An approximation algorithm for finding the zero-forcing number of a graph

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Joint work with Ben Cameron, Rogers Mathew and Zhiyuan (Owen) Zhang

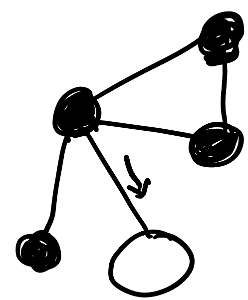
Zero Forcing

AIMS Workshop 2006

Process: Vertices coloured black or white

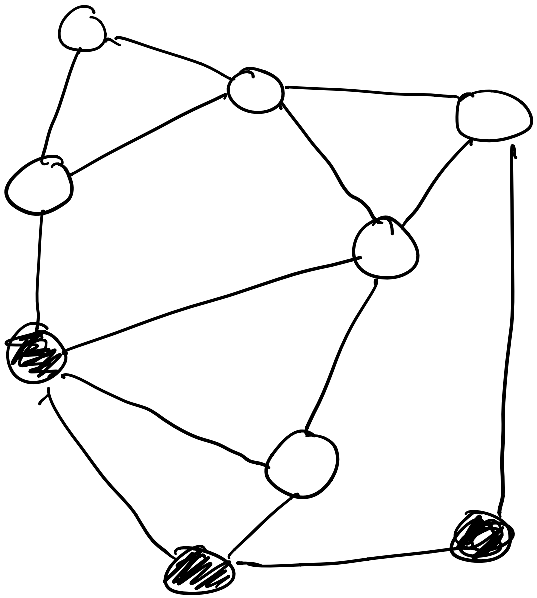
In each step, a black vertex with exactly one white neighbour colours that neighbour black

<u>Zero forcing set</u> Initial set of black vertices so that process ends with all vertices black

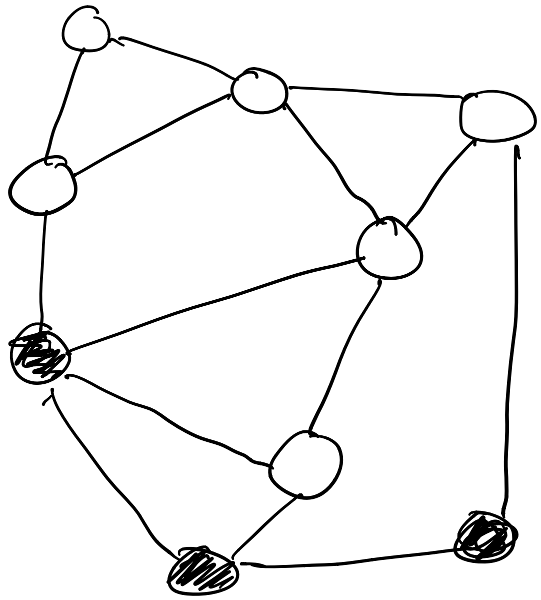


$Z(G)$ = min size of a zero forcing set

Zero Forcing



Zero Forcing

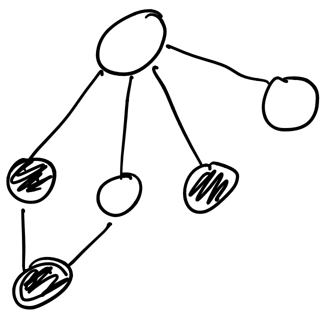


For all graphs G

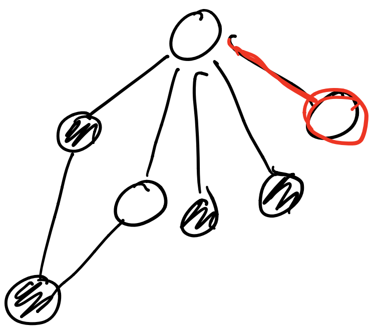
$Z(G) \geq \delta(G)$

Zero Forcing

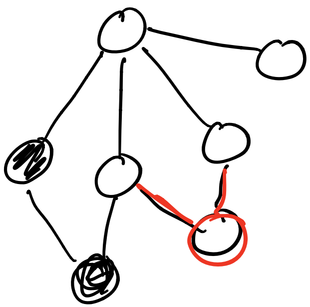
$Z(G)$ not monotone



$Z(G) = 3$



$Z(G) = 4$

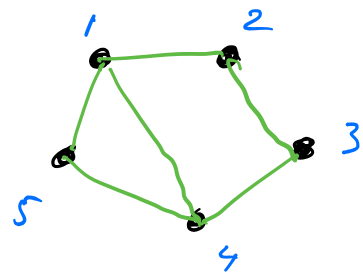


$Z(G) = 2$

Zero Forcing

Combinatorial matrix theory

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & & & & & \\
2 & & & & & \\
3 & & & & & \\
4 & & & & & \\
5 & & & & &
\end{array}
\quad
A = \begin{bmatrix} 0 & 2 & 0 & -1 & 5 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ -1 & 0 & -2 & 1 & 1 \\ 5 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

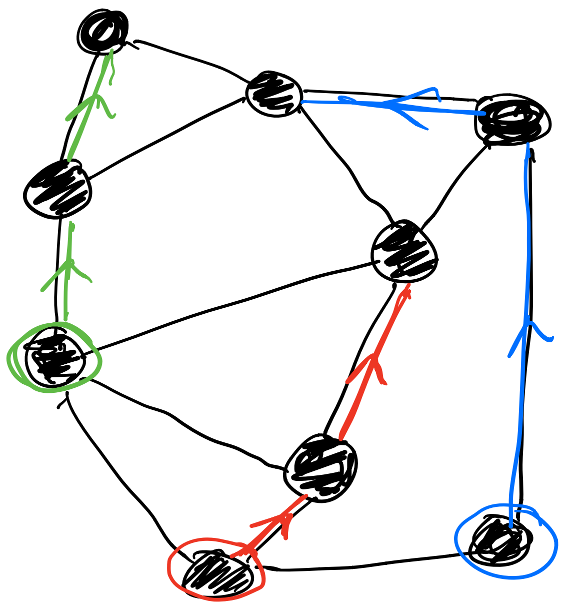


$G(A)$

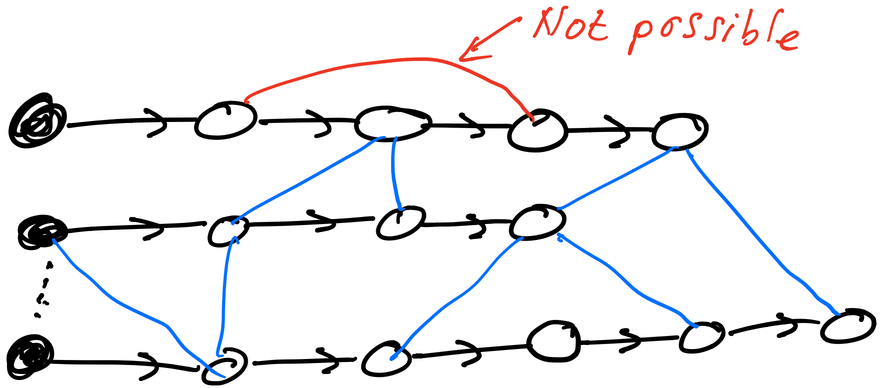
$$M(G) = \text{Max Nullity of } A$$
$$= \operatorname{Max}\{\operatorname{Null}(A) : G(A) = G\}$$

$$M(G) \le Z(G)$$

Forcing chains



Forcing chains



Forcing chains form a partition of the vertices into <u>induced</u> paths

Chain twists

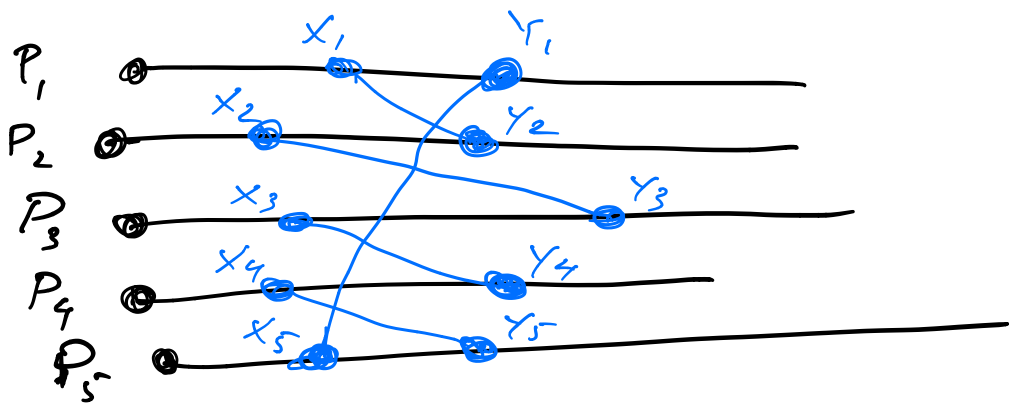
Given a set of (oriented) disjoint paths

$P_1\ P_2\ \ldots\ P_t$, a chain twist

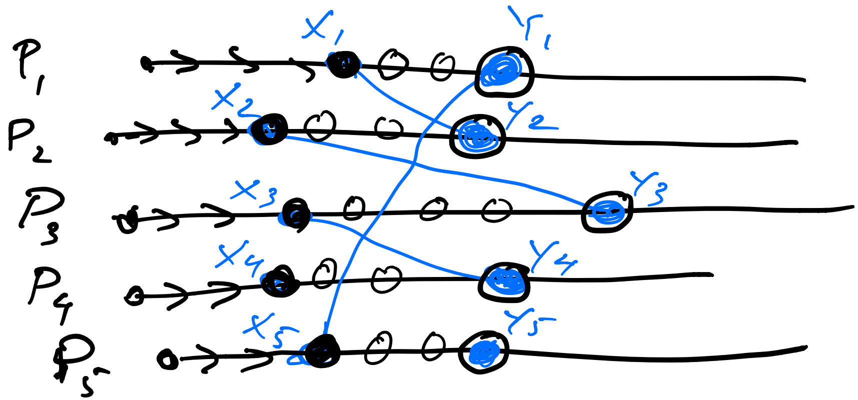
is a set of edges $x_1y_2\ x_2y_3 \ldots x_{t-1}y_t, x_ty_1$

so that x_i, y_i on P_i and y_i is further

from the start vertex of P_i than x_i.



Chain twists



Forcing chains cannot have a chain twist

Proper path partition (ppp)

Definition

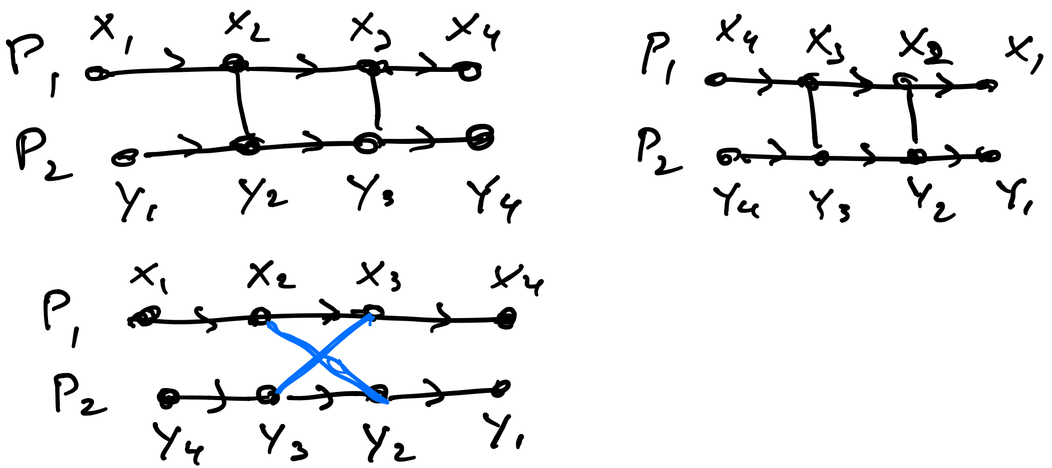
A *proper path partition (ppp)* of a graph G is a partition of the vertices of G into paths so that each path is induced, and an orientation of each path so that there is no chain twist.

Theorem (JJ, Cameron, Zhang)

For each graph G, $z(G) \leq k$ if and only if G has a proper path partition consisting of k paths.

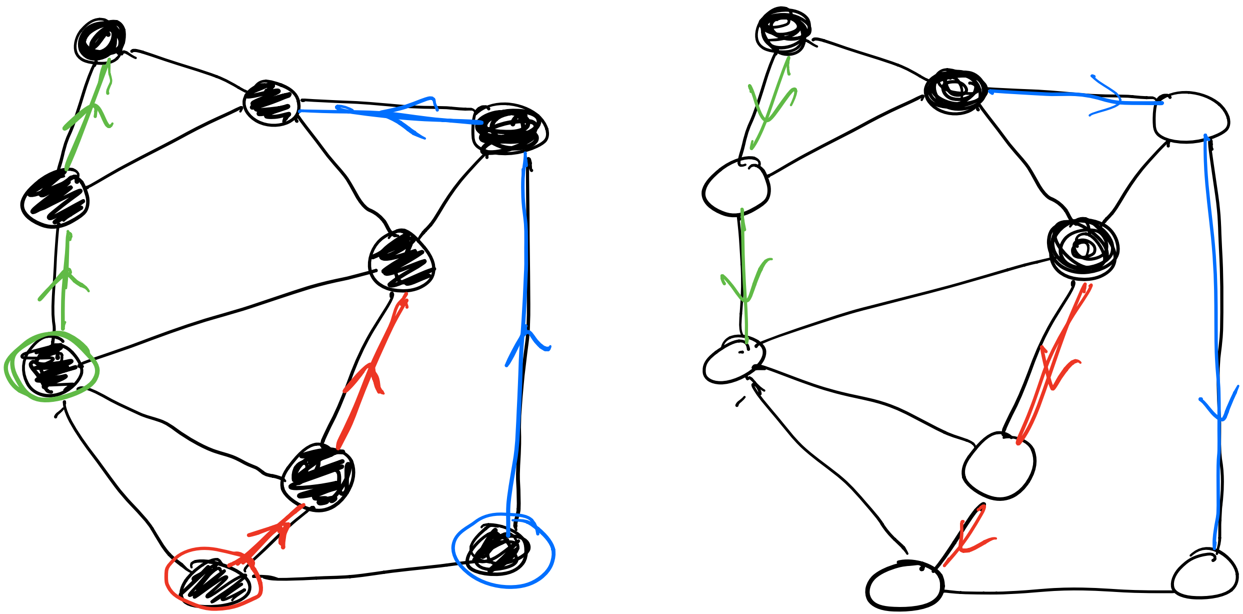
New proofs of old results: forcing chains are reversible

Reversing the orientation of *all* paths cannot create chain twists

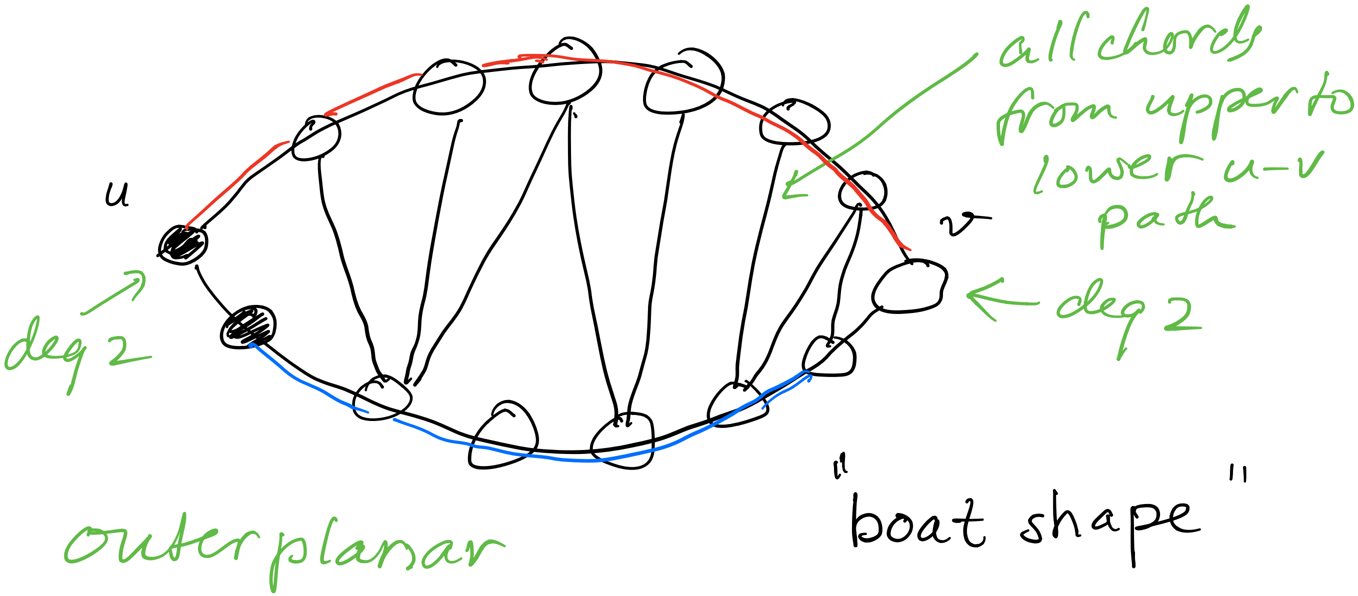


$\Longrightarrow$ End vertices of forcing chains also form a forcing set.

New proofs of old results: forcing chains are reversible



New proofs of old results: $\delta(G) = z(G) = 2$



New proofs of old results: unit interval graphs

Unit interval graph

$v_1 \; v_2 \; \cdots\cdots \; v_n$ vertices

If $v_i \sim v_j$, $i < j$, then $v_i, v_{i+1}, \ldots, v_j$ clique

Edge clique cover: collection of cliques covering all edges.

$cc(G)$ = size of smallest edge clique cover

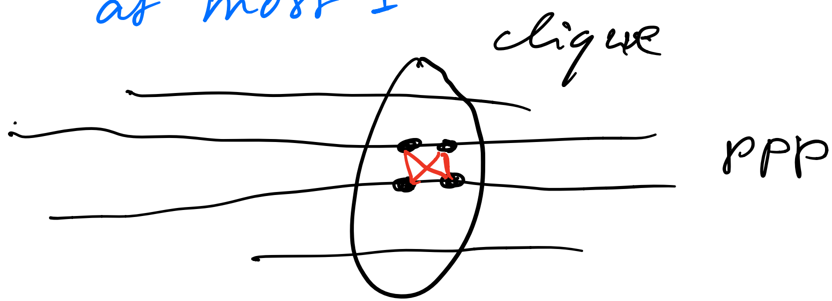
New proofs of old results: unit interval graphs

Huang, Chang, Yeh 2010

For all graphs G :

$$z(G) \geq n - cc(G)$$

Only one path in a ppp
can contain 2 vertices
in a clique – the others
at most 1



New proofs of old results: unit interval graphs

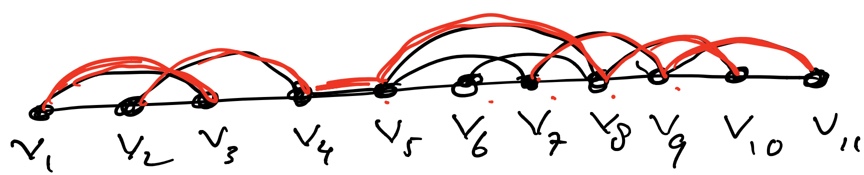
Huang, Chang, Yeh 2010

For unit interval graphs G,

$$z(G) = n - cc(G)$$

Take "furthest" edge in every clique of the cover. These form a PPP of size $n - cc(G)$

$n = 11$
$cc = 7$
4 paths



Approximation algorithm

A. Azaami, 2007

Finding $Z(G)$ is NP-Hard

B. Brimkov, I. Hicks

Efficient algorithms for finding $Z(G)$ of some special classes of graphs

Sean English – visit 2019

JJ – Mathew

Approximation algorithm,

approx. ratio $pw+1$

pw = path-width

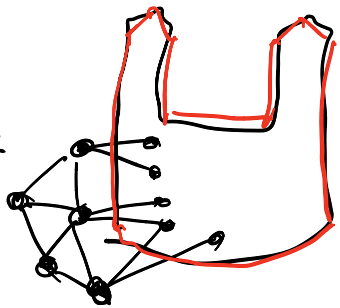
mistake in abstract

Forts

C. Fast 2017

A fort is a set of vertices S so that every vertex not in S has zero or ≥ 2 neighbours in S.

- Every fort must contain a vertex from the forcing set



- Forcing set = set of vertices intersecting all forts

Approximation algorithm

JJ, Mathew

Algorithm which returns

- zero forcing set B
- collection of disjoint forts F

so that $|B| \le (pw+1)\,|F|$

where pw = path width of graph

Approximation ratio $pw+1$

$(Z(G) \geq |F|)$

Path decomposition

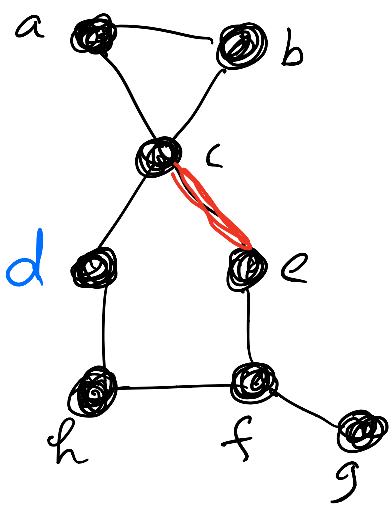
$G = (V, E)$

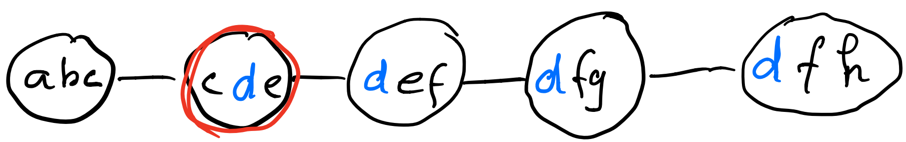
Collection of "bags" $X_1, X_2, \ldots\ldots X_p$

$X_i \subseteq V$

such that

- for all $\{u, v\} \in E$, $\exists i$ $\{u, v\} \subseteq X_i$
- for all $u \in V$, bags containing u are contiguous





Path decomposition

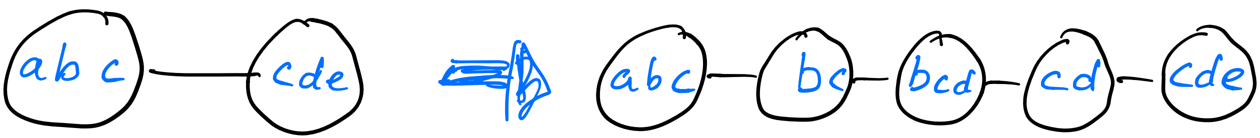
$pw(G)$ = minimum size s so that G has a path decomposition where all bags have size $\leq s+1$.

abc — cde — def — dfg — dfh $pw = 2$

Path decomposition

"Nice" path decomposition:

$$\forall_i \quad X_i = X_{i-1} \cup \{w\} \text{ or } X_i = X_{i-1} \setminus \{w\}$$



$\boxed{\text{Note: every bag is a vertex cut}}$

Approximation algorithm - sketch

Input: graph $G=(V,E)$

Nice path decomposition

$X_1 \dots X_p$

Maintain:

- Current bag X_t

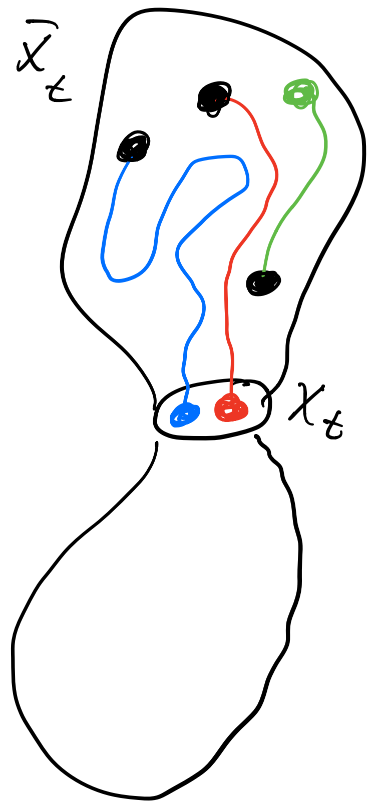
  $\overline{X}_t = \bigcup_{i \le t} X_i$

- Ppp B of $G[\overline{X}_t]$

  so that each vertex in X_t is endpoint of path

- Collection F of disjoint forts in $G[\overline{X}_t]$ so that

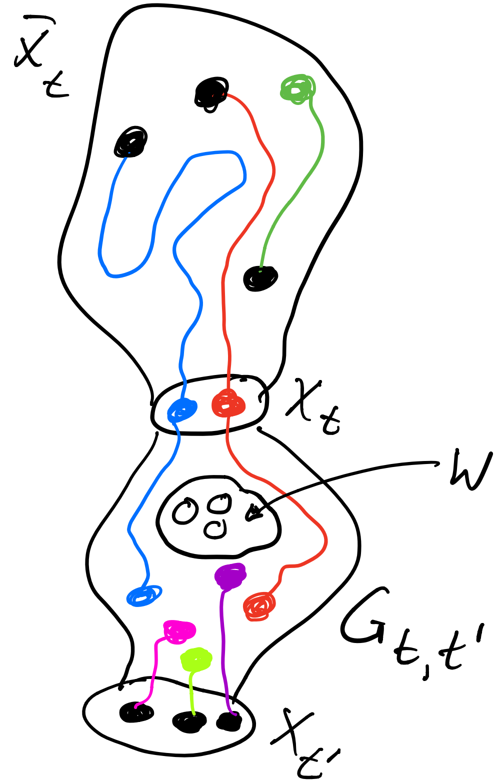
$$|B| \le (pw+1)|F|$$



Approximation algorithm - sketch

Iterative step

- $G_{t,t'} = G\left[\bigcup_{i=t}^{t'} X_i\right]$
- $t' > t$ smallest so that $X_t \cup X_{t'}$ not forcing set in $G_{t,t'}$
- W = vertices in $G_{t,t'}$ remaining white

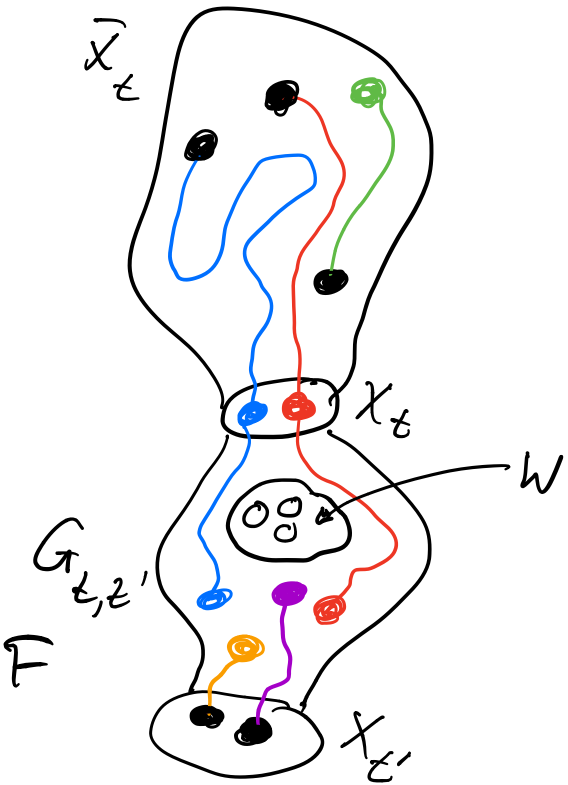


Approximation algorithm - sketch

Note:

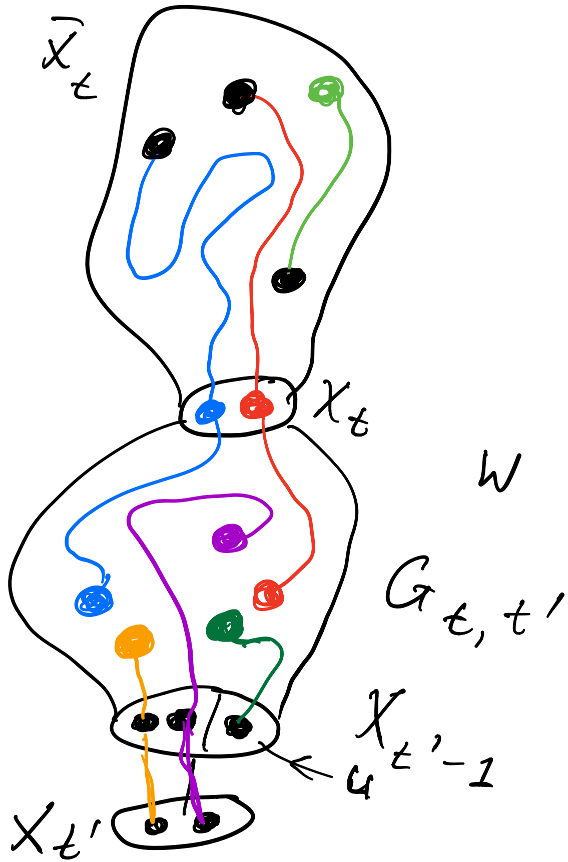
- W is a fort
- W contains no vertices from $\bar{X}_t$
- W disjoint from any fort in F

$\longrightarrow$ Add W to F



Approximation algorithm - sketch

- $X_{t'-1} \cup X_t$ is forcing set in $G_{t,t'}$
- $X_{t'-1} = X_{t'} \cup \{u\}$
- Replace B by forcing chains of this set - <u>reversed</u>
- $|B|$ increases by $|X_{t'-1}| = pw+1$



Further work

- Approximation algorithm derived from tree decomposition.
- Zero forcing number of geometric graphs?
- More applications of proper path partitions.

Thank you!