Families of graphs containing only finitely many vertex-critical graphs.

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(Joint work with Chính Hoàng and Joe Sawada)

Atlantic Graph Theory Seminar

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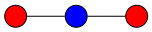
Figure: Asher Benjamin Cameron, the catalyst for this research.

Outline

- Brief definitions.
- Background on complexity of k-Coloring.
- Vertex-critical graphs for solving k-Coloring.
- Bound order of k-vertex-critical $(P_2 + \ell P_1)$-free graphs.
- Give exact number of k-vertex-critical $(P_3 + P_1)$-free graphs for $k \leq 7$.
- Open Problems.

Definitions

- P_n is the path on n vertices (P_3: ).

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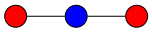
- P_n is the path on n vertices (P_3: ).
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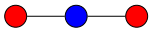
- P_n is the path on n vertices (P_3: ).
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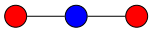
- P_n is the path on n vertices (P_3: ).
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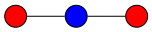
- P_n is the path on n vertices (P_3: ).
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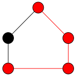
- A graph is H-free if it does not contain H as an induced subgraph.  is P_5-free but not P_4-free.

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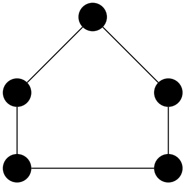

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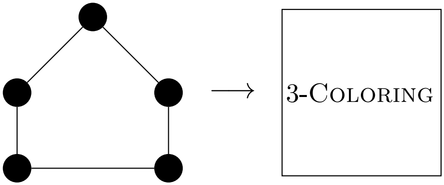

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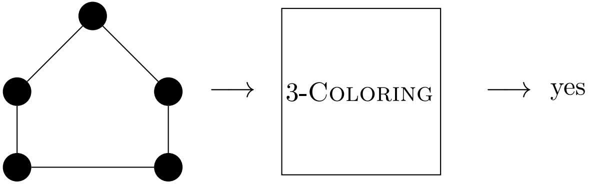

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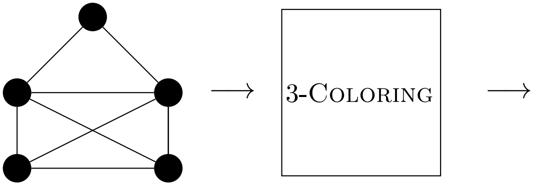

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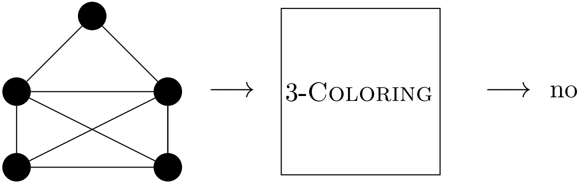

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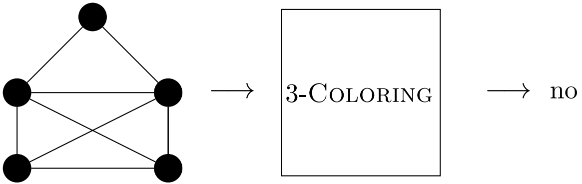

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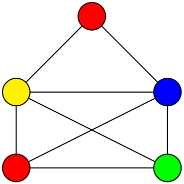
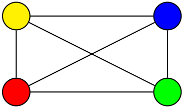
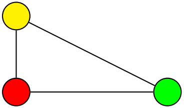
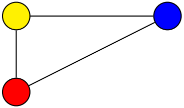
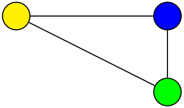
- How can we verify a "no"?

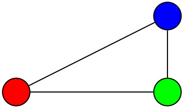


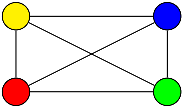












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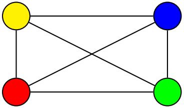

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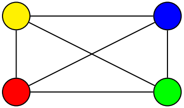

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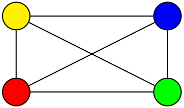

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Issue 1: For $k \geq 3$ there are an infinite number of k-vertex-critical graphs.
Issue 2: k-vertex-critical is a mouthful, so simply k-critical from now on.

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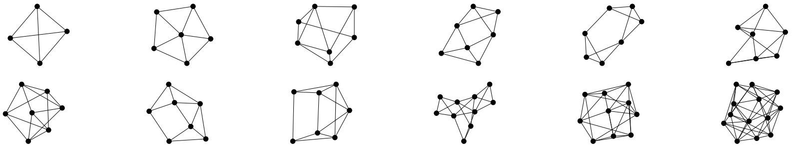

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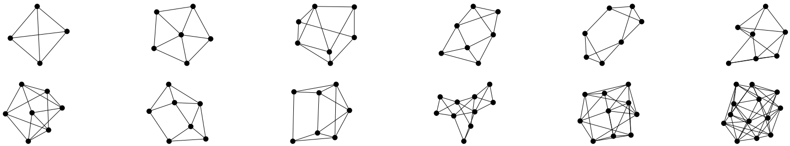

3-Coloring P_5-free graphs in polynomial-time by searching for above graphs as induced subgraphs, returning one as a no-certificate.

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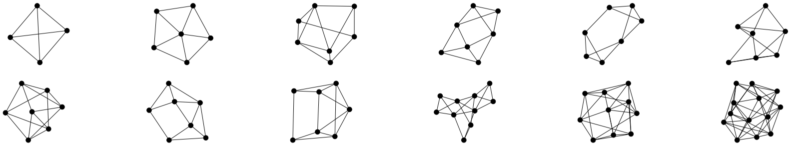

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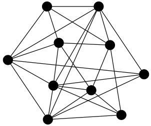

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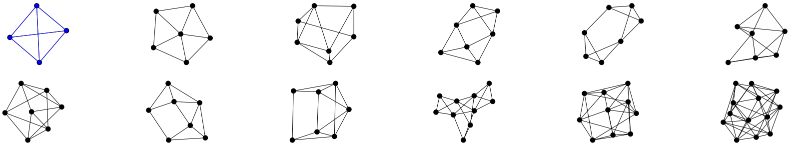

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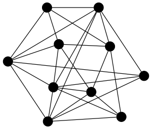

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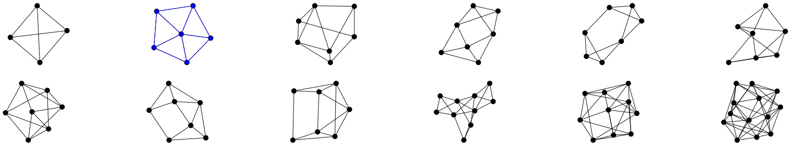

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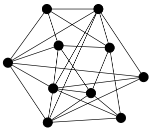

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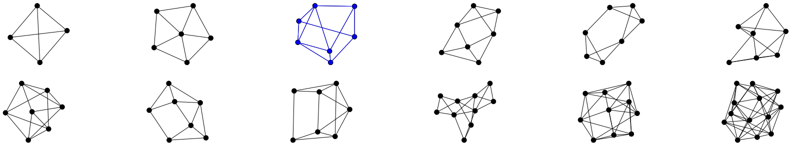

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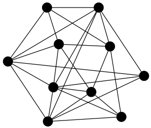

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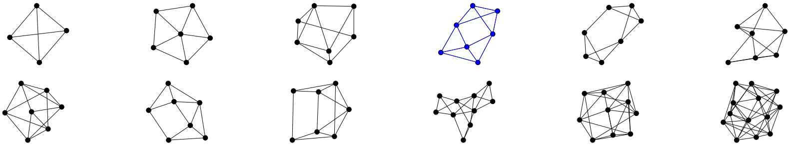

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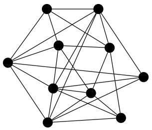

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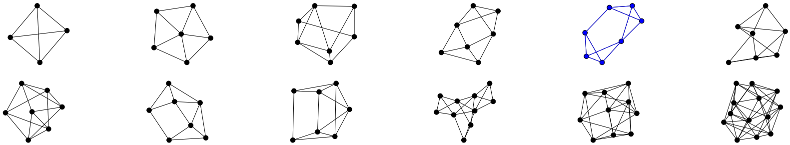

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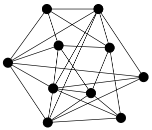

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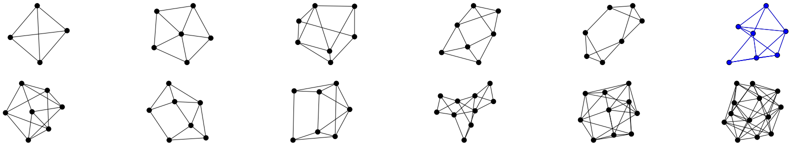

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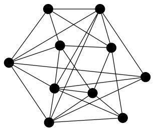

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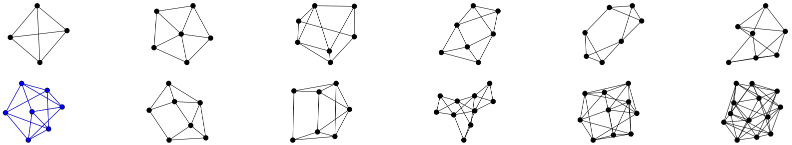

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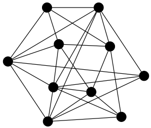

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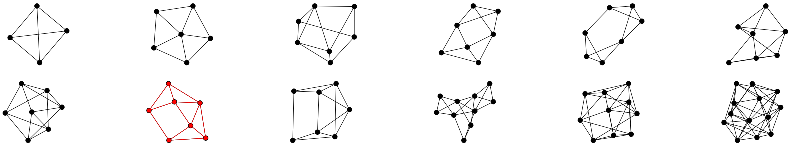

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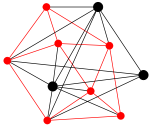

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$P_5, banner$	$k = 6$	(Cai et al. 2019)
$P_5, \overline{P_5}$	all k	(Dhaliwal et al. 2017)
$P_6, banner$	$k = 4, 5$	(Huang et al. 2019)
$P_t, K_{s,s}$	all k	(Kamiński-Pstrucha 2019)
P_7, C_5	$k = 4$	(Goedgebeur-Schaudt 2018)
$P_3 + P_1$	all k	(K. Cameron et al. 2020)

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In each case, a polynomial-time algorithm exists to solve $(k-1)$-Coloring that can also return a no-certificate.

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Theorem (Hoàng et al. 2015): If H contains $2P_2$ as an induced subgraph, then there is an infinite number of k-critical H-free graphs for all $k \geq 5$.

Theorem (Chudnovsky et al. 2020): There is a finite number of 4-critical H-free graphs if and only if H is an induced subgraph of P_6, $2P_3$, or $P_4 + \ell P_1$ for some $\ell \geq 0$.

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H must be one of:

- ℓP_1
- $P_2 + \ell P_1$
- $P_3 + \ell P_1$
- $P_4 + \ell P_1$

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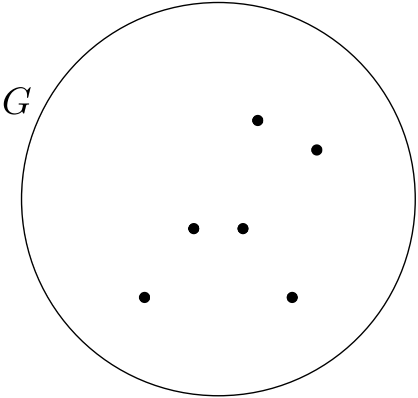

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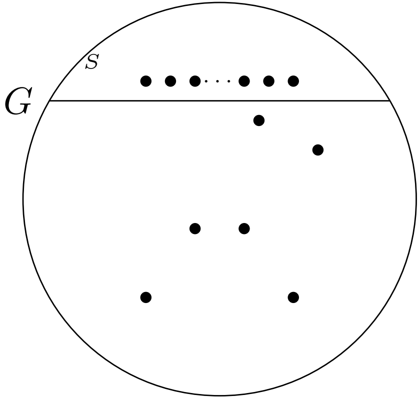

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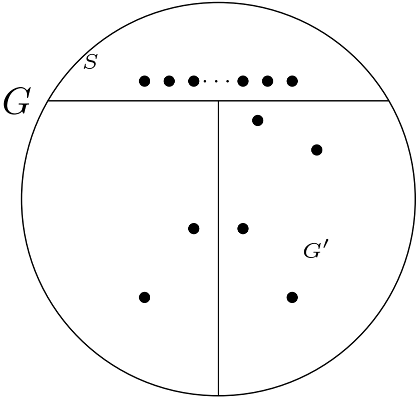

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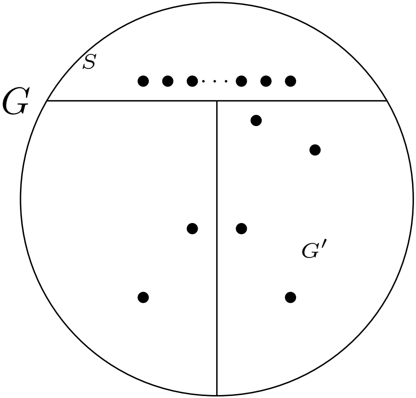

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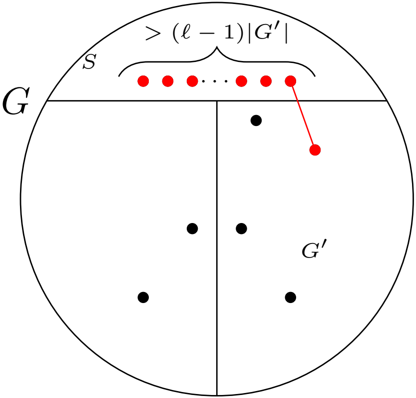

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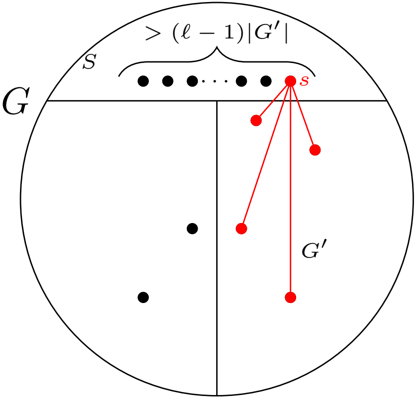

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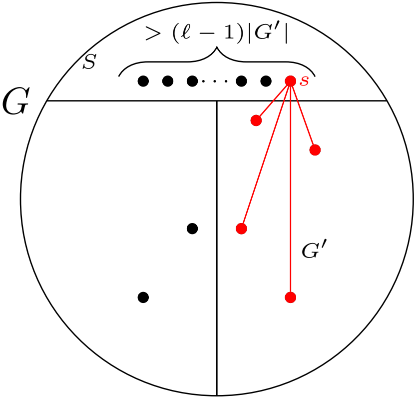

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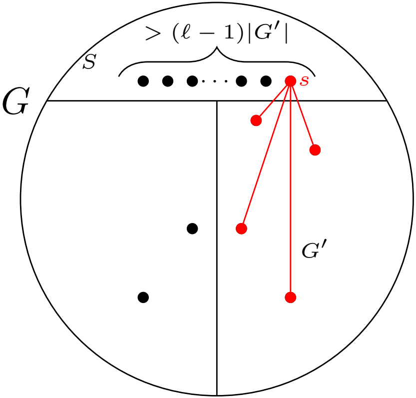

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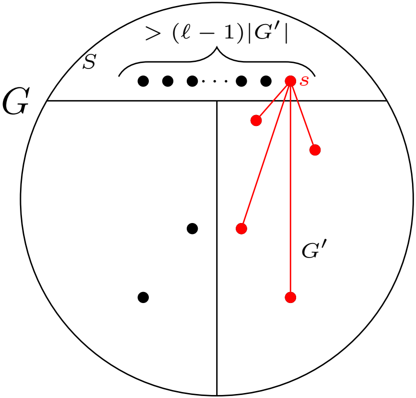

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Reduction for Open Problem: Only remaining forbidden subgraphs are $P_3 + \ell P_1$ and $P_4 + mP_1$ for $\ell > 1$, $m > 0$.

Bounded Order Open Problem: For which graphs H of order 4 are there only finitely many k-critical H-free graphs for *all* k?

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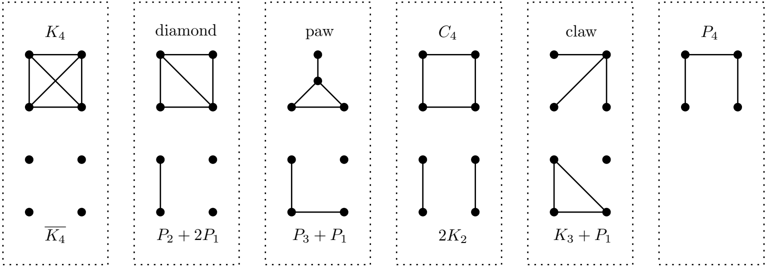

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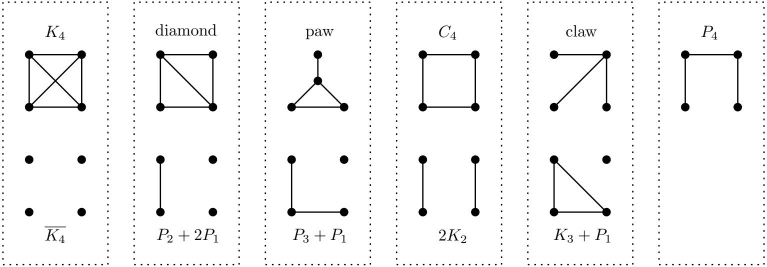

k/H	K_4	$\overline{K_4}$	diamond	$P_2 + 2P_1$	paw	$P_3 + P_1$	$2K_2$	C_4	claw	$K_3 + P_1$	P_4
3	∞	3	∞	2	∞	2	2	∞	∞	∞	1
4	∞	finite	∞	9	∞	8	7	∞	∞	∞	1
5+	∞	finite	∞	finite	∞	finite	∞	∞	∞	∞	1

Theorem (C-Hoàng-Sawada 2020+): Let H be a graph containing at most four vertices. There is a finite number of k-critical H-free graphs for all k if and only if H is an induced subgraph of $\overline{K_4}$, P_4, $P_2 + 2P_1$, or $P_3 + P_1$.

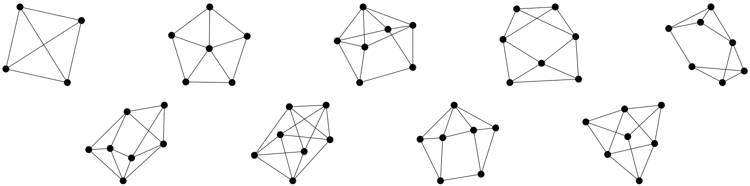

Figure: All 4-critical graphs $(P_2 + 2P_1)$-free graphs.

Theorem (K. Cameron et al. 2020): Let H be a graph of order four and $k \geq 5$ be fixed. Then there is a finite number of k-critical (P_5, H)-free graphs if and only if $H \neq 2P_2$ and $H \neq K_3 + P_1$.

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- Thus certifying algorithm for k-Coloring $(P_3 + P_1)$-free graphs for $k \geq 4$ cannot be implemented.

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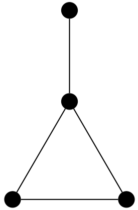

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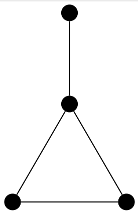

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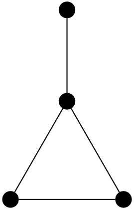

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Further $|V(G)| = 2k - 1$ if and only if $\overline{G}$ is connected

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- Construct the k-critical graphs for smaller orders by joining these graphs together.

$n/\#k$-critical	$k=4$	$k=5$	$k=6$	$k=7$
4	1	0	0	0
5	0	1	0	0
6	1	0	1	0
7	6	1	0	1
8	0	6	1	0
9	0	170	6	2
10	0	0	171	6
11	0	0	17,828	171
12	0	0	0	17,834
13	0	0	0	6,349,629
total	8	178	18,007	6,367,642

Table: Number of k-critical $(P_3 + P_1)$-free graphs of order n for $k \leq 7$.

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Problem 4. Can a tight upper bound on the order of k-critical $(P_2 + \ell P_1)$-free graphs be found for all k?

Thank You!

Figure: Mark Holmes of Platinum Blonde singing "Situation Critical".