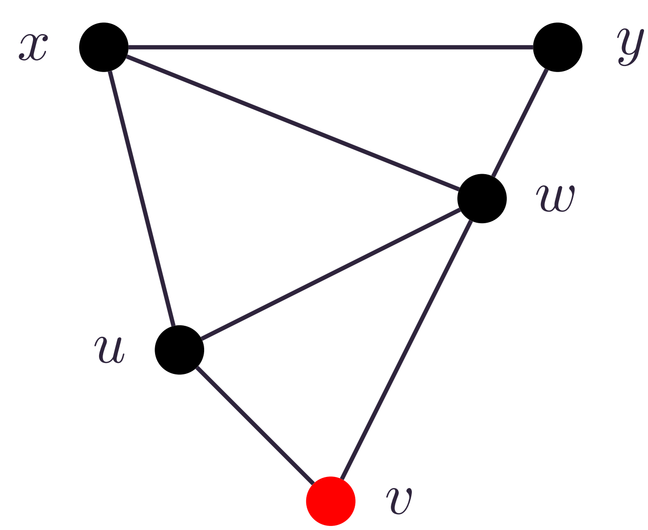
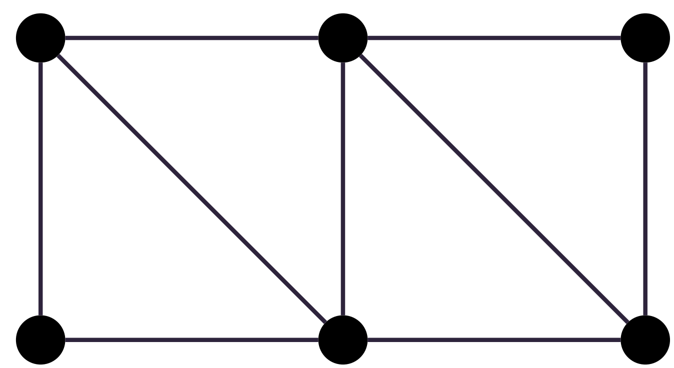


Special Case: Powers of Paths

Theorem: Let P_n^2 be the square of the path P_n. Then the digitally convex sets of P_n^2 satisfy the recurrence

$$n_{\mathscr{D}}(P_n^2) = n_{\mathscr{D}}(P_{n-1}^2) + n_{\mathscr{D}}(P_{n-3}^2)$$

with $n_{\mathscr{D}}(P_3^2) = 2, n_{\mathscr{D}}(P_4^2) = 4, n_{\mathscr{D}}(P_5^2) = 6$. (OEIS sequence A000930, multiplied by 2.)



$$n_{\mathscr{D}}(P_6^2) = n_{\mathscr{D}}(P_5^2) + n_{\mathscr{D}}(P_3^2) = 6 + 2 = 8$$

Special Case: Powers of Paths

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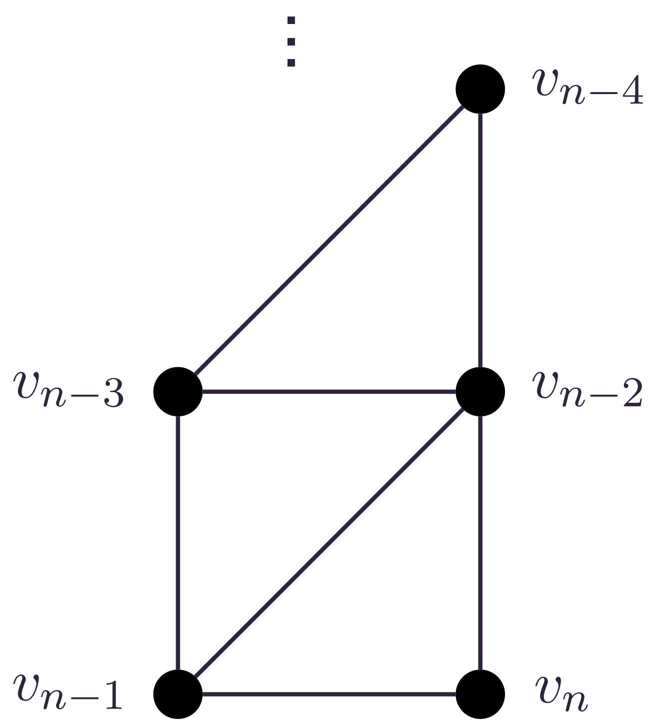

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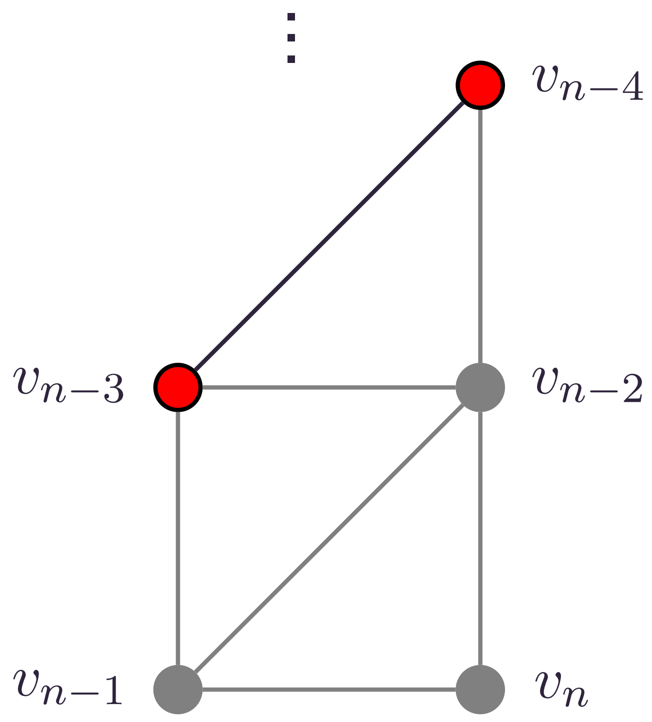

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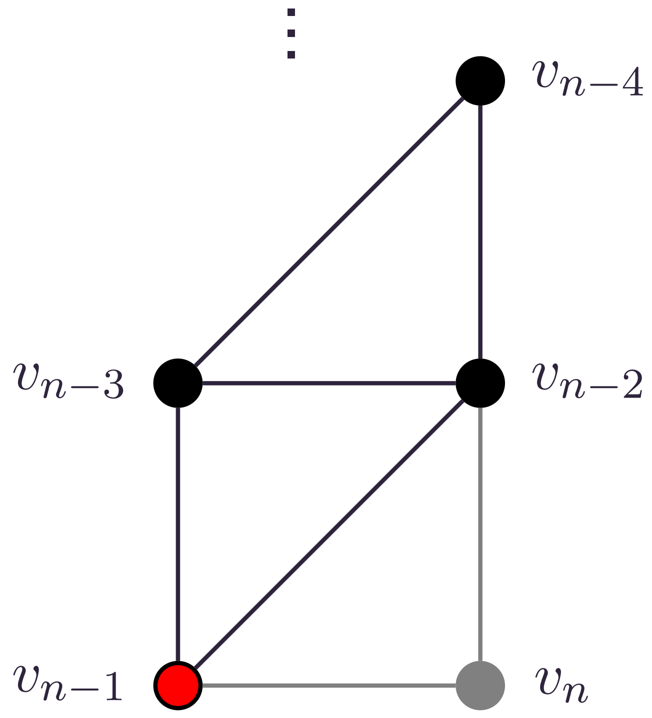

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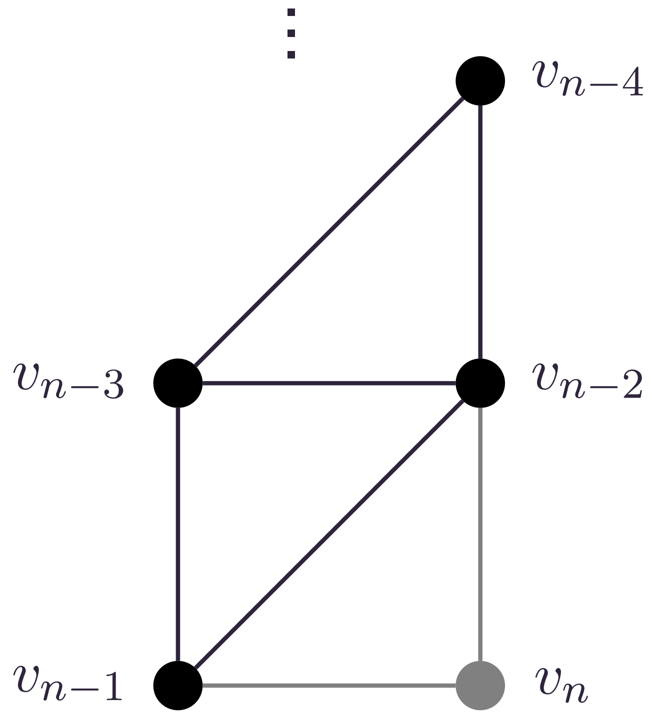

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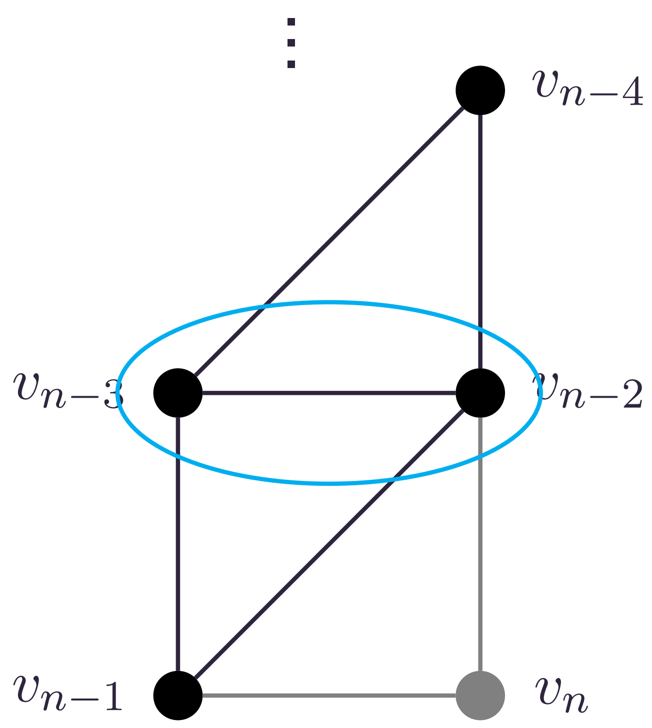

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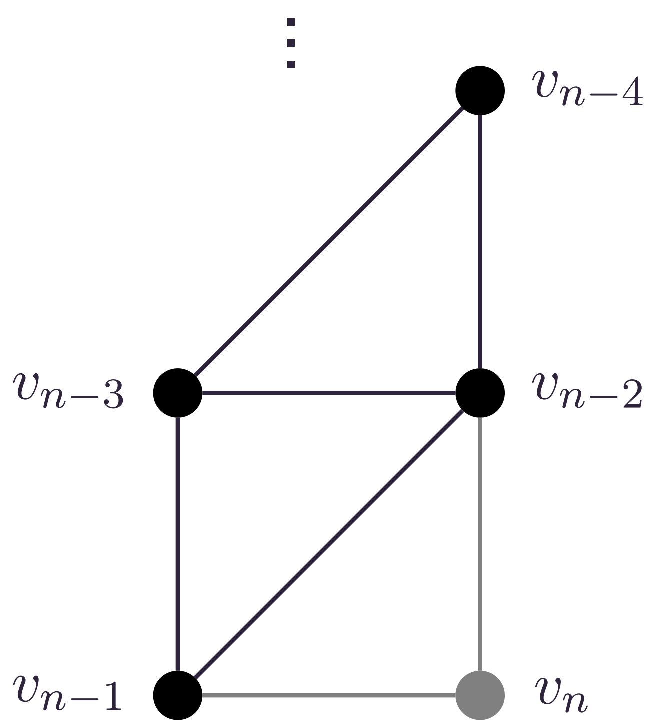


Theorem: Let P_n^k be the k^{th} power of the path P_n. Then the digitally convex sets of P_n^k satisfy the recurrence

$$n_{\mathscr{D}}(P_n^k) = n_{\mathscr{D}}(P_{n-1}^k) + n_{\mathscr{D}}(P_{n-k-1}^k)$$

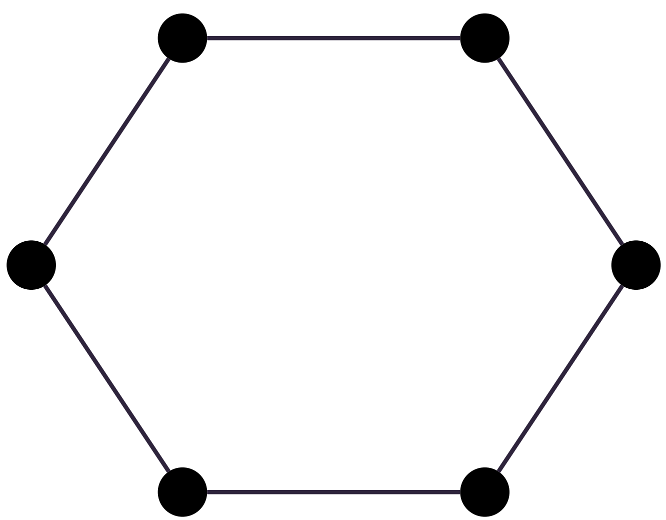
Theorem: Let C_n be the cycle of order n. Then, $n_{\mathscr{D}}(C_3) = 2$, $n_{\mathscr{D}}(C_4) = 6$, $n_{\mathscr{D}}(C_5) = 12$, $n_{\mathscr{D}}(C_6) = 20$ and, for $n \geq 7$,

$$n_{\mathscr{D}}(C_n) = 2n_{\mathscr{D}}(C_{n-1}) - n_{\mathscr{D}}(C_{n-2}) + n_{\mathscr{D}}(C_{n-4}).$$

This is equivalent to the number of cyclic binary n-bit strings with no alternating substring of length greater than 2. (OEIS sequence A007039)

Bijection between $\mathscr{D}(C_n)$ and cyclic binary n-bit strings without 010 or 101:

- $\rightarrow$ Label edges of C_n from 1 to n
- $\rightarrow$ Given a digitally convex set S, construct a cyclic binary string S^* such that bit i is 1 if edge i is incident with a vertex in S, and 0 otherwise



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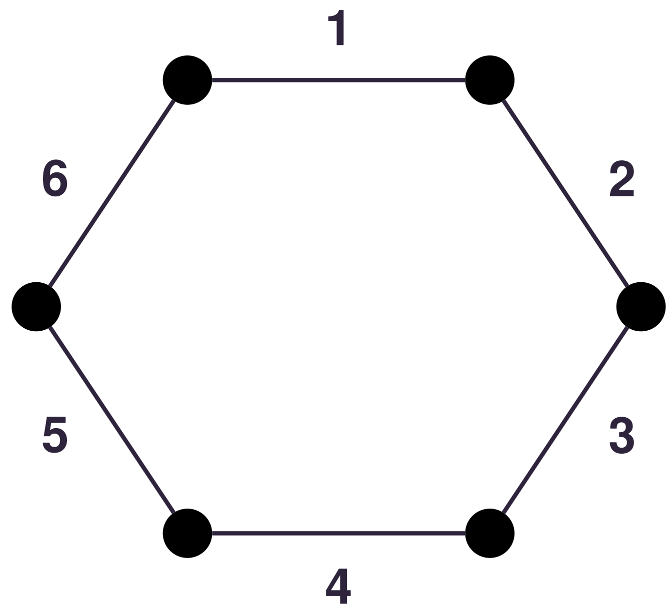

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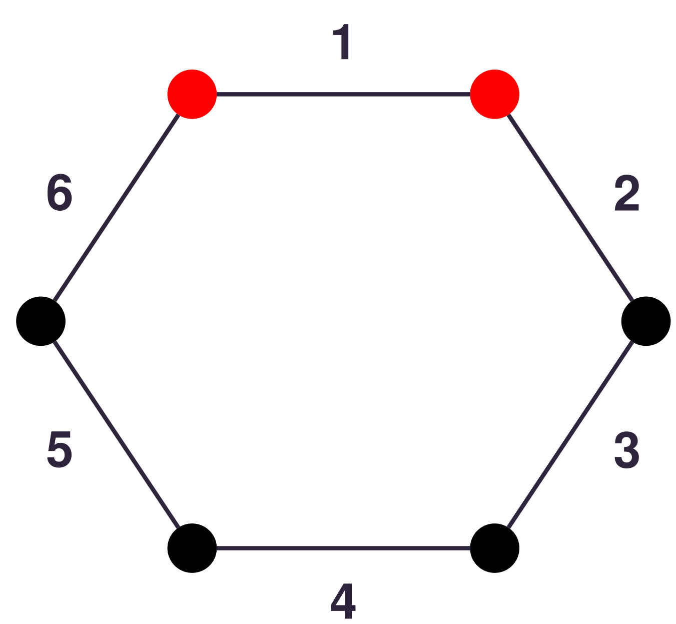

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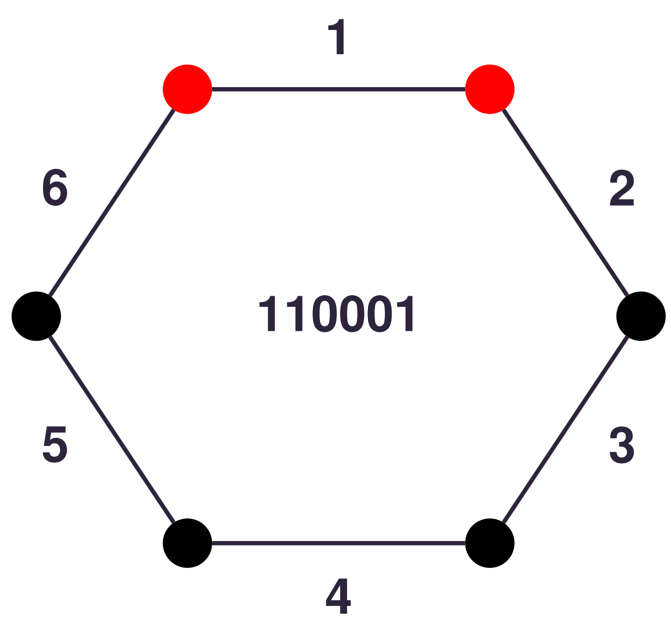
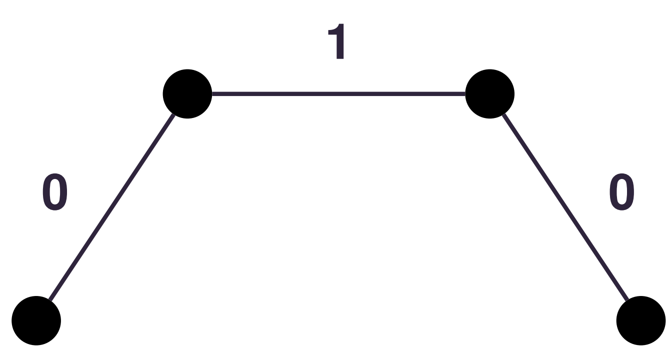


Bijection between $\mathscr{D}(C_n)$ and cyclic binary n-bit strings without 010 or 101:

- $\rightarrow$ Label edges of C_n from 1 to n
- $\rightarrow$ Given a digitally convex set S, construct a cyclic binary string S^* such that bit i is 1 if edge i is incident with a vertex in S, and 0 otherwise
- $\rightarrow$ No substring 010:



Bijection between $\mathscr{D}(C_n)$ and cyclic binary n-bit strings without 010 or 101:

- $\rightarrow$ Label edges of C_n from 1 to n
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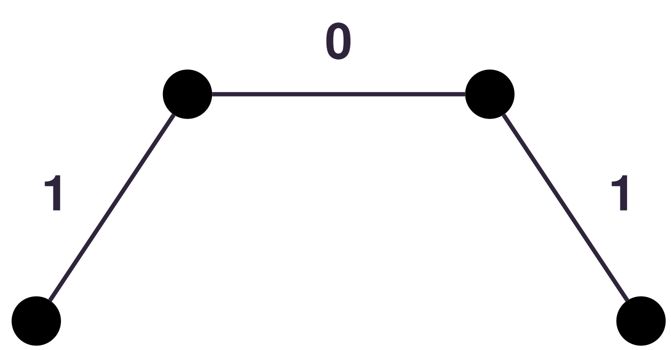
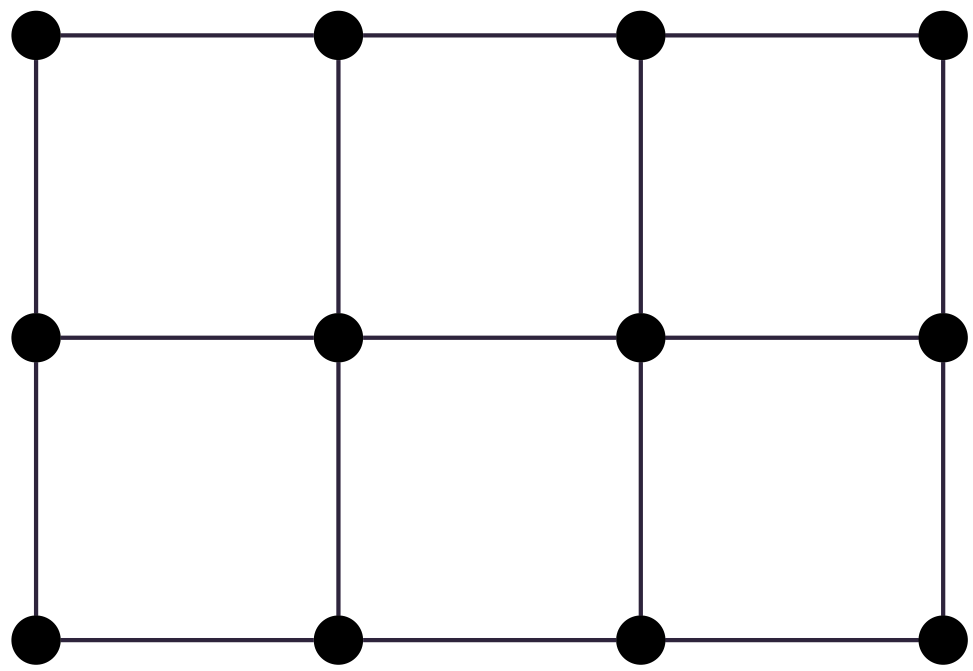


Theorem: Let C_n be the cycle of order n and let $k \geq 1$. Then, $n_{\mathscr{D}}(C_i^k) = 2$ for $3 \leq i \leq 2k+1$, $n_{\mathscr{D}}(C_j^k) = 2 + j(j-2k-1)$ for $2k+2 \leq j \leq 2k+4$ and, for $n \geq 2k+5$,

$$n_{\mathscr{D}}(C_n^k) = 2n_{\mathscr{D}}(C_{n-1}^k) - n_{\mathscr{D}}(C_{n-2}^k) + n_{\mathscr{D}}(C_{n-2k-2}^k).$$

$\rightarrow$ Proof uses a bijection between the sets in $\mathscr{D}(C_n^k)$ and the cyclic binary strings whose blocks (maximal runs of 0's or 1's) each have length at least $k+1$

Definition: The *Cartesian product* of graphs G and H, denoted by $G \square H$ is the graph with vertex set $V(G \square H) = V(G) \times V(H)$ and such that two vertices (x, y) and (u, v) are adjacent in $G \square H$ if and only if $x = u$ in G and $yv \in E(H)$ or $y = v$ in H and $xu \in E(G)$.



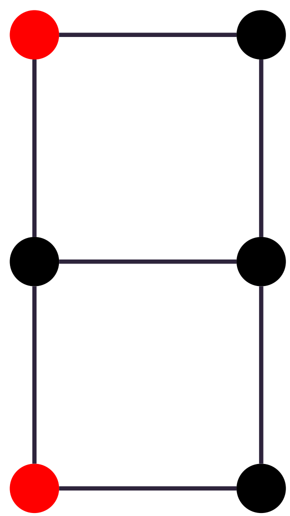
Let A be an $n \times m$ binary array. Then A^* is the $n \times m$ binary array whose entries are the minimum over the closed neighbourhood of the corresponding entry in A

$$A = \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \qquad A^* = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Digital Convexity in Cartesian Product of Paths

Theorem: Let $\mathscr{A}_{n,m}$ be the set of all $n \times m$ binary arrays. Then $n_{\mathscr{D}}(P_n \square P_m) = |\mathscr{A}^*_{n,m}|$.

Outline of proof:



1	**0**
0	**0**
1	**0**

Theorem: Let $\mathscr{A}_{n,m}$ be the set of all $n \times m$ binary arrays. Then $n_{\mathscr{D}}(P_n \Box P_m) = |\mathscr{A}^*_{n,m}|$.

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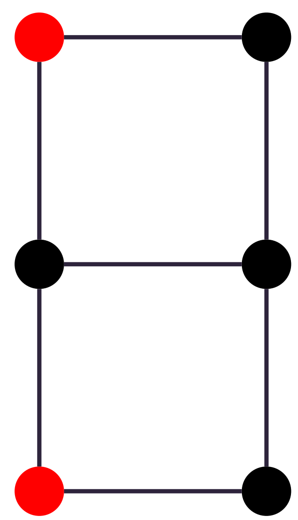

1	1
1	0
1	1

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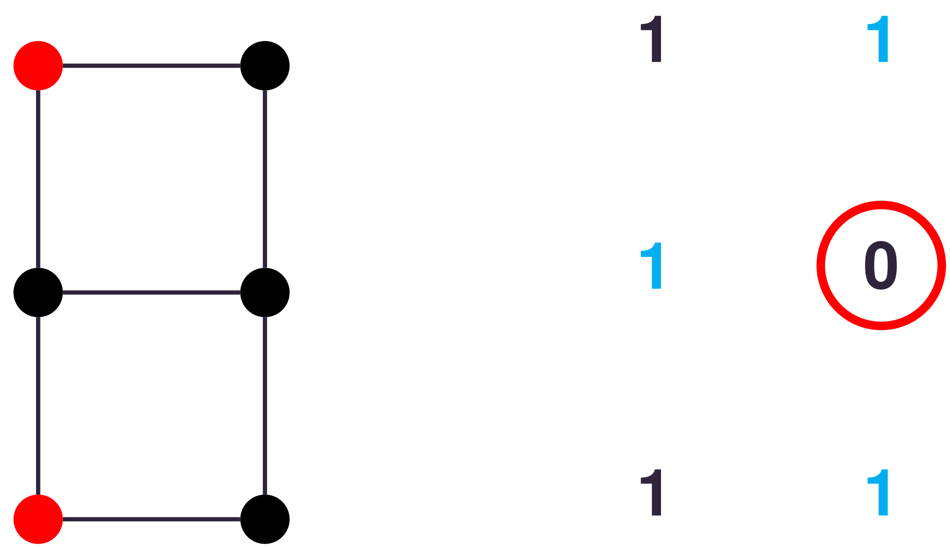

Theorem: Let $\mathscr{A}_{n,m}$ be the set of all $n \times m$ binary arrays. Then $n_{\mathscr{D}}(P_n \Box P_m) = |\mathscr{A}^*_{n,m}|$.

$\rightarrow$ OEIS sequence A217637 — also equal to the number of maximal independence sets in $P_n \Box P_m \Box P_2$

Open Problems

- $\rightarrow$ Is there a formula or upper/lower bounds on $n_{\mathscr{D}}(G\Box H)$ in terms of $n_{\mathscr{D}}(G)$ and $n_{\mathscr{D}}(H)$?
- $\rightarrow$ What do the digitally convex sets look like in other graph products?
- $\rightarrow$ What happens to the number of digitally convex sets in a graph when an edge is added or removed?

Open Problems

→ Is there a formula or upper/lower bounds on $n_{\mathscr{D}}(G \Box H)$ in terms of $n_{\mathscr{D}}(G)$ and $n_{\mathscr{D}}(H)$?

→ What do the digitally convex sets look like in other graph products?

→ What happens to the number of digitally convex sets in a graph when an edge is added or removed?

$$n_{\mathscr{D}}(P_4) = 6$$

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→ What happens to the number of digitally convex sets in a graph when an edge is added or removed?

$$n_{\mathscr{D}}(P_4 - e_1) = 8$$

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- $\rightarrow$ Is there a formula or upper/lower bounds on $n_{\mathscr{D}}(G \square H)$ in terms of $n_{\mathscr{D}}(G)$ and $n_{\mathscr{D}}(H)$?
- $\rightarrow$ What do the digitally convex sets look like in other graph products?
- $\rightarrow$ What happens to the number of digitally convex sets in a graph when an edge is added or removed?

$$n_{\mathscr{D}}(P_4 - e_2) = 4$$

THANK YOU!