The basics of the deduction game

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Atlantic Graph Theory Seminar
October 13, 2021

The chaser and runner model

The chasers. . .

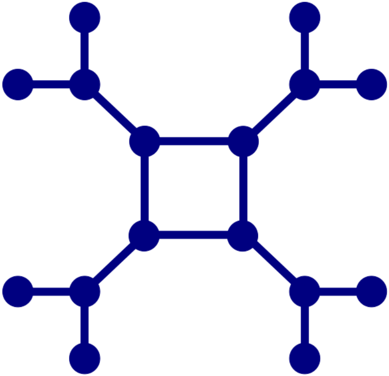
- . . . have complete knowledge of the graph.
- . . . move slowly, from vertex to vertex.
- . . . can see the runner.
- . . . can all simultaneously move.
- . . . can remain in their position.

The runner. . .

- . . . has complete knowledge of the graph.
- . . . moves slowly, from vertex to vertex.
- . . . can see the chasers.
- . . . can remain in its position.

In a graph G, the minimum number of chasers needed to guarantee capture of the runner in a finite number of turns is the **chase number** $c(G)$.

A chaser and runner example



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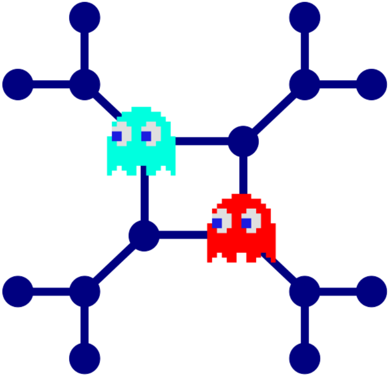

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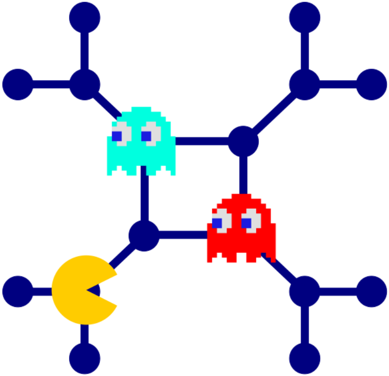

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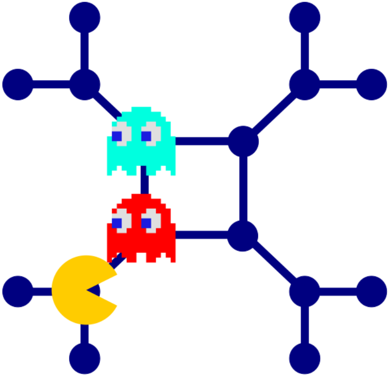

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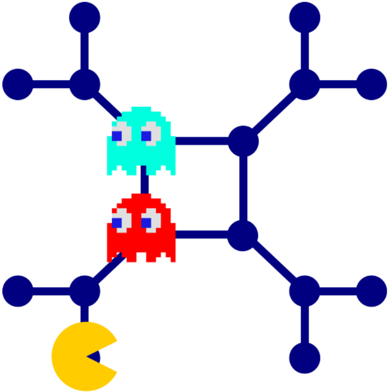

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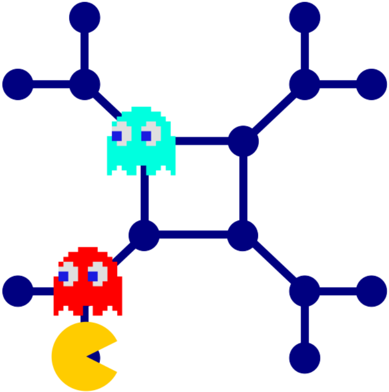

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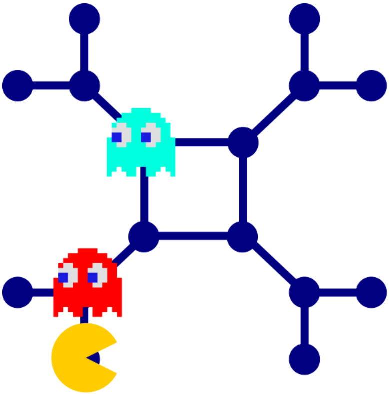
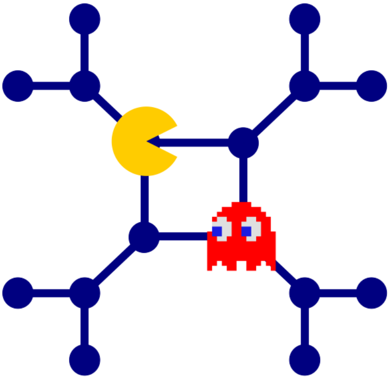


So, $c(Y) \leq 2$.

A chaser and runner example



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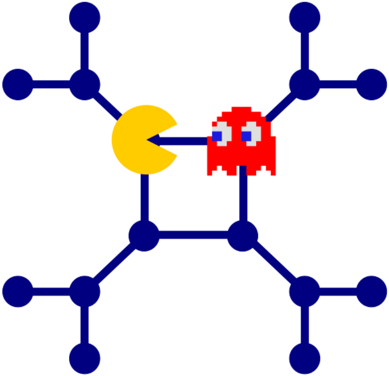

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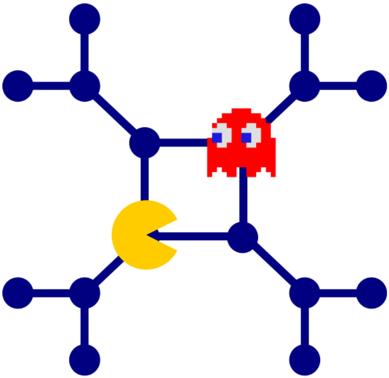

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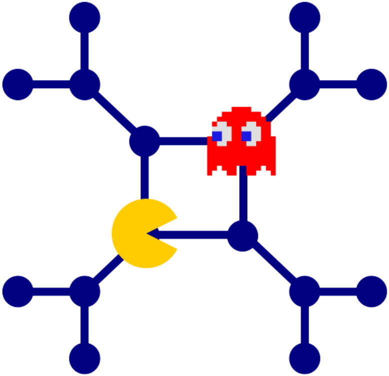


So, $c(Y) \leq 2$. In fact, $c(Y) = 2$.

The **zero-visibility** chaser and runner model

The chasers...

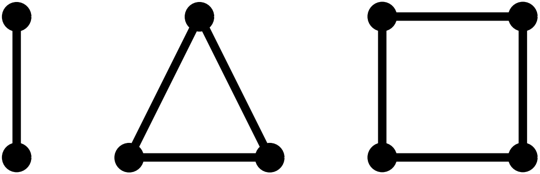
- ...have complete knowledge of the graph.
- ...move slowly, from vertex to vertex.
- ...CANNOT see the runner.
- ...can all simultaneously move.
- ...can remain in their position.

The runner...

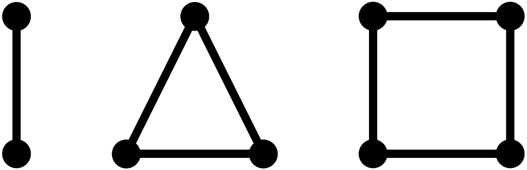
- ...has complete knowledge of the graph.
- ...moves slowly, from vertex to vertex.
- ...can see the chasers.
- ...can remain in its position.

In a graph G, the minimum number of chasers needed to guarantee capture of the runner in a finite number of turns is the zero visibility chase number $c_0(G)$.

Basic differences

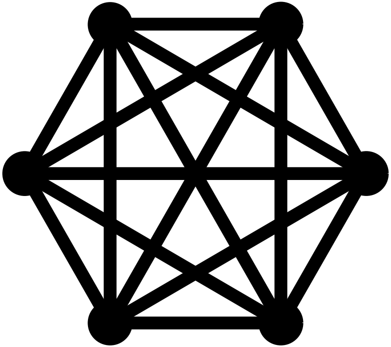


$c(K_2) = 1$ $c(K_3) = 1$ $c(C_4) = 2$



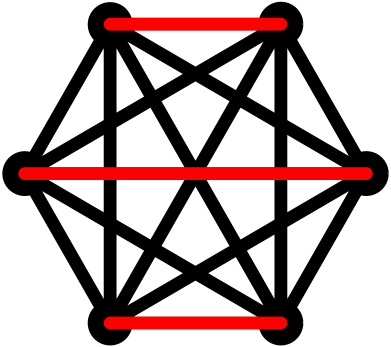
$c_0(K_2) = 1$ $c_0(K_3) = 2$ $c_0(C_4) = 2$

Basic differences



So, $c(K_n) = 1$.

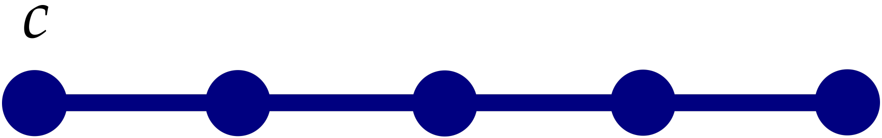
Basic differences



So, $c(K_n) = 1$. But $c_0(K_n) = \lceil \frac{n}{2} \rceil$ – that is, $\dfrac{c_0(G)}{c(G)}$ can be arbitrarily large.
(Tošić 1985, Tang 2004)

What if you wanted to capture the runner **quickly**?

In a graph G with $k \geq c(G)$ chasers, the *length* of a game is the number of moves it takes to capture the runner. The capture time is the minimum length of a game.



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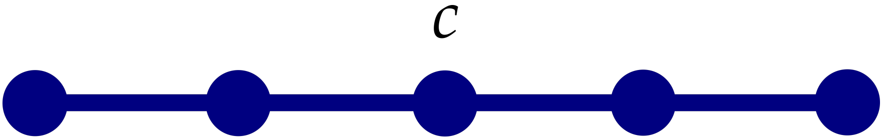

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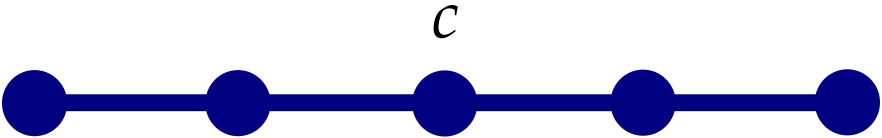


So, the capture time for P_5 is 2, with only a single chaser.

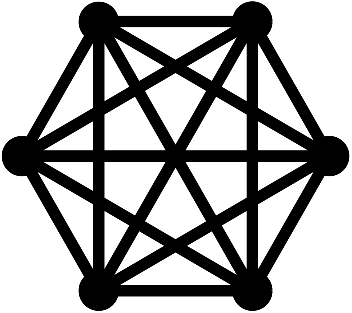
What if you wanted to capture the runner **very quickly**?

Problem swap!

In a graph G, with a fixed length t, what is the minimum number of chasers needed such that the capture time is t?

We define the 1-tick chase number of a graph G, denoted $1\text{-}c(G)$, to be the minimum number of chasers needed to capture a runner in only 1 move. Similarly, $1\text{-}c_0(G)$ is the minimum number of chasers needs to capture an invisible runner in only 1 move.

An example



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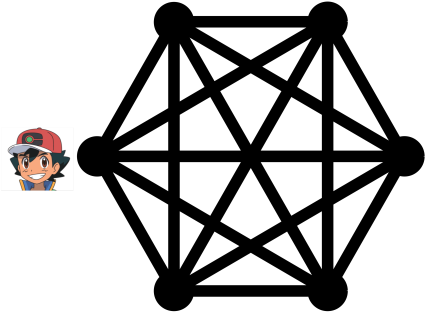
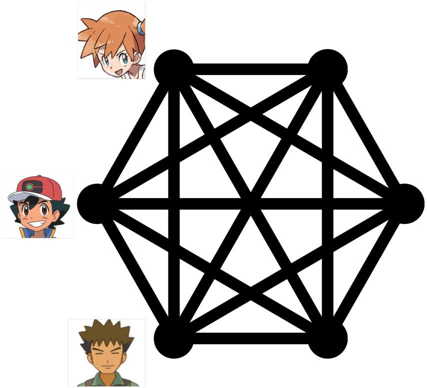


So, $1\text{-}c(K_6) = 1$.

An example



So, $1\text{-}c(K_6) = 1$. But $1\text{-}c_0(K_6) = 3$.

Time constraints

Recall that $\gamma(G)$ is the domination number of G.

Theorem (Alspach, Dyer, Hanson, Yang 2008)

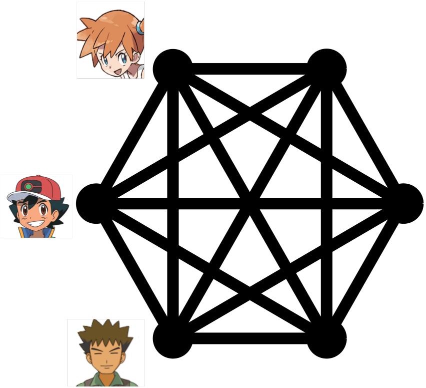
If G is a graph, then 1-$c(G) = \gamma(G)$.

Recall that a minimum edge cover of a graph G is a set $E' \subseteq E(G)$ with the fewest edges for which every vertex of G is an end of at least one edge. We denote size of such a set as $\beta'(G)$.

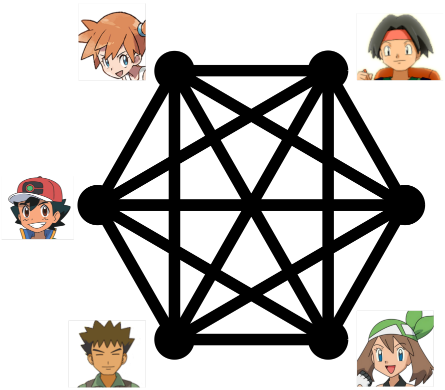
Theorem (ADHY 2008)

If G is a graph, then 1-$c_0(G) = \beta'(G)$.

No communication and an invisible runner



No communication and an invisible runner



Now we need 5 chasers!

The deduction game

The chasers. . .

- . . . have complete knowledge of the graph.
- . . . move slowly, from vertex to vertex.
- . . . CANNOT see the runner.
- . . . can all simultaneously move.
- . . . can remain in their position.
- . . . CANNOT communicate, unless they're on the same vertex.
- . . . MUST capture the runner in at most one move.

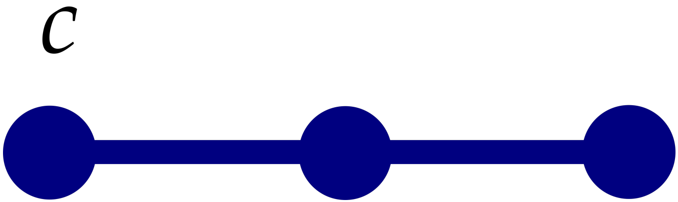
The runner. . .

- . . . has complete knowledge of the graph.
- . . . moves slowly, from vertex to vertex.
- . . . can see the chasers.
- . . . can remain in its position.

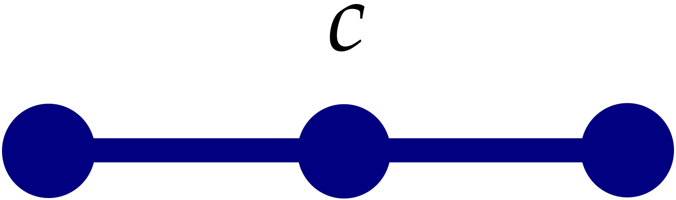
More terminology

A **layout** is a chasers' arrangement on vertices of a graph G, denoted by $L(G)$. A **successful layout** is one in which the chasers can deduce how to capture the runner, and the deduction number, $d(G)$, is the minimum number of chasers possible in a successful layout.

An unsuccessful layout



Another unsuccessful layout



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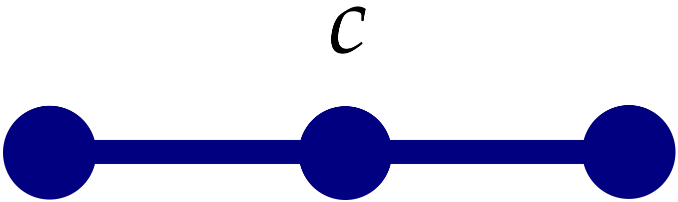
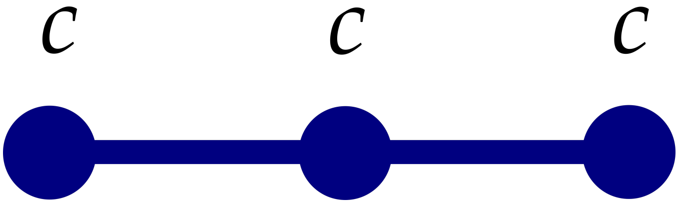
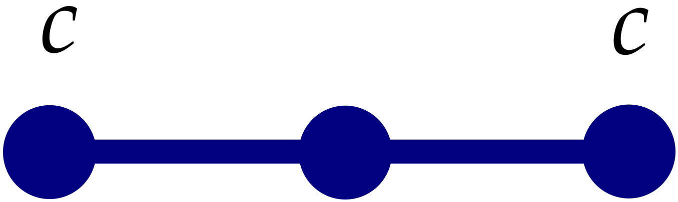


This chaser is flummoxed.

A successful, but dull, layout



An optimal successful layout



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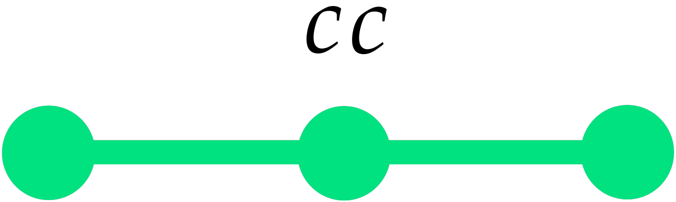
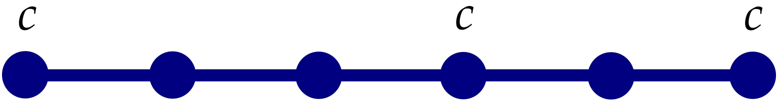


So, $d(P_3) = 2$.

But where's the deduction?



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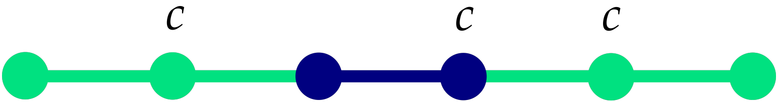


But where's the deduction?

Some bounds

Theorem (Burgess, Dyer, Farahani 2021+)

If G is a graph, then 1-$c(G) \leq$ 1-$c_0(G) \leq d(G)$.

Theorem (BDF 2021+)

If G is a graph of order $n \geq 2$, then $\lceil \frac{n}{2} \rceil \leq d(G) \leq n - 1$.

Theorems (BDF 2021+)

1. If P_n is a path of order n, then $d(P_n) = \lceil \frac{n}{2} \rceil$.
2. If C_n is a cycle of order $n \geq 3$, then $d(C_n) = \lceil \frac{n}{2} \rceil$.
3. If K_n is a complete graph of order $n \geq 2$, then $d(K_n) = n - 1$.
4. If S_n is a star of order $n \geq 2$, then $d(S_n) = n - 1$.

With a little more work...

Theorem (BDF 2021+)

If $m \geq n \geq 2$, *then* $d(K_{m,n}) = m + n - 2$.

Theorem (BDF 2021+)

If G *and* H *are graphs, then*
$d(G \square H) \leq \min\{|V(G)| \cdot d(H), |V(H)| \cdot d(G)\}$.

Subgraphs

A basic question: If H is a subgraph of G, is $d(H) \leq d(G)$?
Tricky, since local changes in a layout can have far reaching effects.

Theorem (BDF 2021+)

If K_m is a subgraph of a graph G, then $d(K_m) \leq d(G)$.

Recall that $\omega(G)$ is the clique number of G.

Corollary (BDF 2021+)

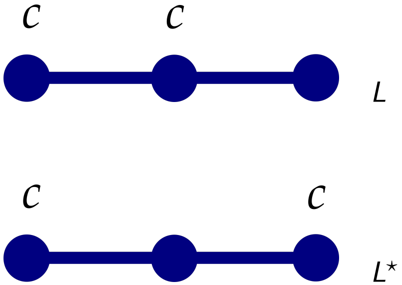
If G is a graph, then $\omega(G) - 1 \leq d(G)$.

Reversability

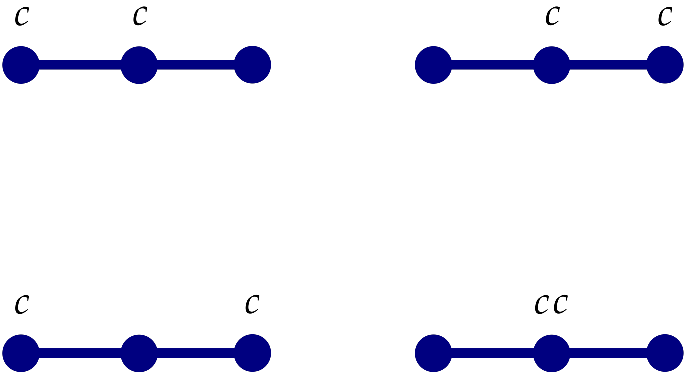
The layout obtained from the movement of chasers in a successful layout L is the dual of L, denote $L^\star$.

A basic question: If L is successful, is $L^\star$ successful?

We don't know, but it seems so.



Who cares?



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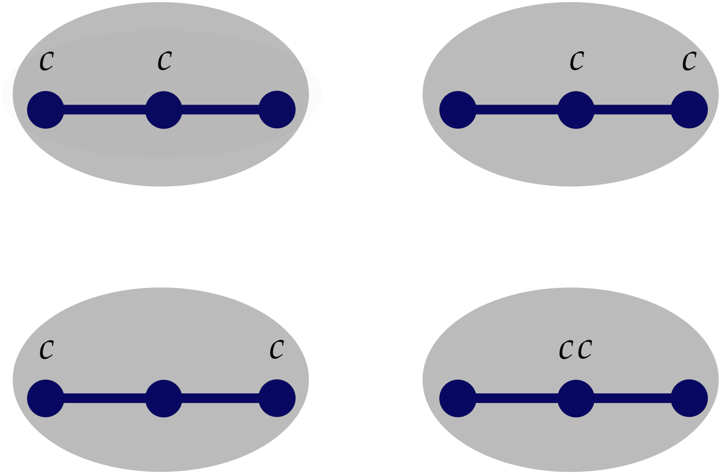

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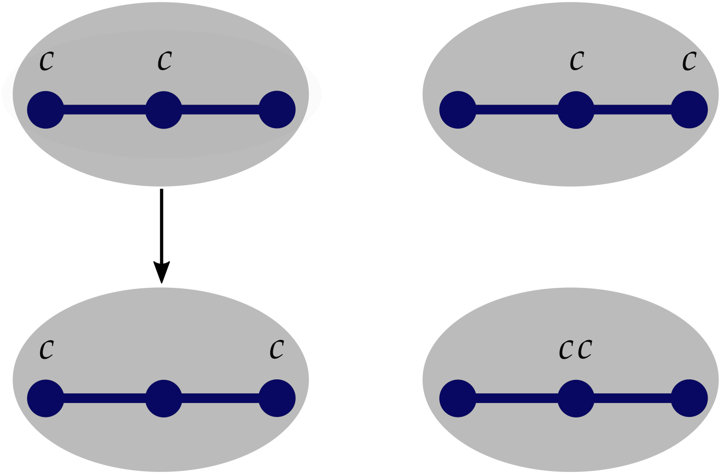

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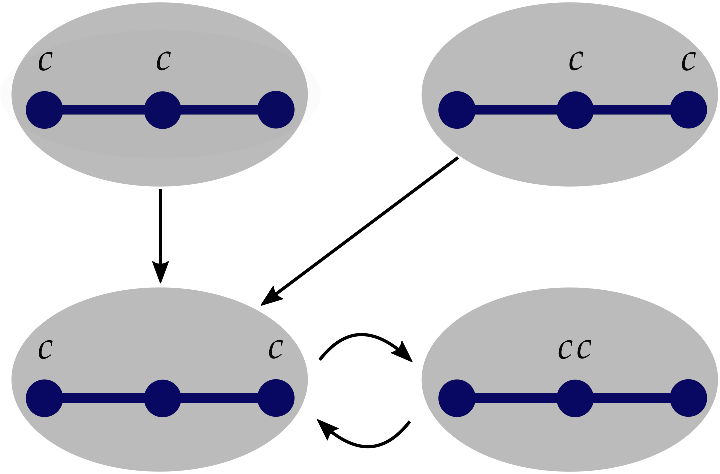
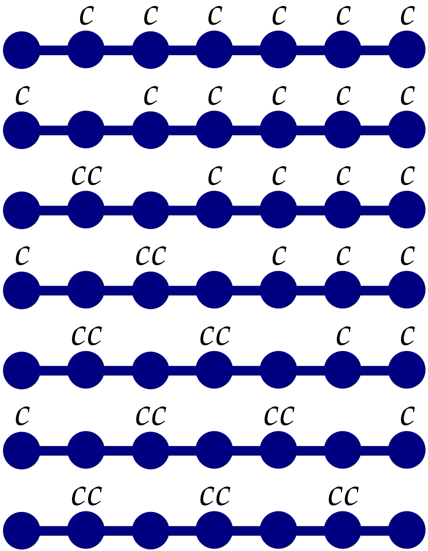


This can take a long time.



Open questions

- What conditions do I need on a subgraph H of G to guarantee that $d(H) \leq d(G)$?
- Is the dual of a successful layout successful?
  - What can we say about the metagraph of successful layouts?
- How good are our results on graph products?
  - (For hypercubes, solved.)
  - What about other kinds of products?
- This grew out of "1-tick" zero visibility. What about the 2-tick version?
  - Cops get to make a set of deductions, then a further set of deductions.
  - Does the robber move in between? Much closer to a two player game.

Graph Searching in Alantic Canada CRG

Speaker: Dr. Petr Golovach (University of Bergen)
Title: Can Romeo and Juliet Meet? Or Rendezvous Games with Adversaries on Graphs

https://sites.google.com/view/graphsearchingonline2020/home

Questions? Comments?

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My theme: https://bit.ly/2XGQeZ9

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THANKS!