Improved bounds for zeros of the chromatic polynomial on bounded degree graphs

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Based on joint work with Maurizio Moreschi, Viresh Patel and Ayla Stam

Introduction: The chromatic polynomial

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- For positive integer k, $\chi_G(k)$ equals the number of proper k-colorings of G.
- Introduced by Birkhoff in 1912.
- χ_G is a monic polynomial of degree $|V(G)|$.
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This talk: location of complex zeros of χ_G for bounded degree graphs G.

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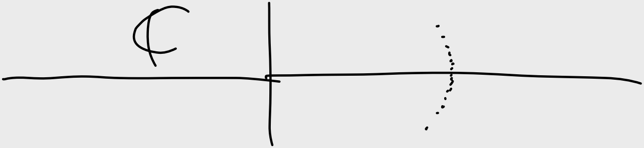

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- Statistical physics: relation with phase transitions of the zero-temperature limit of the anti-ferromagnetic Potts model.
- Algorithms: Absence of complex zeros implies efficient approximation algorithms for computing evaluations of χ_G via Barvinok's interpolation method.

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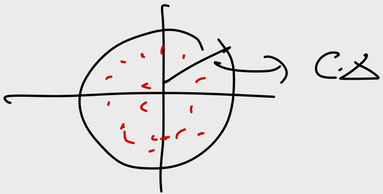
- The constant C is at most 6.91 (Fernandéz and Procacci 2008).



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Theorem (Moreschi, Patel, R. Stam, 2021+)

The constant C is at most 5.02.

Overview of the rest of the talk

- Revisit Sokal's approach
  (a) Expres the chromatic polynomial as a multivariate independence polynomial.
  (b) Use known conditions that guarantee zero-freeness of multivariate independence polynomials.
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- Improving on Sokal's approach
  (a') Expressing the chromatic polynomial as a multivariate block polynomial.
  (b') Prove conditions that guarantee zero-freeness of multivariate block polynomials.
  (c') Verify these conditions.
- Concluding remarks and questions

(a) From chromatic to independence

Look at

$$\hat{\chi}_G(x) := \sum_{F \subseteq E(G)} (-1)^{|F|} x^{|V(G)|-k(F)} = x^{|V(G)|} \chi_G(1/x)$$

$$= \sum_{F \subseteq E} \prod_{C \text{ cpt of } F} (-1)^{|C|} x^{|V(G)|-1}$$

$$= \sum_{\mathcal{P} \text{ partition of } V} \prod_{S \in \mathcal{P}} \sum_{\substack{F \subseteq E(S) \\ (S,F) \text{ conn.}}} (-1)^{|F|} \cdot x^{|S|-1}$$

notice that we can ignore singletons in $\mathcal{P}$!

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Define for $S \subseteq V(G)$ such that $|S| \geq 2$.

$$w(S) := \sum_{\substack{F \subseteq E(S) \\ (S,F) \text{ connected}}} (-1)^{|F|} x^{|S|-1}$$

and set $w(S) = 0$ otherwise. Then

$$\hat{\chi}_G(x) = \sum_{k \geq 0} \sum_{\substack{S_1,\ldots,S_k \subseteq V(G) \\ S_i \cap S_j = \emptyset \text{ if } i \neq j}} \prod_{i=1}^{k} w(S_i).$$

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Suppose there exists $a > 0$ such that for all $v \in V(G)$:

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$e^a - 1 = a + \frac{a^2}{2} +$

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"$T(1,0) \leq T(1,1)$"

$$|w(S)| = \Big| \sum_{\substack{F \subseteq E(S) \\ (S,F) \text{ connected}}} (-1)^{|F|} x^{|S|-1} \Big|$$

$$\leq \#\textit{spanning trees on } S \cdot |x|^{|S|-1}.$$

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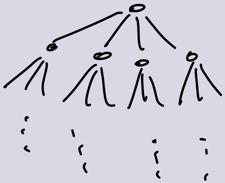

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- $t_k(T_\Delta) \leq (e\Delta)^{k-1}$.

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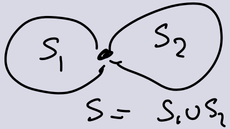
Observation: w is multiplicative over the blocks of $G[S]$.



$$x^{|S|-1} = x^{|S_1|-1} \cdot x^{|S_2|-1}$$

(a') Improvement: From chromatic to block

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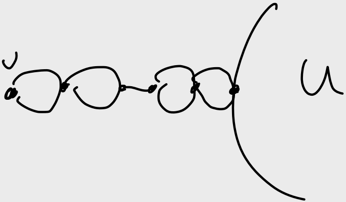

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Theorem (Moreschi, Patel, R. Stam, 2021+)

Suppose there exists $a > 0$ such that for all $v \in V(G)$ and connected sets $U \subseteq V(G) \setminus \{v\}$:

$$\sum_{B \in \mathcal{B}(v,U)} |w(B)| e^{a(|B|-1)} \leq e^a - 1,$$

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- The method also applies to other polynomials. In particular to the partition function of the Ising model.
- Plan to look at applications to the partition function of the Potts model.
- Block polynomials can be extended to matroids and a similar zero-free result can be proved in that setting. (Joint with Vincent Schmeits)

Thank you for your attention!