Perfect 1-Factorisations

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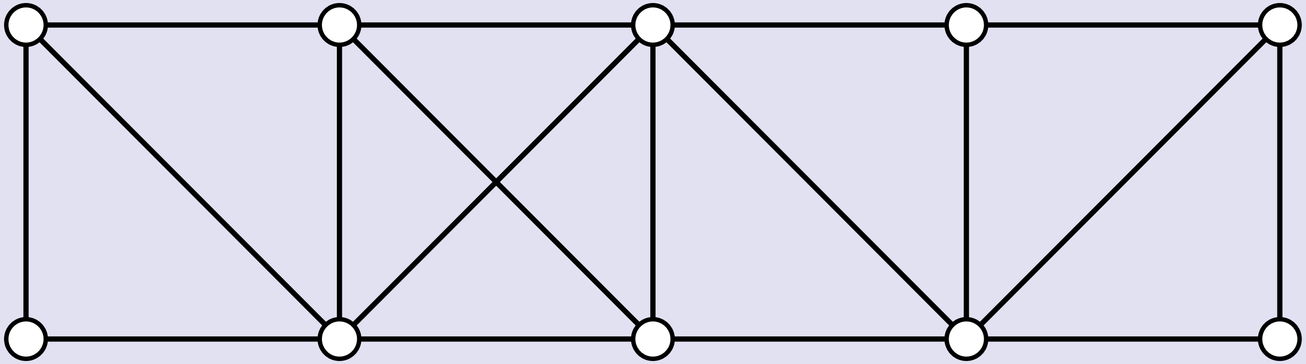
Definition:

A matching in a graph G is a subset $M \subseteq E(G)$ of the edge set of G such that no two edges of M share a vertex.

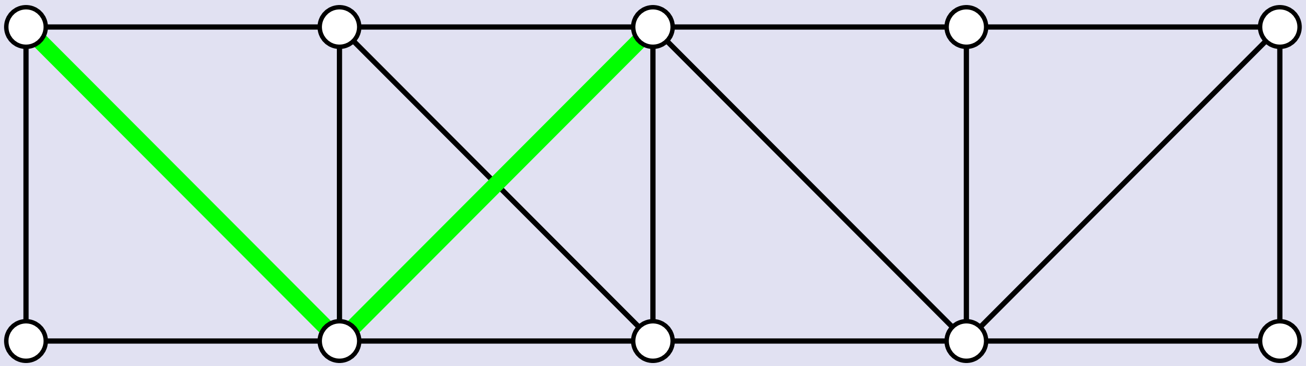
A matching M is called a perfect matching if every vertex of G is in one of the edges of M.

Examples:



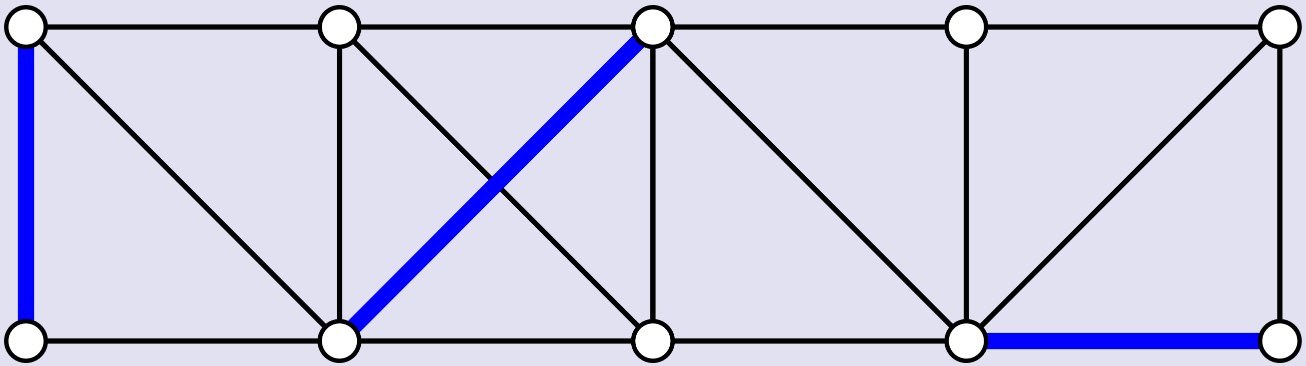
Let G be the graph shown above.

Examples:



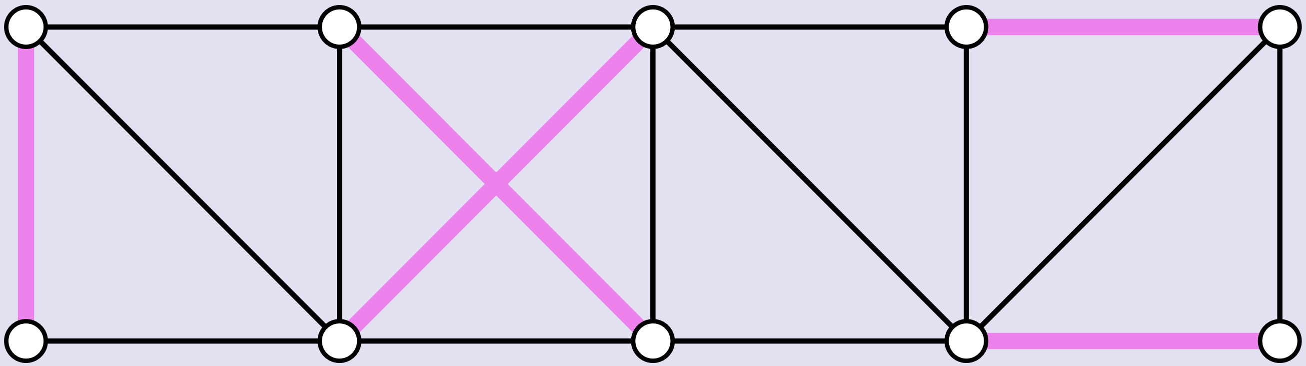
This is not a matching.

Examples:



This matching is not perfect.

Examples:



This is a perfect matching. It saturates every vertex.

Indeed, it is a 1-regular spanning subgraph of G.

Definition:

A k-factor of a graph G is a k-regular spanning subgraph of G.

So perfect matchings and 1-factors are the same as each other.

A few slides from now, the adjective “perfect” will also get used for something else.

NB: Any graph that has a 1-factor must have an even number of vertices.

Tutte's 1-Factor Theorem (Tutte, 1947)

A graph G has a 1-factor if and only if $o(G - S) \leqslant |S|$ for every $S \subseteq V(G)$, where $o(H)$ denotes the number of components of H that have odd order.

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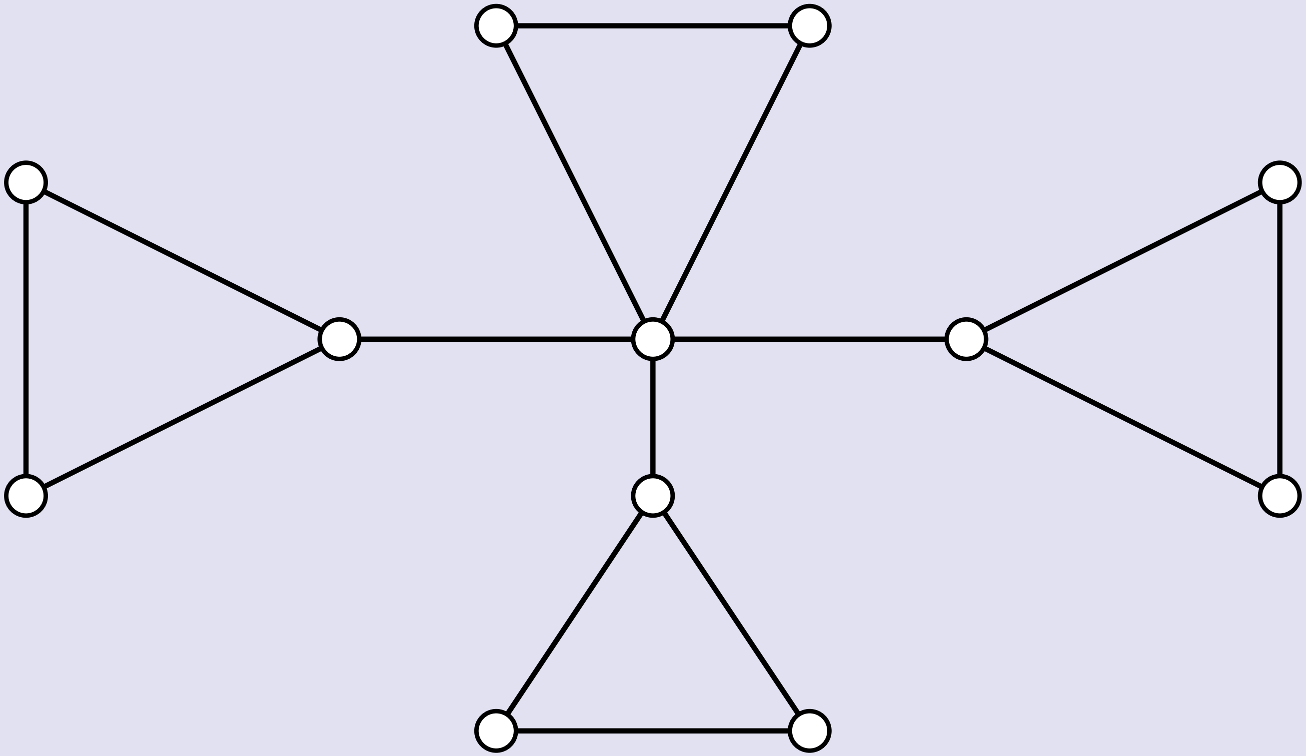

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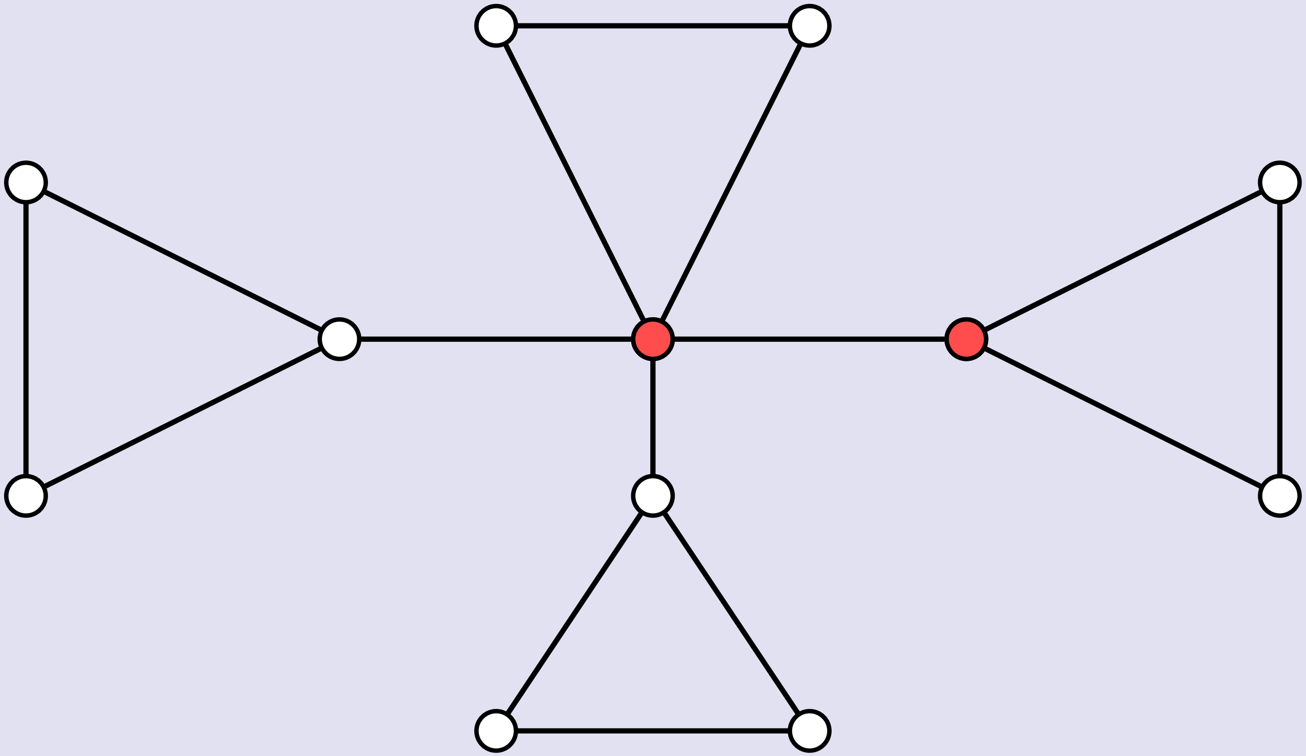

$$|S| = 2 = o(G - S)$$

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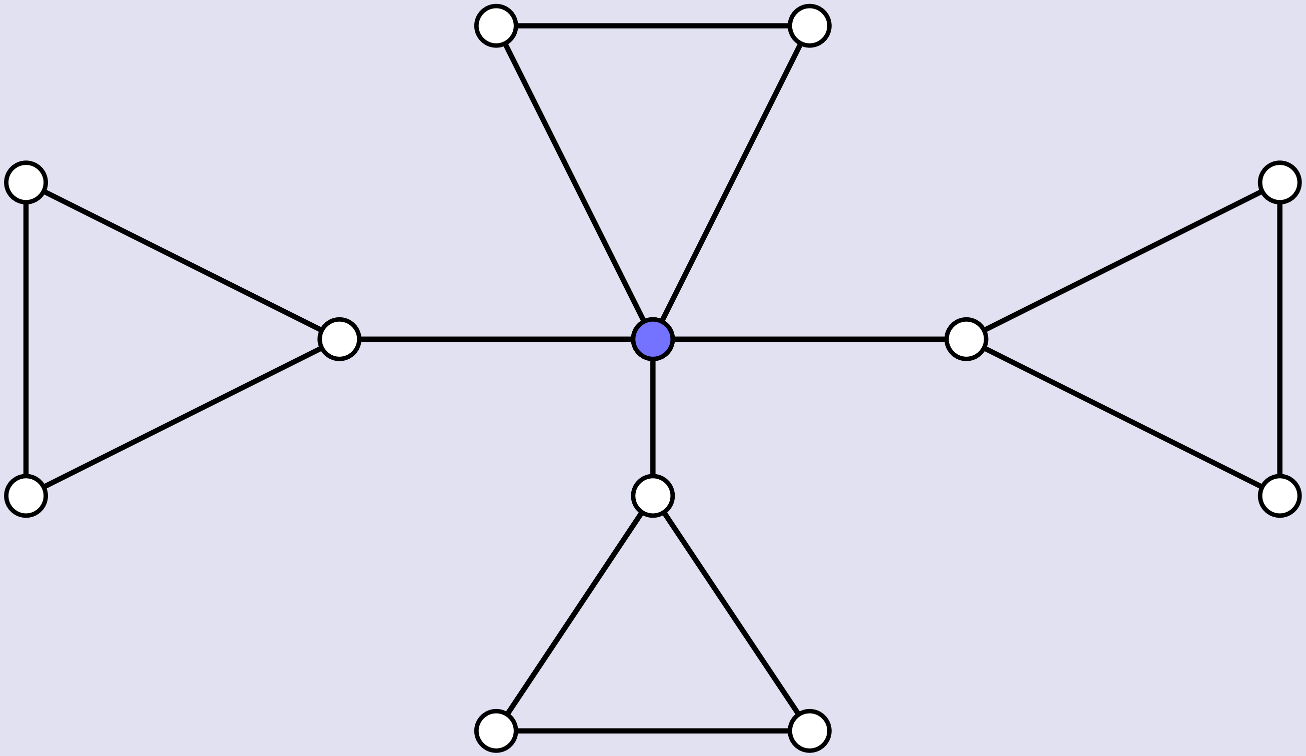


$$|S| = 1 < 3 = o(G - S)$$

Definition:

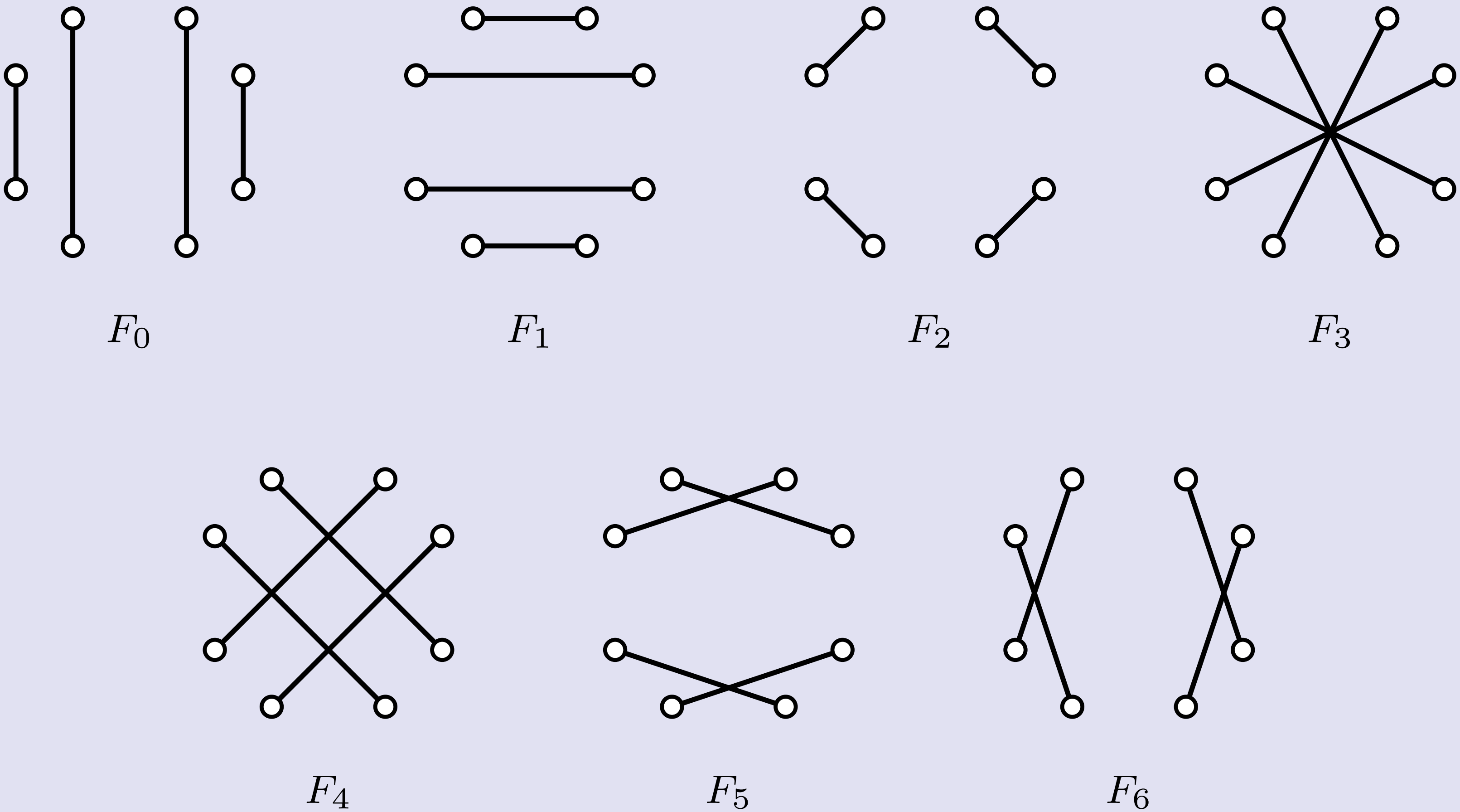
A k-factorisation of a graph G is a partition of the edge set $E(G)$ of G into k-factors.

Observe: if G admits a k-factorisation then G is regular and k must divide the degree Δ.

An application of 1-factorisations:

A 1-factorisation of the complete graph K_{2m} can be used to schedule the games of a round robin tournament with only $2m - 1$ rounds.

Example: a 1-Factorisation of K_8



1-Factorisation Conjecture (Dirac?, 1950s?)

Suppose G is a regular graph of even order.
If $\Delta \geqslant \frac{1}{2}|V(G)|$ then G has a 1-factorisation.

Theorem (Chetwynd and Hilton, 1989)

Suppose G is a regular graph of even order.
If $\Delta \geqslant \frac{1}{2}(\sqrt{7}-1)|V(G)|$ then G has a 1-factorisation.

Theorem (Walecki, 1890s)

For each $m \geqslant 2$, the complete graph K_{2m} has a 1-factorisation.

Hence a resolvable $(2m, 2, 1)$-BIBD exists for each $m \geqslant 2$.

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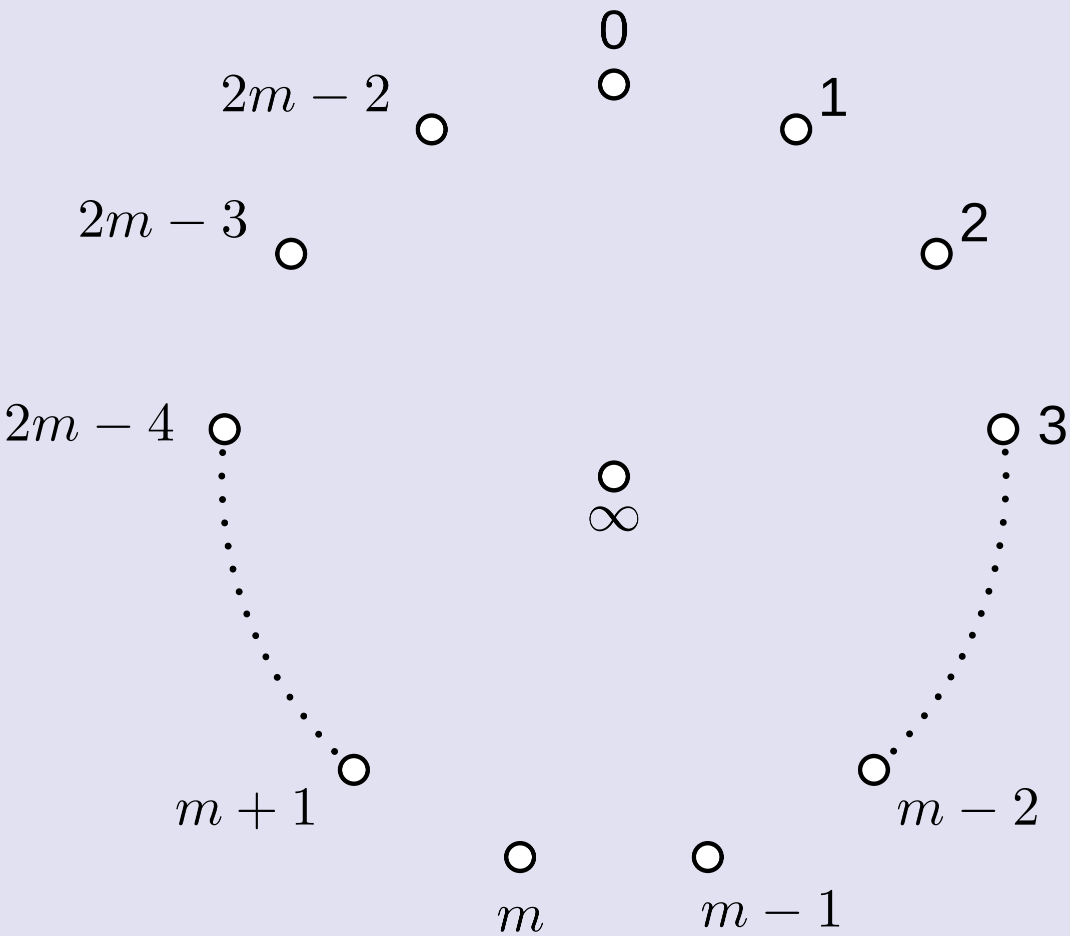

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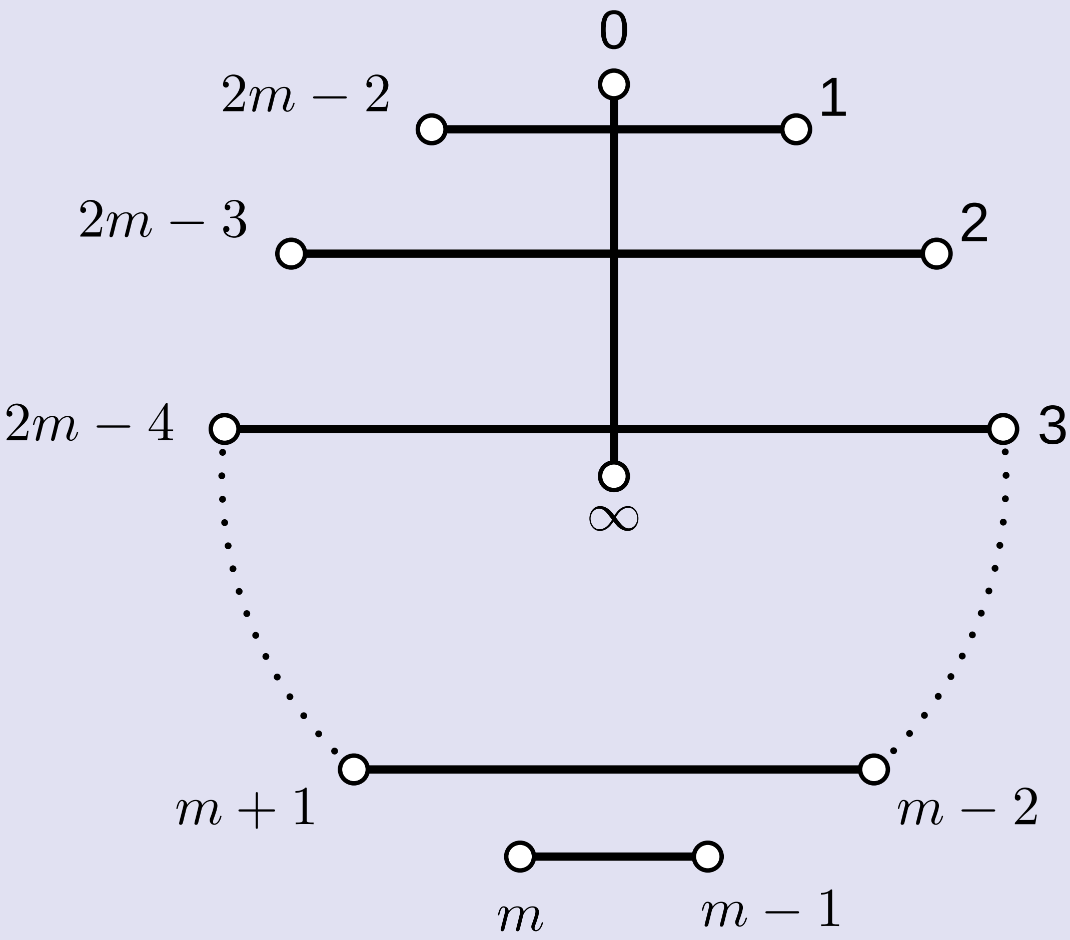

F_0

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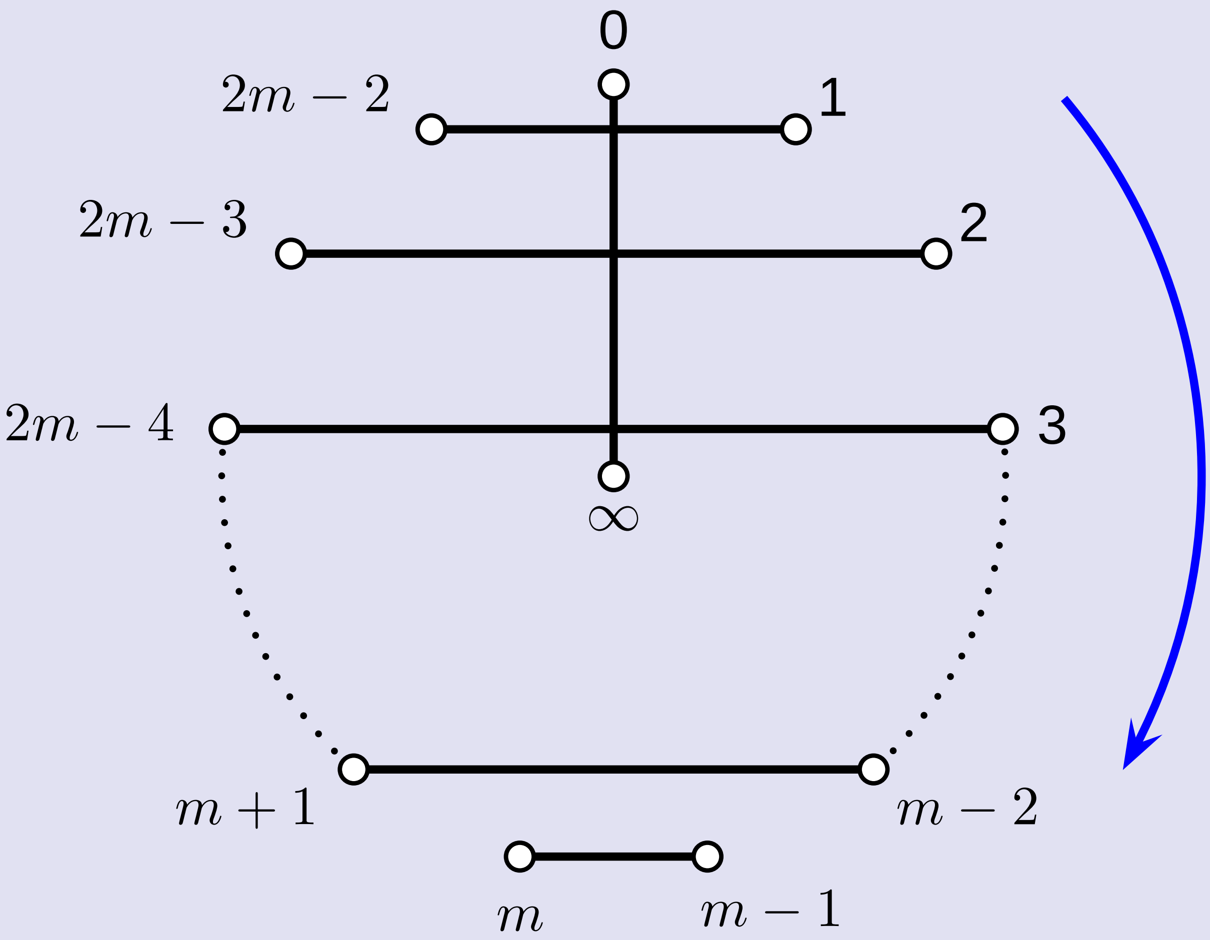

F_0

For each $i = 1, 2, \ldots, 2m-2$,
let $F_i = F_0 + i$.

Theorem (Walecki, 1890s)

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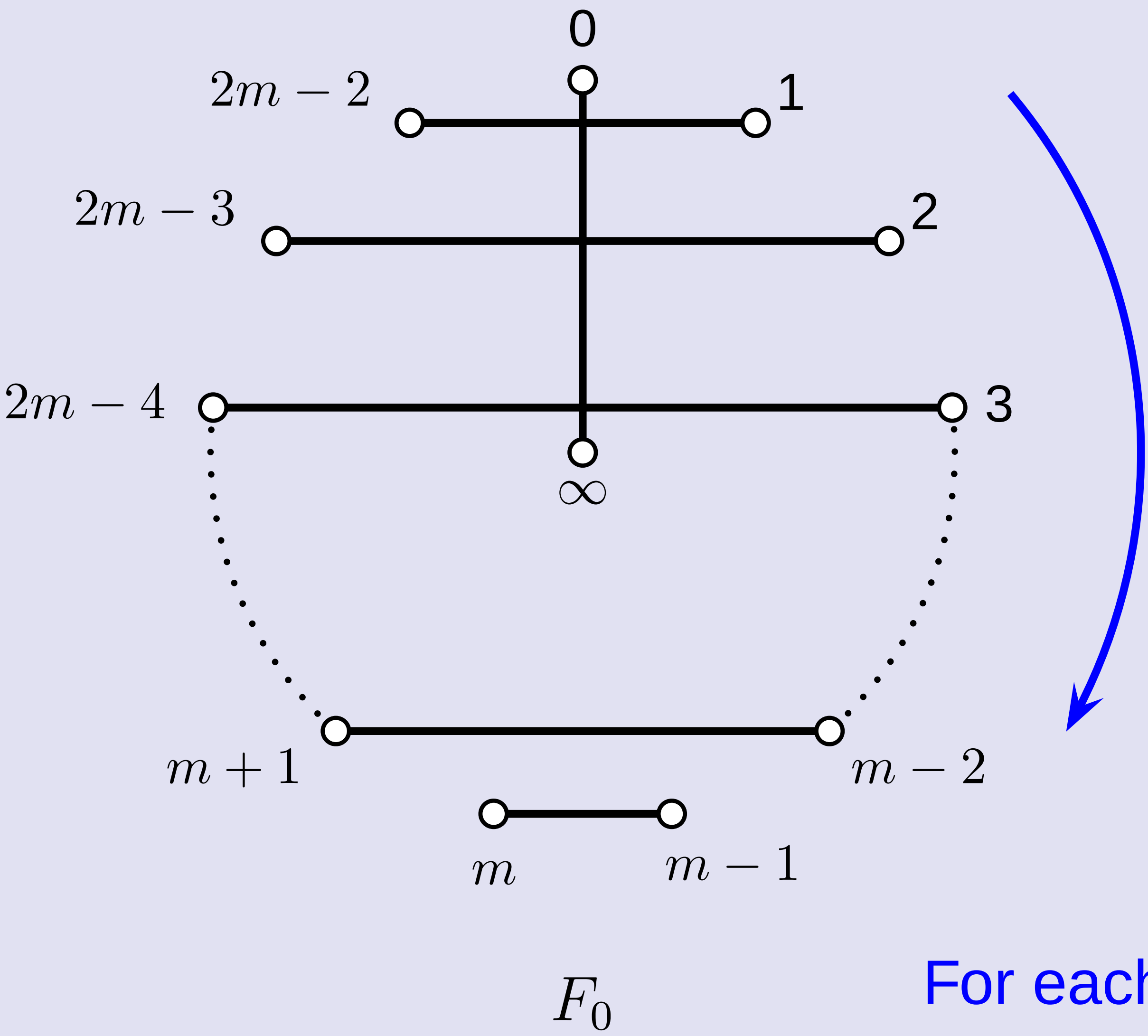
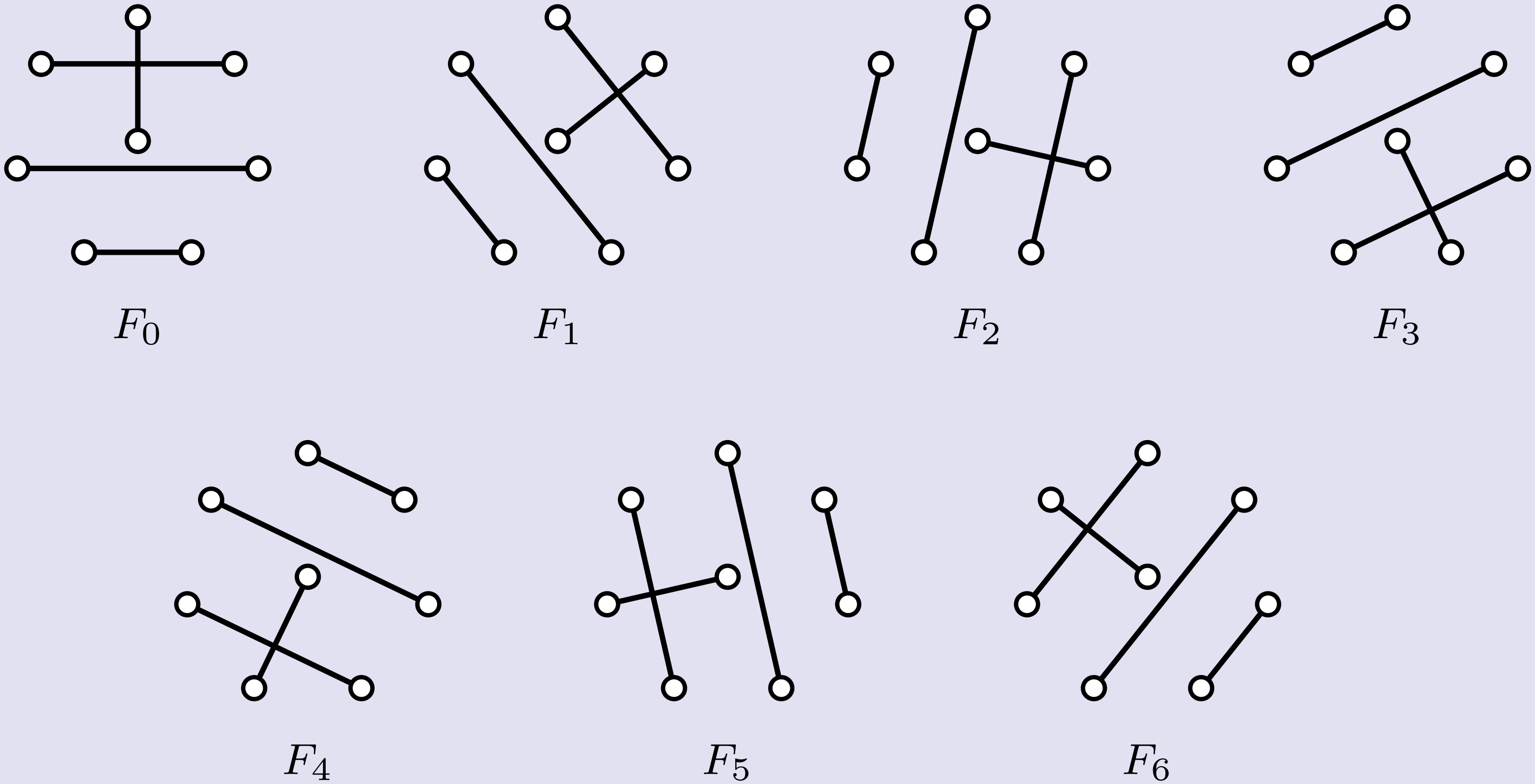


This 1-factorisation is denoted GK_{2m}

For each $i = 1, 2, \ldots, 2m-2$, let $F_i = F_0 + i$.

The 1-Factorisation of K_8 known as GK_8



Definition:

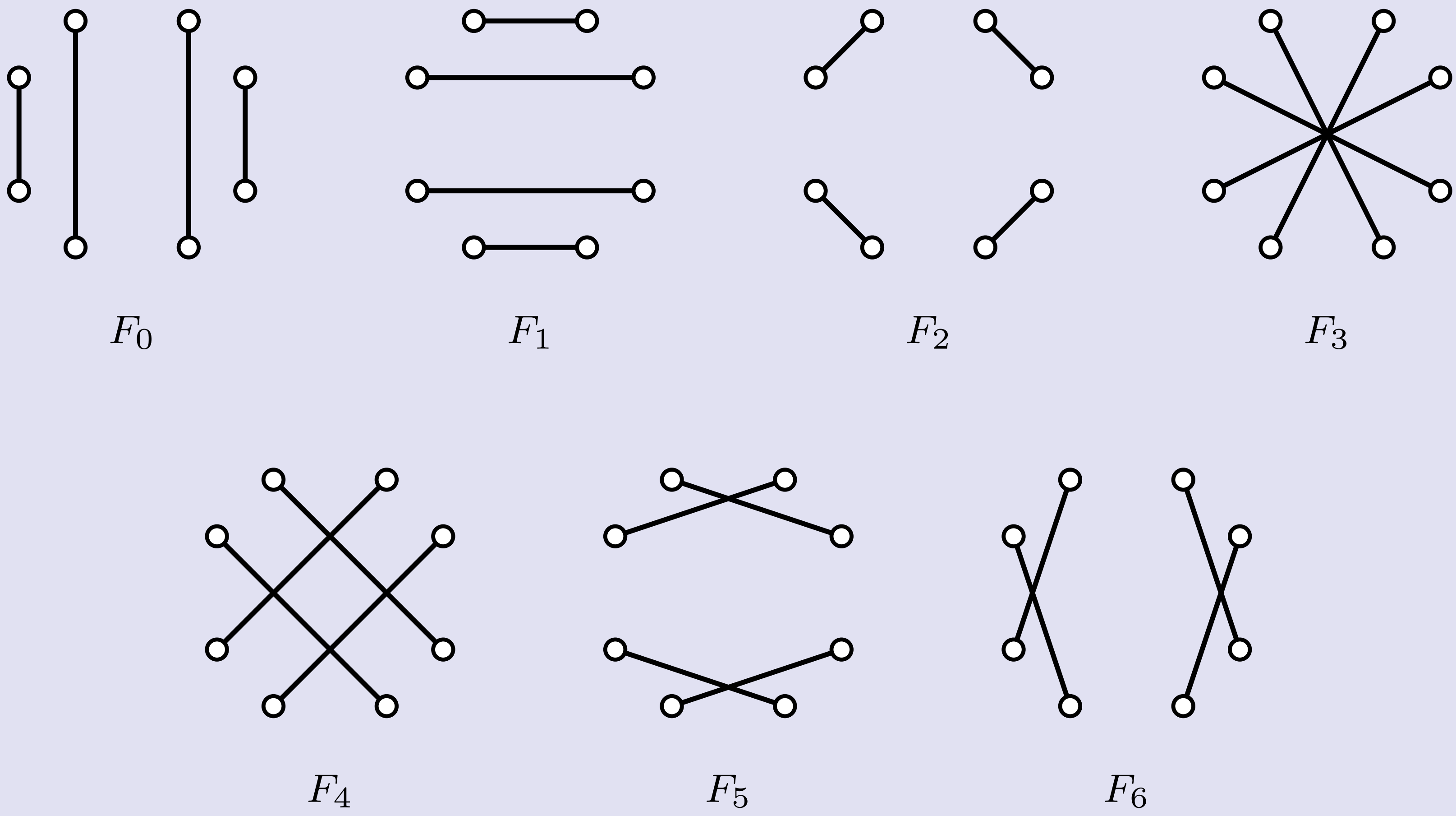
A 1-factorisation of a graph G is called perfect if the union of any two of its 1-factors yields a Hamilton cycle of G.

Every 1-factorisation of K_4 is perfect.

Every 1-factorisation of K_6 is perfect.

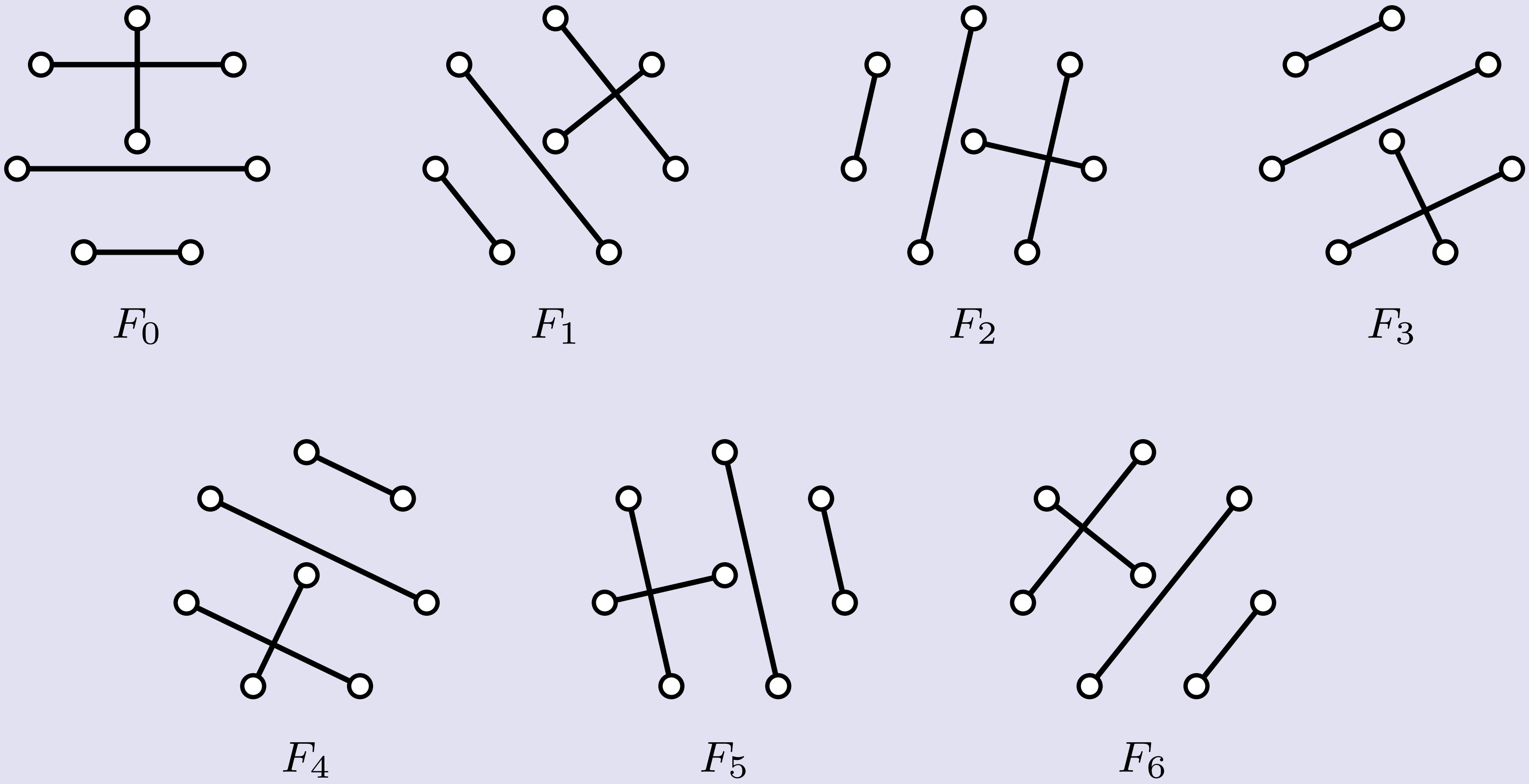
But not every 1-factorisation of K_8 is perfect.

Example: our first 1-factorisation of K_8



Observe that $F_0 \cup F_1$ yields a pair of 4-cycles
(actually, no $F_i \cup F_j$ is Hamiltonian for this 1-factorisation).
Hence this 1-factorisation is not perfect.

GK_8



Every pair of 1-factors yields an 8-cycle.

Hence this 1-factorisation is perfect.

Perfect 1-Factorisation Conjecture (Kotzig, 1963)

For each $m \geqslant 2$, K_{2m} admits a perfect 1-factorisation.

Theorem (Kotzig, 1963)

GK_{2m} is perfect if and only if $2m - 1$ is prime.

Theorem (Anderson, 1973)

If m is prime, then the 1-factorisation GA_{2m} is perfect.

Corollary

K_{2m} has a perfect 1-factorisation if $2m$ is one of 4, 6, 8, 10, 12, 14, 18, 20, 22, 24, 26, 30, 32, 34, 38, 42, 44, 46, 48, 54, 58, 60, 62, 68, 72, 74, 80, 82, 84, 86, 90, 94, 98, etc.

This leaves unsettled: 16, 28, 36, 40, 50, 52, 56, 64, 66, 70, 76, 78, 88, 92, 96, 100, etc.

Other known P1Fs of K_{2m} for small m

16: Kotzig and Anderson, 1974

There are 3155 nonisomorphic P1Fs of K_{16} (Gill and Wanless, 2020; Meszka, 2020)

20: Anderson, 1974

36: Seah and Stinson, 1988

40: Seah and Stinson, 1989

50: Ihrig, Seah and Stinson, 1987

52: Wolfe, 2009

This leaves unsettled: 56, 64, 66, 70, 76, 78, 88, 92, 96, 100, etc.

Some other known P1Fs of K_{2m}

126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860, 12168, 29792 have been known since at least 1991. For references, see the survey by Seah in the *Bulletin of the ICA*, volume 1 (1991).

Several instances where $2m = p^t + 1$ have been established. The most recent examples (by Wanless, 2005) include 530, 2810, 4490, 6890, 11450, 11882, 15626, 22202, 24390, 24650, 26570, 29930, etc. For more details see Wanless' website.

Also see the survey by Alex Rosa in *Mathematica Slovaca*, volume 69 (2019).

Still unsettled: 56, 64, 66, 70, 76, 78, 88, 92, 96, 100, etc.

Applications of P1Fs of K_{2m}

They can be used in the design of RAID (redundant array of independent disks) schemes for distributed data storage.

Theorem (Laufer, 1980)

If K_{2m} has a perfect 1-factorisation, then $K_{2m-1,2m-1}$ has a perfect 1-factorisation.

Theorem (Wanless, 1999)

If $K_{n,n}$ has a perfect 1-factorisation, then there exists a Latin square of order n with no proper subsquares.

Dudeney's Round Table Problem (Dudeney, 1917)

A group of n people want to sit around a large round table for several meals so that each person sits between each pair of other people exactly once. Can this be accomplished?

If $n = 2m$ is even, then a perfect 1-factorisation of K_n yields a solution. Given $n - 1$ 1-factors $F_0, F_1, \ldots, F_{n-2}$, each of the $\binom{n-1}{2}$ pairs $F_i \cup F_j$ provides a seating arrangement.

Theorem (Kobayashi, Kiyasu and Nakamura, 1993)

For even n, Dudeney's problem has a solution.

For odd n, Dudeney's problem remains open.

Starters:

A starter in $\mathbb{Z}_{2t-1}$ consists of a set S of $t-1$ disjoint unordered pairs $\{x_i, y_i\} \subset \{0, 1, \ldots, 2t-2\}$ such that for each $d \in \{1, 2, \ldots, 2t-2\}$, one of the $t-1$ pairs $\{x_i, y_i\}$ satisfies either $x_i - y_i \equiv d \pmod{2t-1}$ or $y_i - x_i \equiv d \pmod{2t-1}$.

Example:

For $t = 7$ consider the set S with these pairs:

$\{0, 1\}$ produces d values of 1 and 12 (mod 13)

$\{4, 6\}$ produces d values of 2 and 11 (mod 13)

$\{9, 12\}$ produces d values of 3 and 10 (mod 13)

$\{7, 11\}$ produces d values of 4 and 9 (mod 13)

$\{5, 10\}$ produces d values of 5 and 8 (mod 13)

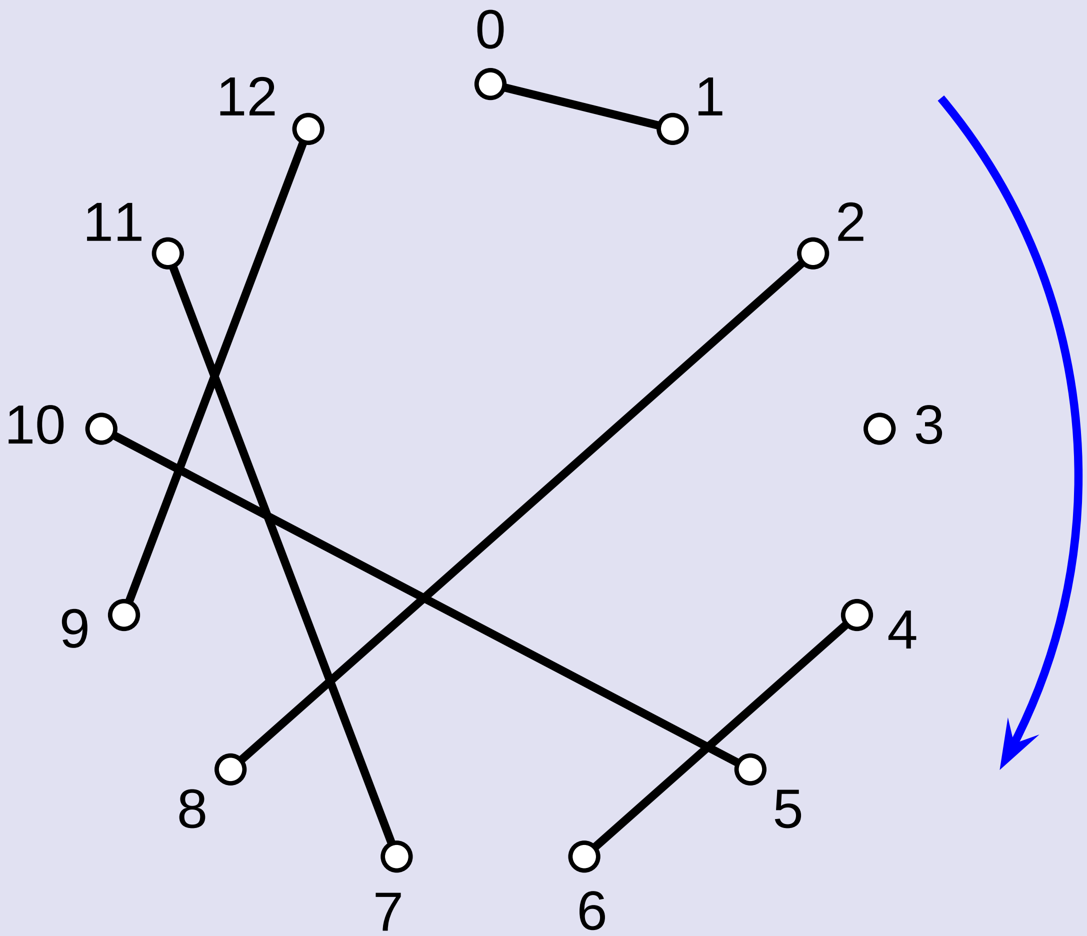
$\{2, 8\}$ produces d values of 6 and 7 (mod 13)

Starters:

Observe that a starter in $\mathbb{Z}_{2t-1}$ yields a near 1-factorisation of K_{2t-1}.

Example:

$t = 7$

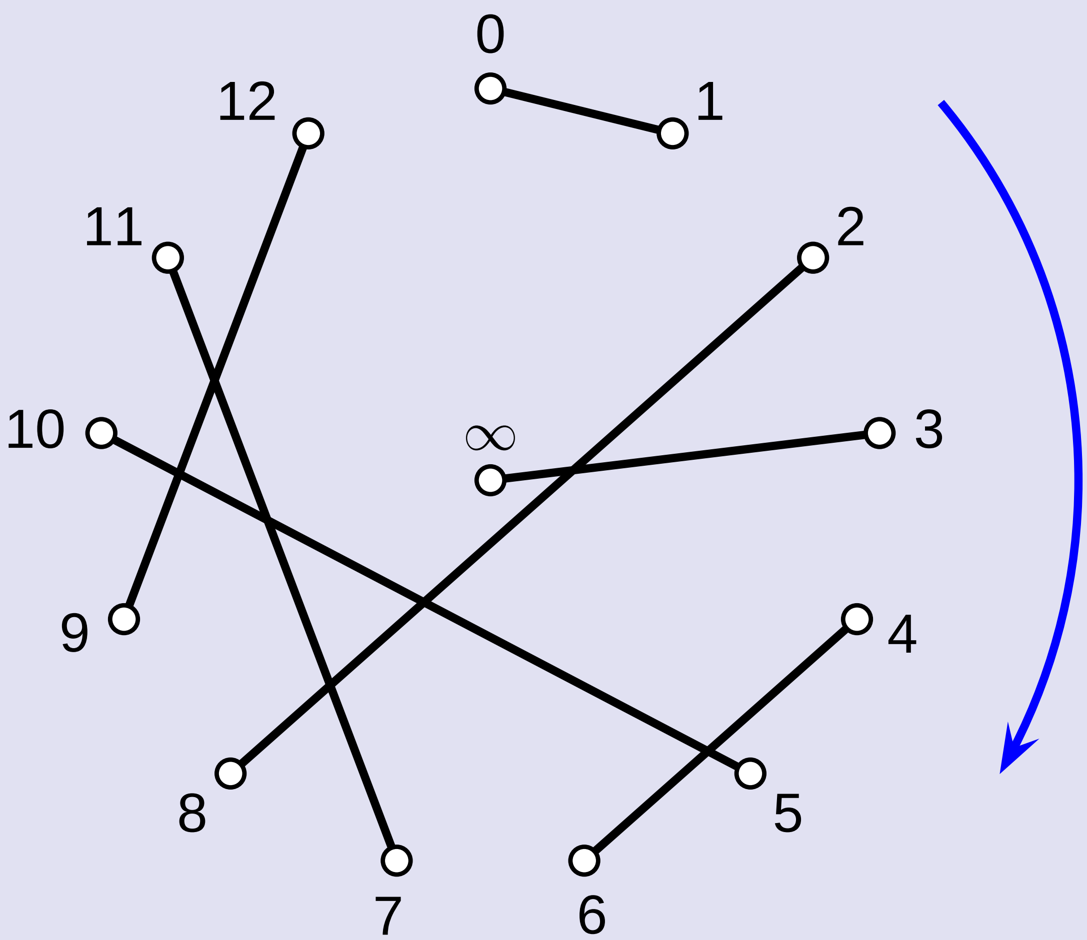


Starters:

Observe that a starter in $\mathbb{Z}_{2t-1}$ yields a near 1-factorisation of K_{2t-1} and also a 1-factorisation of K_{2t}.

Example:

$t = 7$



Even Starters:

An even starter in $\mathbb{Z}_{2t-2}$ consists of a set E of $t-2$ disjoint unordered pairs $\{x_i, y_i\} \subset \{0, 1, \ldots, 2t-3\}$ such that for each $d \in \{1, 2, \ldots, 2t-3\} \setminus \{t-1\}$, one of the $t-2$ pairs $\{x_i, y_i\}$ satisfies either $x_i - y_i \equiv d \pmod{2t-2}$ or $y_i - x_i \equiv d \pmod{2t-2}$.

Example:

For $t = 8$ consider the set E with these pairs:

$\{10, 11\}$ produces d values of 1 and 13 (mod 14)

$\{6, 8\}$ produces d values of 2 and 12 (mod 14)

$\{2, 5\}$ produces d values of 3 and 11 (mod 14)

$\{3, 7\}$ produces d values of 4 and 10 (mod 14)

$\{4, 13\}$ produces d values of 5 and 9 (mod 14)

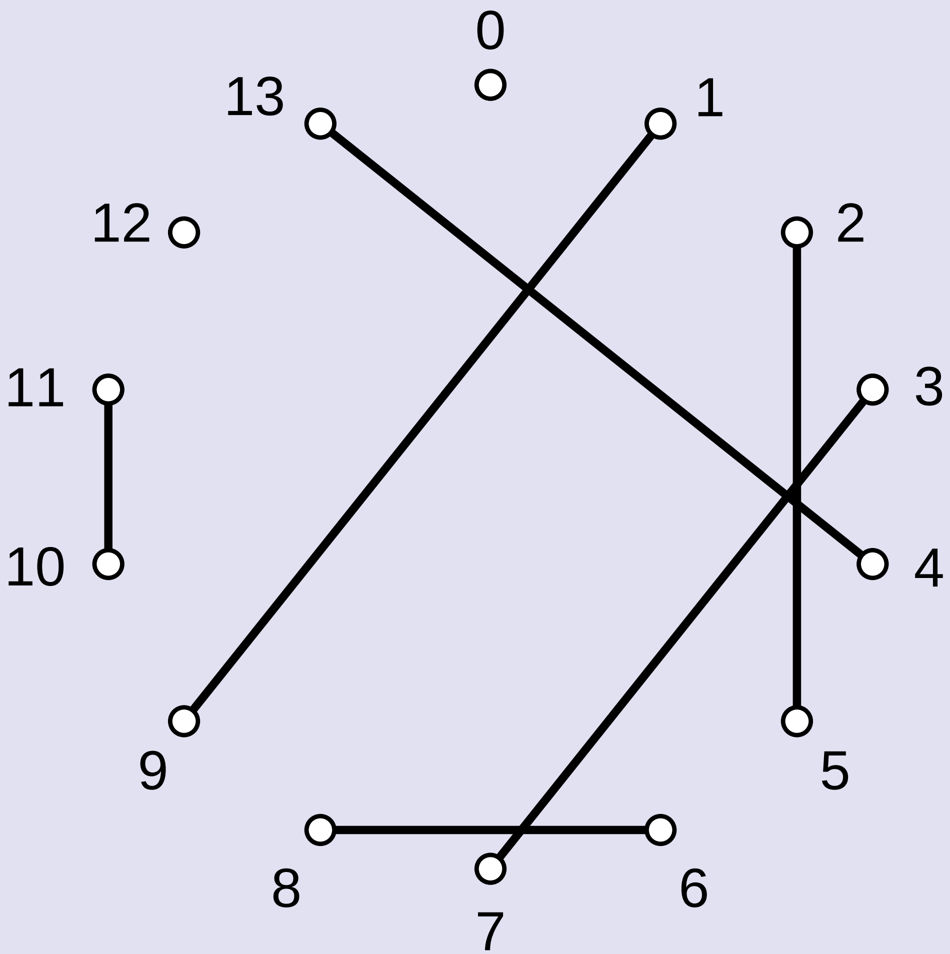
$\{1, 9\}$ produces d values of 6 and 8 (mod 14)

Even Starters:

Observe that an even starter in $\mathbb{Z}_{2t-2}$ yields a 1-factorisation of K_{2t}

Example:

$t = 8$



Even Starters:

Observe that an even starter in $\mathbb{Z}_{2t-2}$ yields a 1-factorisation of K_{2t}

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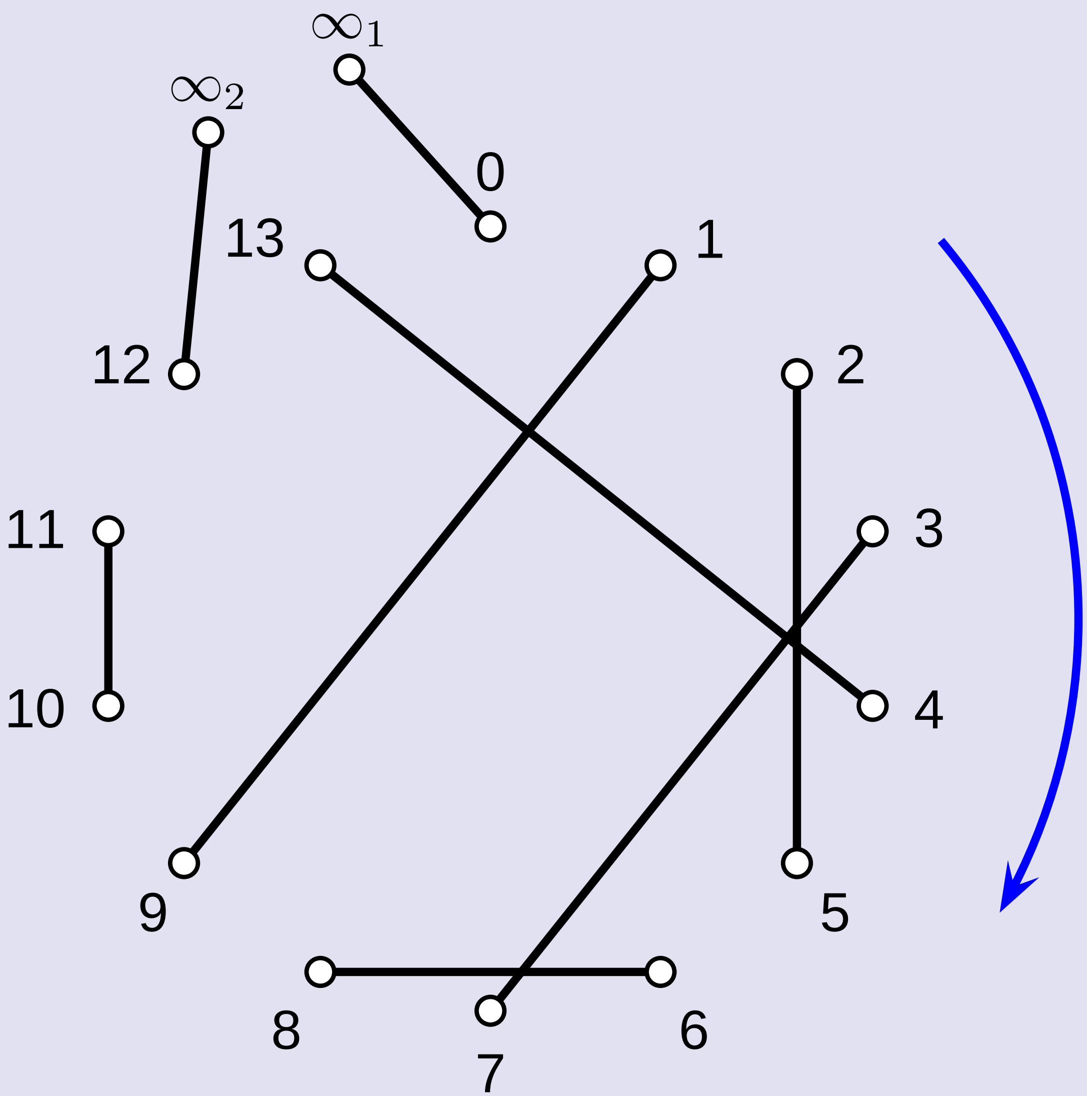
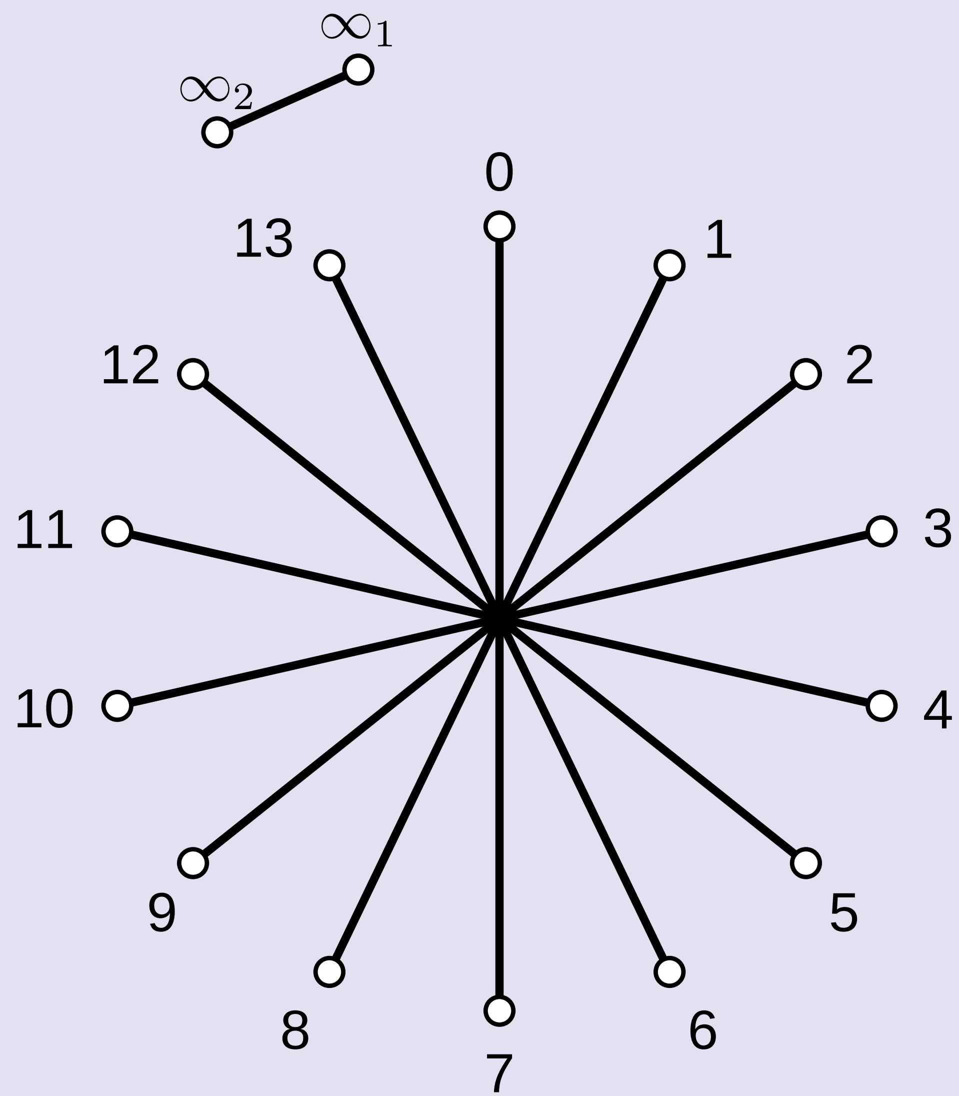


Even Starters:

Observe that an even starter in $\mathbb{Z}_{2t-2}$ yields a 1-factorisation of K_{2t} when combined with this 1-factor.

Example:

$t = 8$



Wolfe's Approach for K_{4m}

Begin by finding a pair of starters in $\mathbb{Z}_{2m-1}$.

Merge them to build an even starter in $\mathbb{Z}_{4m-2}$.

Each of the $m-1$ pairs of each starter is given a high/low designation as part of the construction.
So in fact 2^{m-1} even starters can be built from each pair of starters.

Use the even starter to build a 1-factorisation for K_{4m}.

Test the 1-factorisation for perfection.

Do this many times.

Wolfe found a P1F for K_{52}, published in 2009

He tested 7.494 billion pairs of starters in $\mathbb{Z}_{25}$.

Each pair produced $2^{12} = 4096$ even starters in $\mathbb{Z}_{50}$.

To find one that yielded a P1F of K_{52} took 10,000 hours of computing time (i.e., about 166 days) on a cluster.

Real time was 5 days.

It had been about 20 years since the previous smallest open case of the Perfect 1-Factorisation Conjecture was settled.

A P1F for K_{56}, published in 2019

Wolfe's approach was used on a cluster with one director task and 1023 worker tasks running in parallel.

Each worker built pairs of starters in $\mathbb{Z}_{27}$, merged them in $2^{13} = 8192$ ways, and tested the resulting 1-factorisations.

The worker that found a P1F had compared 7,730,443 pairs of starters in a time span of 33 days 6 hours.

The other 1022 workers were terminated after 43 days 9 hours.

Estimated total number of pairs of starters: 10.3 billion

Total computing time for workers: 1,064,700 hours
(i.e., a bit more than 121 years)

A P1F for K_{56}, published in 2019

The even starter of $\mathbb{Z}_{54}$ shown below yields a P1F for K_{56}.

$$\{36,17\}, \{44,12\}, \{39,45\}, \{18,35\}, \{8,50\}, \{23,15\},$$
$$\{42,32\}, \{5,46\}, \{19,49\}, \{22,37\}, \{10,6\}, \{33,30\}, \{3,41\},$$
$$\{14,21\}, \{48,43\}, \{16,52\}, \{25,34\}, \{7,38\}, \{11,31\}, \{4,2\},$$
$$\{29,28\}, \{1,27\}, \{0,40\}, \{13,24\}, \{51,26\}, \{53,20\}$$

The smallest open case of the Perfect 1-Factorisation Conjecture is now K_{64}.

Thank you.

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