Path decompositions of random directed graphs

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Dalhousie Graph Theory Seminar

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Introduction

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- (perfect) matchings
- paths (of fixed length)
- (Hamilton) cycles
- trees

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$\chi'(G) =$ min $\#$ matchings needed to decompose $E(G)$

- $\chi'(G) \geq \Delta(G)$
- $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ (Vizing)
- Almost all G satisfy $\chi'(G) = \Delta(G)$ (Erdős-Wilson).

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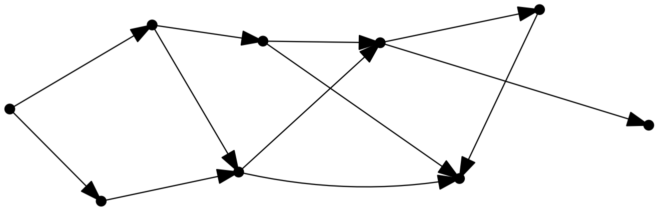

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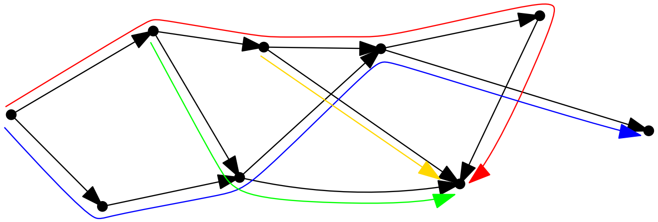

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- Any path decomposition of D has at least
  - $d^+(v) - d^-(v)$ paths starting at vertex v
  - $d^-(v) - d^+(v)$ paths ending at vertex v

  Hence
$$\mathrm{pn}(D) \geq \frac{1}{2} \sum_{v \in V(D)} |d^+(v) - d^-(v)|.$$

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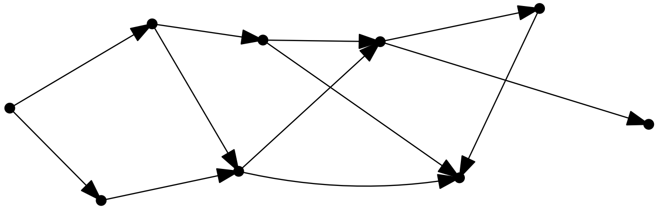

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Given directed graph D, and $v \in V(D)$,

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Conjecture (Alspach, Mason, Pullman, 1976)

$\mathrm{pn}(T) = \mathrm{ex}(T)$ *for every even tournament* T.

Theorem (Patel-Lo-Skokan-Talbot (2020) + Girão-Granet-Künh-Lo-Osthus (2021+))

The conjecture holds (asymptotically).

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- Start with n isolated vertices.
- Add each directed edge independently with probability p.

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Theorem (Espuny Díaz, Patel, Stroh (2021+))

Suppose

$$n^{-1/3}\log^4 n \le p \le 1 - n^{-1/5}\log^3 n$$

and let $D \sim D_{n,p}$. *Then*

$$\mathbb{P}\left(\mathrm{pn}(D) = \mathrm{ex}(D) \right) \to 1 \quad \textit{as} \quad n \to \infty.$$

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- $p = 1/2$: almost all D satisfy $\mathrm{pn}(D) = \mathrm{ex}(D)$.
- Some upper bound on p necessary.
- Show $\mathrm{pn}(D) = \mathrm{ex}(D)$ for many “nonrandom-like” D.