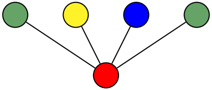
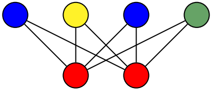


Extremal questions for vertex colorings of graphs

John Engbers

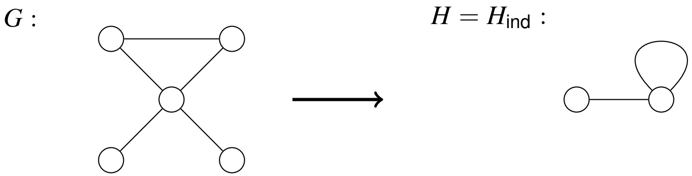
Department of Mathematical and Statistical Sciences
Marquette University

Atlantic Graph Theory Seminar, April 2022



H-colorings

Graph homomorphism (H-coloring): A map from $V(G)$ to $V(H)$ that preserves edge adjacency.



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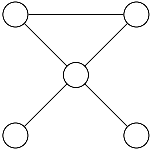

$\longrightarrow$

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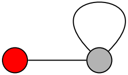

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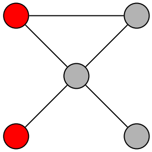

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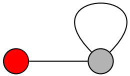

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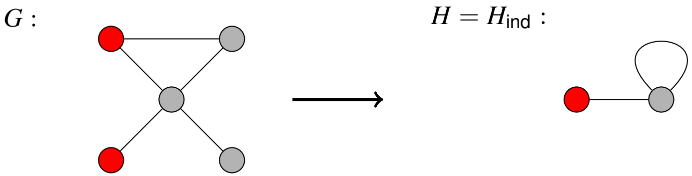

Examples: independent sets,

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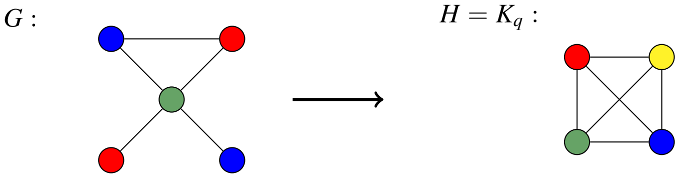

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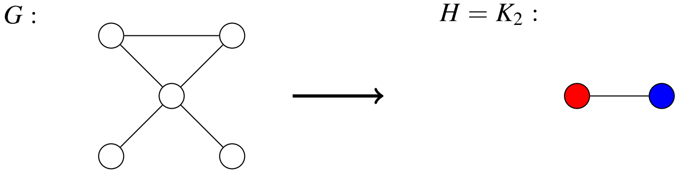

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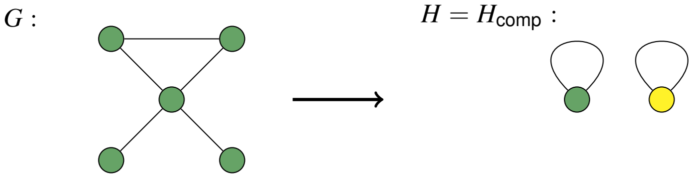

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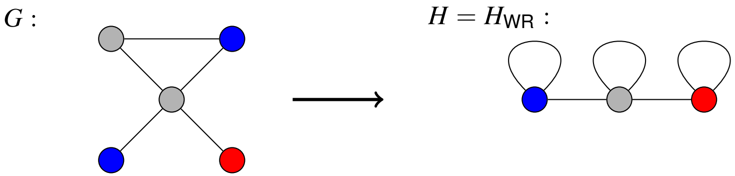

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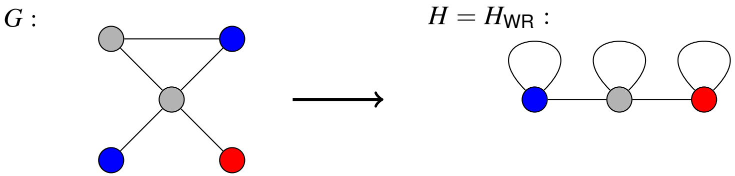

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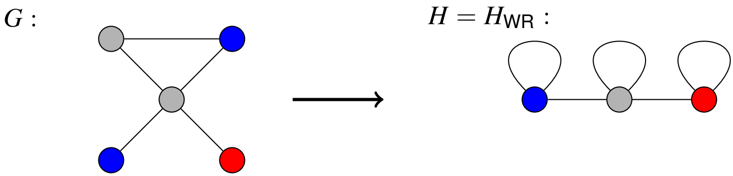

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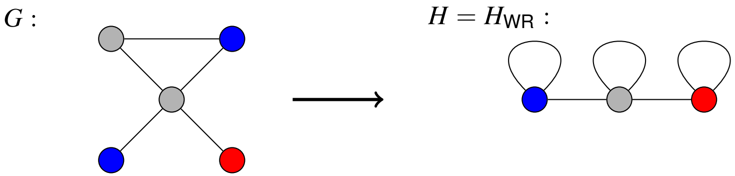

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- Terminology: map/color the vertices of G
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- Natural for H to have loops

Notation and conventions

Notations:

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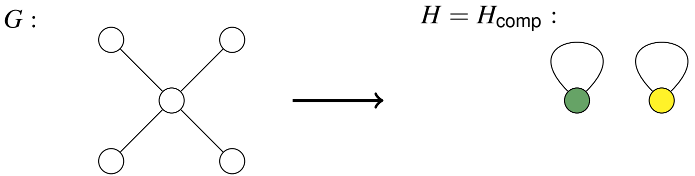

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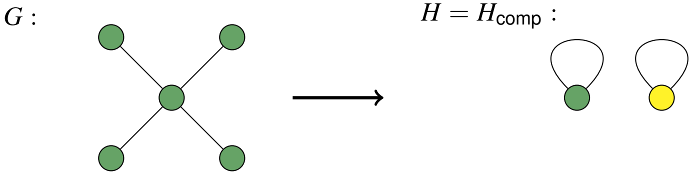

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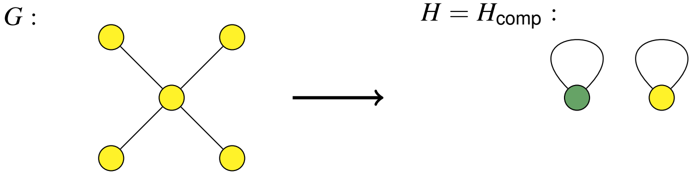

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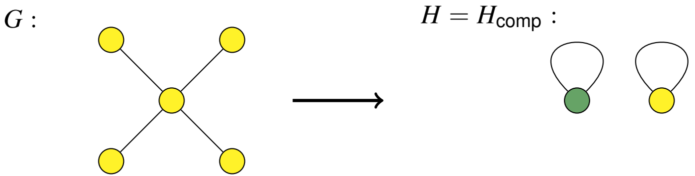

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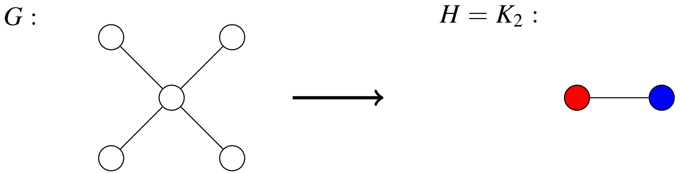

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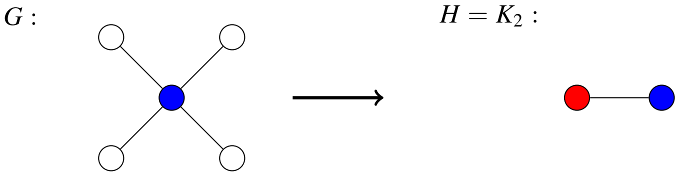

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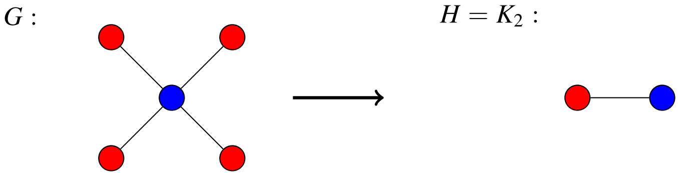

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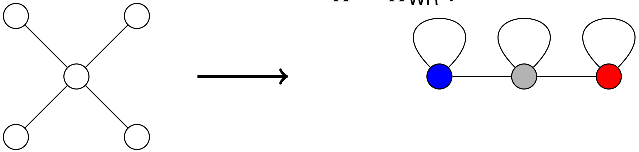
G : $\quad\quad H = H_{\text{WR}}$:



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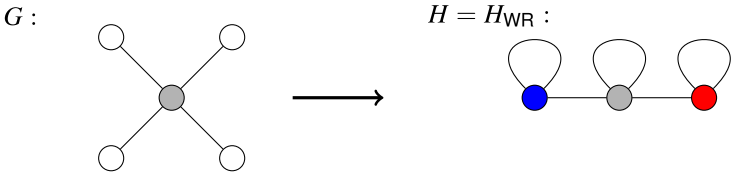

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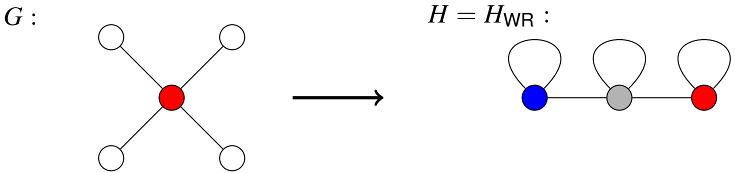

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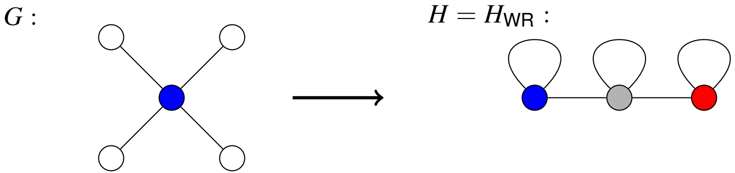

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Also: $d(v)$ is the degree of v (where loops count *once*)

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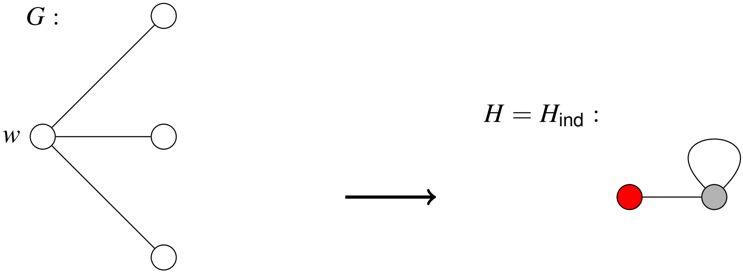

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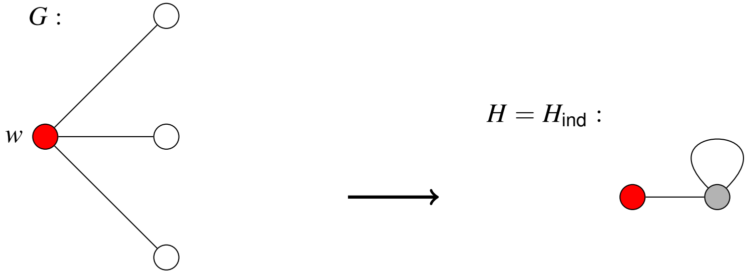

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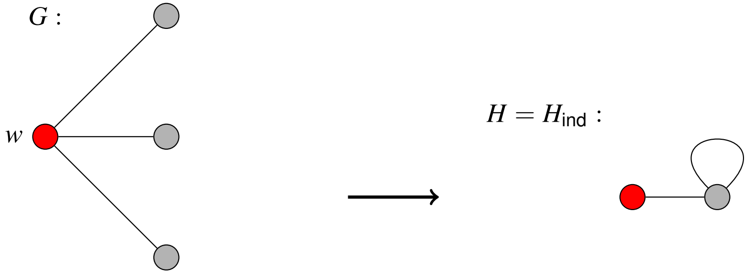

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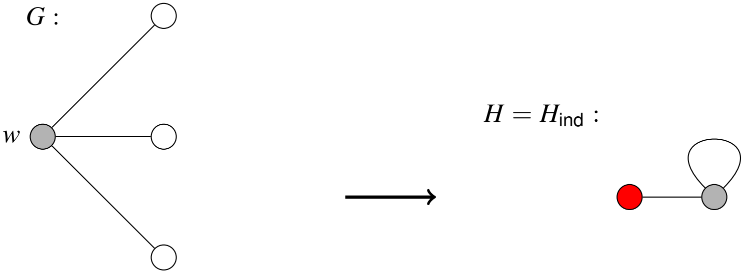

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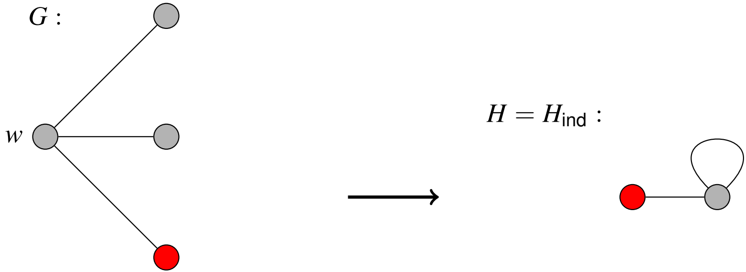

- w is red $\implies$ each neighbor of w has 1 choice ($d(\text{red}) = 1$)
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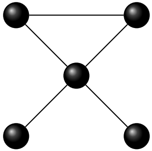
Statistical physics interpretation

Hard constraint spin systems:

Imagine $V(G)$ = particles, $E(G)$ = adjacency (e.g. spatial proximity)

G :



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Imagine $V(G)$ = particles, $E(G)$ = adjacency (e.g. spatial proximity)

Place spins on those particles so that adjacent particles receive 'compatible' spins

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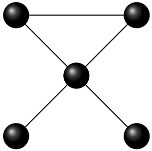

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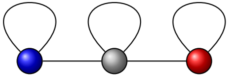

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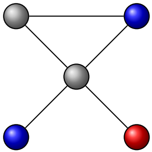

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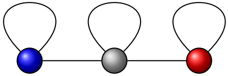

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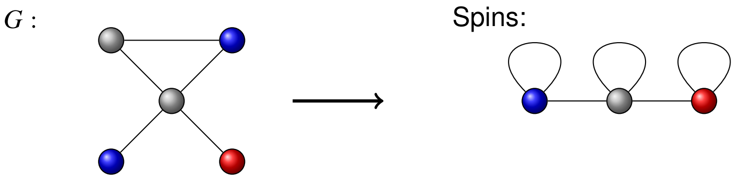

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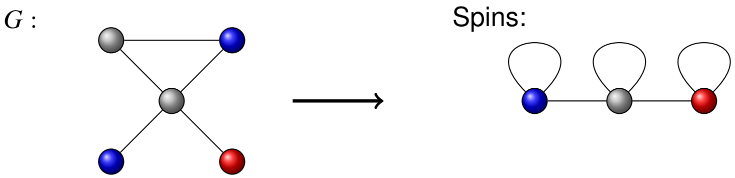

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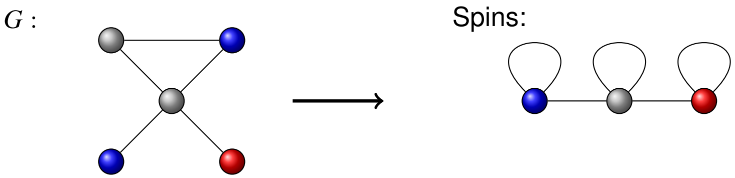

- Spins = colors; a spin configuration is an H-coloring
- Can put weights on the spins
- This idea generalizes to putting objects (with relationships) into classes with hard rules

An extremal question

Question

Fix H. Given a family $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes/minimizes $\hom(G, H)$?

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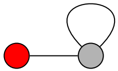

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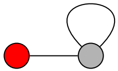

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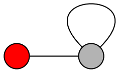

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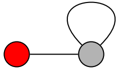

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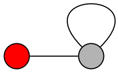

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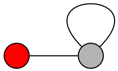

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- Note: edges in G create coloring restrictions; interesting families force each graph G to have a large number of edges.

Family: Trees

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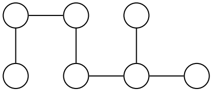
Question

$\mathcal{G} = \{$*n-vertex trees*$\}$. *Which* $T \in \mathcal{G}$ *maximizes/minimizes* $\hom(T, H)$*?*



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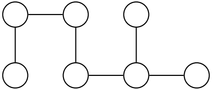

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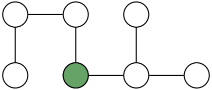

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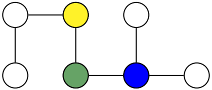

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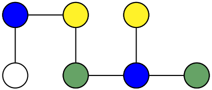

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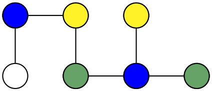

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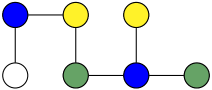

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(Can also see this inductively by considering a leaf.)

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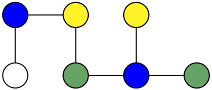

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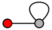
Also: This same argument shows $\hom(T, H)$ is constant on n-vertex trees T and any *regular* H.

Trees — independent sets

Next: What about independent sets $H = H_{\text{ind}} =$  ?

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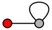
Next: What about independent sets $H = H_{\text{ind}} =$  ?

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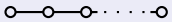
If T is an n-vertex tree, then with $P_n =$  *and $S_n =$* 

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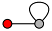
Next: What about independent sets $H = H_{\text{ind}} =$  ?

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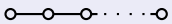
If T is an n-vertex tree, then with $P_n =$  *and $S_n =$* 

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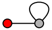
Next: What about independent sets $H = H_{\text{ind}} =$  ?

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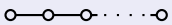
If T is an n-vertex tree, then with $P_n =$  and $S_n =$ 

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Theorem (Hoffman-London, late 1960's)

For all H and $n \ge 1$,

$$\hom(P_n, H) \le \hom(S_n, H).$$

Thought: Star and path maximize/minimize $\hom(T, H)$ for *all H*?

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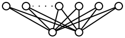
Note: Ideas from the latter proof use stability; have extended to results on n-vertex ℓ-connected | k-chromatic | min-degree ℓ[1] graphs

- Joint work with Galvin, Erey, Keough, Short, Fox, He
- Maximizer: "$K_{\ell,n-\ell}$:" 

[1] More on this in a few minutes

Trees — minimizer

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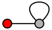
So Far: H regular (e.g. $H = K_q =$ ) and $H = H_{\text{ind}} =$ 

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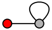
So Far: H regular (e.g. $H = K_q =$ ) and $H = H_{\text{ind}} =$ 

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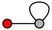
So Far: H regular (e.g. $H = K_q =$ ) and $H = H_{\text{ind}} =$ 

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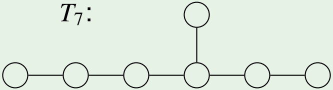

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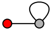
So Far: H regular (e.g. $H = K_q =$ ) and $H = H_{\text{ind}} =$ 

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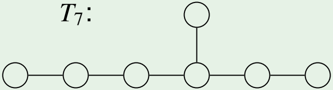

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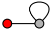
So Far: H regular (e.g. $H = K_q =$ ) and $H = H_{\text{ind}} =$ 

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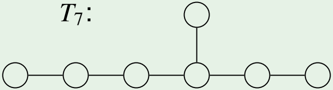

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Question: What happens with $H = H_{\text{WR}}$?

Trees — minimizer

Note: H_{ind}: isolated vertex plus 1 looped dominating vertex.

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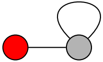

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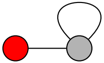

Also: H_{WR}: two looped vertices plus 1 looped dominating vertex.

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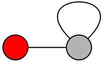

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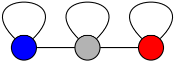

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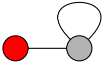

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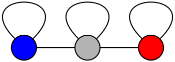


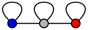
Theorem (E.-Galvin 2017)

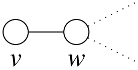
Suppose H is constructed from a regular graph H' by adding $\ell \geq 1$ looped dominating vertices. Then for any n-vertex tree T we have

$$\hom(P_n, H) \leq \hom(T, H).$$

$\ell = 1$; H'-regular: P_n minimizes *partial* H'-colorings of n-vertex trees.

Idea of Proof

Idea of Proof: Use $H_{WR} =$  ; proceed by induction



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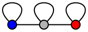
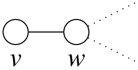
Idea of Proof: Use $H_{WR} =$  ; proceed by induction



First, for any tree T, compute (inductively) $\hom(T, H_{\text{WR}})$:

$$\hom(T, H_{\text{WR}}) = \hom(T, H_{\text{WR}}|v) + \hom(T, H_{\text{WR}}|v) + \hom(T, H_{\text{WR}}|v)$$

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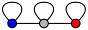
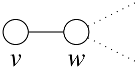
Idea of Proof: Use $H_{WR} =$  ; proceed by induction



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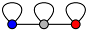
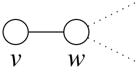
Idea of Proof: Use $H_{WR} =$  ; proceed by induction



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&= 2\hom(T - v, H_{\mathsf{WR}}) + \hom(T - v - w, H_{\mathsf{WR}})
\end{aligned}$$

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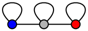
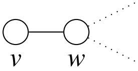
Idea of Proof: Use $H_{WR} =$  ; proceed by induction



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\end{aligned}$$

so in particular

$$\hom(P_n, H_{\mathsf{WR}}) = 2\hom(P_{n-1}, H_{\mathsf{WR}}) + \hom(P_{n-2}, H_{\mathsf{WR}})$$

Note: We minimize over n-vertex *forests* to deal with $T - v - w$

Math pause

Engbers family on/off paths; saw trees and night stars; kids colored in restaurants

Zion NP, Utah, USA, March 2022.

Other families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

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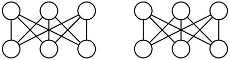
Theorem (Kahn)

$\mathcal{G}$ = n-vertex d-regular bipartite graphs

- *(Independent Sets) $\hom(G, H_{ind})$ maximized by $\frac{n}{2d}K_{d,d}$ (divisibility?)*

$\frac{n}{2d}K_{d,d}$  $\ldots$

Other families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G,H)$?

$\mathcal{G}$ = n-vertex, m-edge graphs **A:** Some results, not one maximizer G

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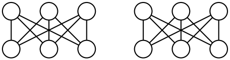
$\frac{n}{2d}K_{d,d}$  ...

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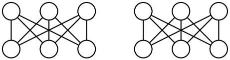
$\frac{n}{2d}K_{d,d}$  ...

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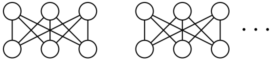
$\frac{n}{2d}K_{d,d}$ 

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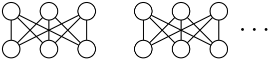
$\frac{n}{2d}K_{d,d}$ 

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A: (Sah-Sawhney-Stoner-Zhao, 2020) Holds for ~~bipartite~~ triangle-free graphs in $\mathcal{G}$ for all H. (Best possible)

Various families

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G,H)$?

Theorem (

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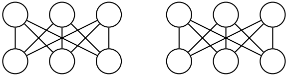
$\frac{n}{2d} K_{d,d}$  $\dots$

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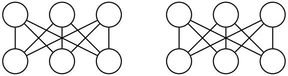
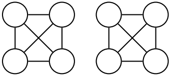
$\frac{n}{2d}K_{d,d}$  . . .

$\frac{n}{d+1}K_{d+1}$  . . .

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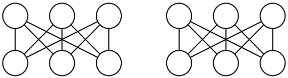
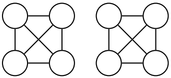
$\frac{n}{2d}K_{d,d}$  $\dots$

$\frac{n}{d+1}K_{d+1}$  $\dots$

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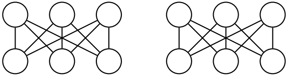
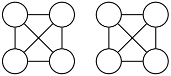
$\frac{n}{2d}K_{d,d}$  $\dots$ $\frac{n}{d+1}K_{d+1}$  $\dots$

Conjecture: Maximizer: copies of graph with $d+1 \le |V| \le c(d)$? ($c(d) = 2d$?)

(**Note:** True for $d=1$ (trivial), $d=2$ (E.))

Fixed Min Degree

Question

Fix H. Given a family of graphs $\mathcal{G}$, which $G \in \mathcal{G}$ maximizes $\hom(G, H)$?

Theorem ( )

$\mathcal{G}(n)$ = n-vertex graphs with minimum degree δ (δ fixed, small)

Fixed Min Degree

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Theorem (Cutler-Radcliffe)

$\mathcal{G}(n)$ = *n-vertex graphs with minimum degree δ (δ fixed, small)*

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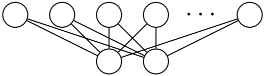
$K_{\delta,n-\delta}$:



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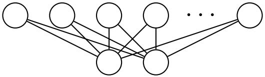

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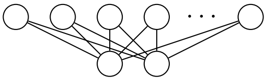

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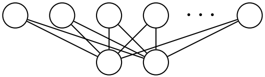

Conjecture: Max: copies of graph ($\delta + 1 \le |V| \le c(\delta)$) or $K_{\delta,n-\delta}$?

Fixed Min Degree

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Theorem (E., 2022+)

Fix δ and H.
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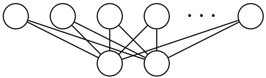
Next: Find conditions on H to make $K_{\delta,n-\delta}$ maximizer

$K_{\delta,n-\delta}$:



Ideally: Condition on H so all "small" G have $\hom(G,H)$ "small".

Partial Progress...

Fixed Min Degree

Theorem (E., 2022+)

Fix H with $\hom(K_{1,\delta}, H) < (\Delta_H)^{1+\delta}$. For n large and G n-vertex with min degree δ,

$$\hom(G, H) \leq \hom(K_{\delta,n-\delta}, H).$$

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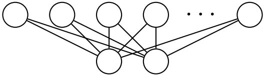

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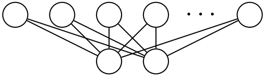
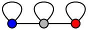
$K_{\delta,n-\delta}$: 

Ex: $H_{WR} =$  : $\hom(K_{1,\delta}, H_{WR}) = 3^{\delta} + 2^{\delta+1} < 3^{\delta+1}$ ✓

Ex: $H = P_3 =$ : $\hom(K_{1,\delta}, P_3) = 2^{\delta} + 2 < 2^{\delta+1}$ $(\delta > 1)$ ✓

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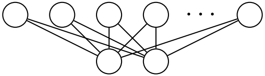
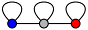


Ex: $H_{WR} =$  : $\hom(K_{1,\delta},H_{WR}) = 3^{\delta} + 2^{\delta+1} < 3^{\delta+1}$ ✓

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Idea: Use greedy coloring (on an ordering of vertices)

Stability technique:

Fixed Min Degree

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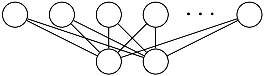
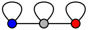
$K_{\delta,n-\delta}$: 

Ex: $H_{WR} =$  : $\hom(K_{1,\delta}, H_{WR}) = 3^\delta + 2^{\delta+1} < 3^{\delta+1}$ ✓

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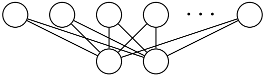
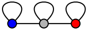
$K_{\delta,n-\delta}$: 

Ex: $H_{WR} =$  : $\hom(K_{1,\delta}, H_{WR}) = 3^\delta + 2^{\delta+1} < 3^{\delta+1}$ ✓

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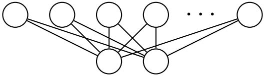
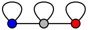
$K_{\delta,n-\delta}$: 

Ex: $H_{WR} =$  : $\hom(K_{1,\delta}, H_{WR}) = 3^\delta + 2^{\delta+1} < 3^{\delta+1}$ ✓

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- **Step 2:** Small blemishes added to $K_{\delta,n-\delta}$ can't be extremal

Idea of proof

Step 0: $\hom(K_{\delta,n-\delta},H) \geq (\Delta_H)^{n-\delta} = \underline{(1/\Delta_H)^\delta}(\Delta_H)^n$

Step 1: Extremal graph: structurally close to $K_{\delta,n-\delta}$

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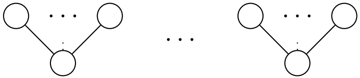
Step 1: Extremal graph: structurally close to $K_{\delta,n-\delta}$

- Extremal $\implies$ not many disjoint $K_{1,\delta}$'s

  (Each copy has $\leq \Delta_H^{\delta+1} - 1$ colorings $\implies \leq e^{\frac{-\#K_{1,\delta}}{(\Delta_H)^{\delta+1}}}(\Delta_H)^n$ col's)



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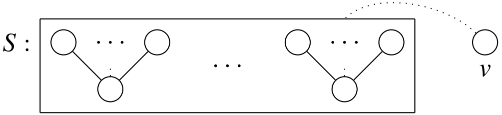

- Fix a maximum set S; Color S
- Look at $v \in V(G) \setminus S$; has neighbor in S.

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Step 1: Extremal graph: structurally close to $K_{\delta,n-\delta}$

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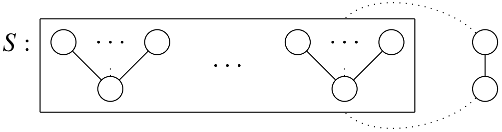
  (Each copy has $\leq \Delta_H^{\delta+1} - 1$ colorings $\implies \leq e^{\frac{-\#K_{1,\delta}}{(\Delta_H)^{\delta+1}}}(\Delta_H)^n$ col's)



- Fix a maximum set S; Color S
- Look at $v \in V(G) \setminus S$; has neighbor in S.
  - Key lemma: $< (\Delta_H)^2$ ways to color adjacent vertices in $V(G) \setminus S$

Idea of proof

Step 0: $\hom(K_{\delta,n-\delta}, H) \geq (\Delta_H)^{n-\delta} = \underline{(1/\Delta_H)^\delta}(\Delta_H)^n$

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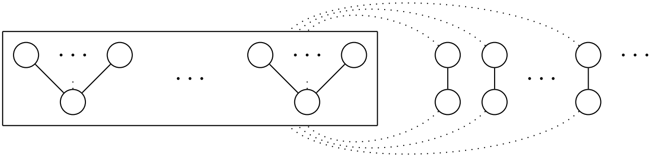
$S :$ 

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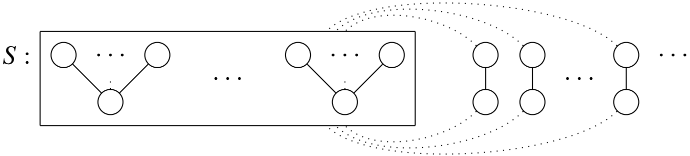

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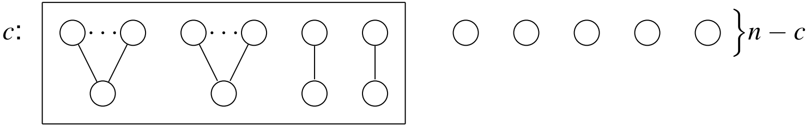

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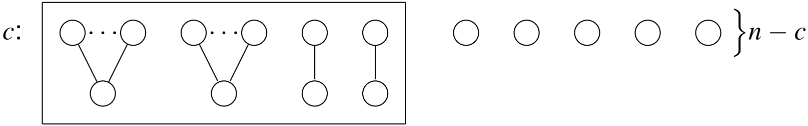

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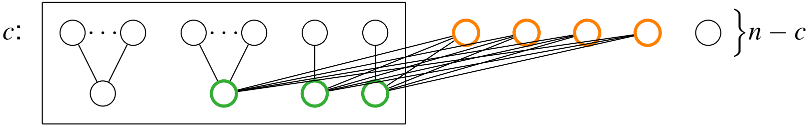

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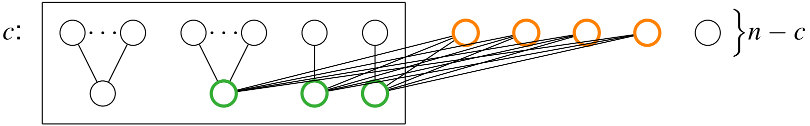

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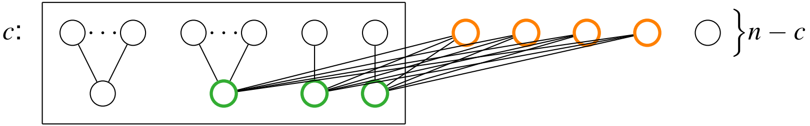

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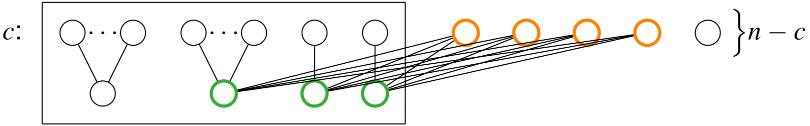

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