The Average Order of Dominating Sets of a Graph

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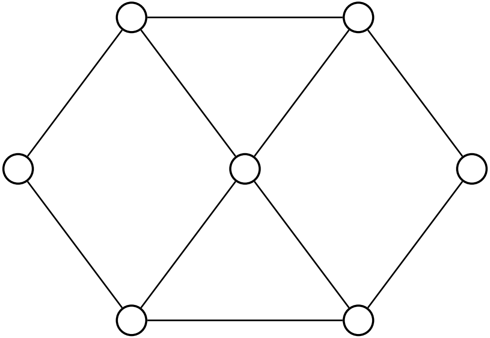
Joint work with Jason I. Brown

Dominating Set

A dominating set is a subset $S \subseteq V$ such that each vertex in $V - S$ has an neighbour in S.



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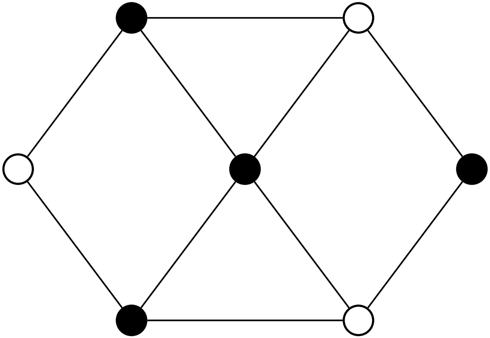

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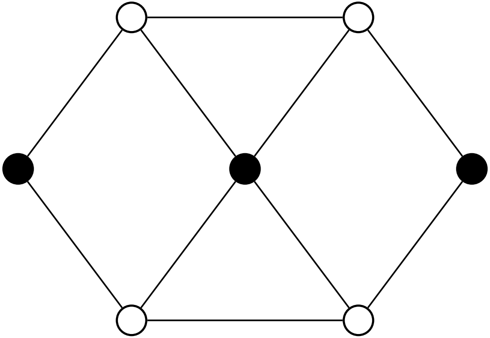

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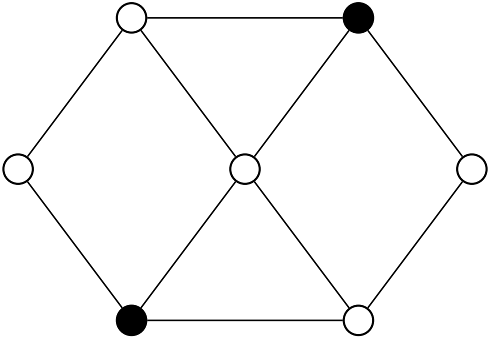
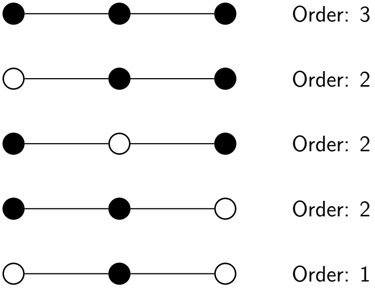


$\mathrm{avd}(G)$

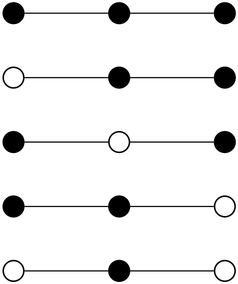
The **average order of the dominating sets of a graph** G, denoted $\mathrm{avd}(G)$, is simply the sum of the orders of each dominating set divided by the number of dominating sets in G.



$$\mathrm{avd}(P_3) = \frac{3+2+2+2+1}{5} = 2$$

$\widehat{\mathrm{avd}}(G)$

The **normalized average order of the dominating sets of a graph** G, denoted $\widehat{\mathrm{avd}}(G)$, is simply $\mathrm{avd}(G)$ divided by the number of vertices in G.



Order: 3

Order: 2

Order: 2

Order: 2

Order: 1

$$\widehat{\mathrm{avd}}(P_3) = \frac{\mathrm{avd}(P_3)}{3} = \frac{2}{3}$$

The Domination Polynomial

Let d_k denote the number of dominating sets of order k in a graph G with n vertices. Then

$$\operatorname{avd}(G) = \frac{\sum_{k=0}^{n} k \cdot d_k}{\sum_{k=0}^{n} d_k}$$

The **domination polynomial** $D(G, x)$ of G is defined as

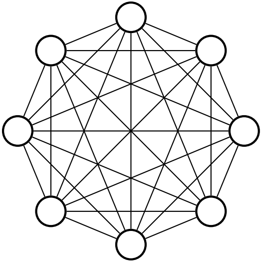
$$D(G, x) = \sum_{k=0}^{n} d_k x^k \;\Rightarrow\; \operatorname{avd}(G) = \frac{D'(G, 1)}{D(G, 1)}$$

$\mathrm{avd}(G)$ and the Domination Polynomial

For graphs G and H, $D(G \cup H, x) = D(G, x) \cdot D(H, x)$.

Lemma (B., Brown, 2020+)

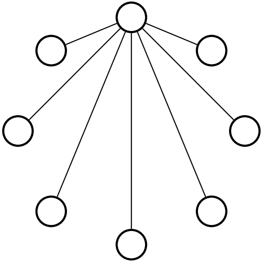
Let G and H be graphs. Then $\mathrm{avd}(G \cup H) = \mathrm{avd}(G) + \mathrm{avd}(H)$.



$$D(K_n, x) = (1+x)^n - 1$$

$$\mathrm{avd}(K_n) = \frac{n2^{n-1}}{2^n-1}$$

$$\mathrm{avd}(K_8) \approx 4.01569\ldots$$



$$D(K_{1,n-1}, x) = x(1+x)^{n-1} + x^{n-1}$$

$$\mathrm{avd}(K_{1,n-1}) = \frac{(n+1)2^{n-2}+n-1}{2^{n-1}+1}$$

$$\mathrm{avd}(K_{1,7}) \approx 4.51938\ldots$$

Other Average Graph Parameters

There has been work done on the averages of several other graphs invariants:

- The mean distance between vertices in a graph was introduced in 1977 by Doyle and Graver.
- The mean subtree order of a graph was introduced in 1983 by Jamison.
- The average size of an independent set in a graph was introduced in 2019 by Andriantiana et. el..
- Andriantiana et. el. also introduced the average size of a matching in a graph In 2020.

Other Average Graph Parameters

A common question is for each order n, which graphs maximize and minimize the given average graph parameter.

Average Parameter	Family	Maximum Graph	Minimum Graph	Author
Distance	Connected graphs	P_n	K_n	Doyle et. el. (1977)
Subtree order	Trees	???	P_n	Jamison (1983)
Independent sets and Matchings	All graphs	$\overline{K_n}$	K_n	Andriantiana et. el. (2019,2020)
Independent sets and Matchings	Trees	P_n	$K_{1,n-1}$	Andriantiana et. el. (2019,2020)

Extremal Graphs

For a graph G of order n, the upper bound for $\mathrm{avd}(G)$ is fairly straightforward:

$$\mathrm{avd}(G) \leq n = \mathrm{avd}(\overline{K_n}).$$

The lower bound for $\mathrm{avd}(G)$ is as you would expect:

Theorem (B., Brown, 2020+)

Let G be a graph of order n then $\mathrm{avd}(G) \geq \frac{n2^{n-1}}{2^n-1}$ with equality if and only if $G \cong K_n$.

Sketch of Proof

Proposition (B., Brown, 2020+)

For a graph G of order n, $d_k \leq d_{n-k}$ for all $k \leq \frac{n}{2}$.

Intuitively:

$$\overset{\text{“pivot”}}{\Downarrow}$$

$$d_0, d_1, \ldots, \underbrace{d_k, \ldots, d_{\frac{n}{2}-1}, d_{\frac{n}{2}}, d_{\frac{n}{2}+1}, \ldots, d_{n-k}}_{d_k \leq d_{n-k}}, \ldots, d_{n-1}, d_n$$

Algebraically it means $k \cdot d_k + (n-k) \cdot d_{n-k} \geq \frac{n}{2}(d_k + d_{n-k})$,
Therefore

$$\mathrm{avd}(G) = \frac{\sum_{k=0}^{n} k \cdot d_k}{\sum_{k=0}^{n} d_k} \geq \frac{\sum_{k=0}^{n} \frac{n}{2} \cdot d_k}{\sum_{k=0}^{n} d_k} = \frac{n}{2}$$

Extremal Trees

A similar, but more involved approach works to find the tree T on n vertices which minimizes $\mathrm{avd}(T)$.

Lemma (B., Brown, 2020+)

If T is a tree with n vertices then $d_{n-k} \geq d_{k+1}$ for all $k+1 \leq \frac{n+1}{2}$.

Theorem (B., Brown, 2020+)

If T is a tree with n vertices $\mathrm{avd}(T) \geq \mathrm{avd}(K_{1,n-1})$ with equality if and only if $T \cong K_{1,n-1}$.

But which tree maximizes $\mathrm{avd}(T)$?

A natural guess would be paths...

$\mathrm{avd}(P_n)$

Theorem (Alikhani, Peng, 2009)

For all $n \geq 4$, $D(P_n, x) = x(D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x))$.

The general solution for the recurrence $D(P_n, x)$ satisfies is:

$$\lambda_1(x) = \frac{x}{3} + p(x) + q(x),$$

$$\lambda_2(x) = \frac{x}{3} - p(x) - q(x) + \frac{\sqrt{3}}{2}\left(p(x) - q(x)\right) i,$$

$$\lambda_3(x) = \frac{x}{3} - p(x) - q(x) - \frac{\sqrt{3}}{2}\left(p(x) - q(x)\right) i,$$

where

$$p(x) = \sqrt[3]{\frac{x^3}{27} + \frac{x^2}{6} + \frac{x}{2} + \sqrt{\frac{x^4}{36} + \frac{7x^3}{54} + \frac{x^2}{4}}}$$

$$q(x) = \sqrt[3]{\frac{x^3}{27} + \frac{x^2}{6} + \frac{x}{2} - \sqrt{\frac{x^4}{36} + \frac{7x^3}{54} + \frac{x^2}{4}}}$$

Theorem (B., Brown, 2020+)

Suppose functions $f_n(x)$ satisfy

$$f_n(x) = \alpha_1(x)(\lambda_1(x))^n + \alpha_2(x)(\lambda_2(x))^n + \cdots + \alpha_k(x)(\lambda_k(x))^n$$

where $\alpha_i(x)$ and $\lambda_i(x)$ are fixed non-zero analytic functions, such that $|\lambda_1(1)| > |\lambda_i(1)|$ for all $i > 1$. Then

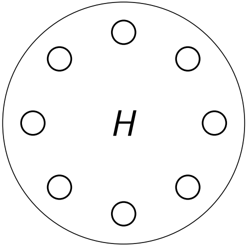
$$\lim_{n\to\infty} \frac{f_n'(1)}{nf_n(1)} = \frac{\lambda_1'(1)}{\lambda_1(1)}.$$

Theorem (B., Brown, 2020+)

$$\lim_{n\to\infty} \frac{\mathrm{avd}(P_n)}{n} = \lim_{n\to\infty} \widehat{\mathrm{avd}}(P_n) \approx 0.618419922.$$

Paths are **NOT** the upper bound for trees.

A graph is called **sunlike** if it is formed by adding one or two pendant vertices to every vertex some base graph H.



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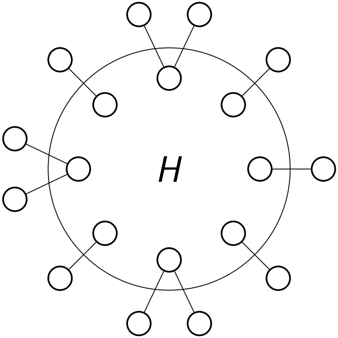


All sunlike graphs G have $\widehat{\mathrm{avd}}(G) = \frac{2}{3}$.

It remains an open question if there is a tree T with $\widehat{\mathrm{avd}}(T) > \frac{2}{3}$.

We know $\frac{n}{2} < \mathrm{avd}(G) \leq n$, but can we do better?

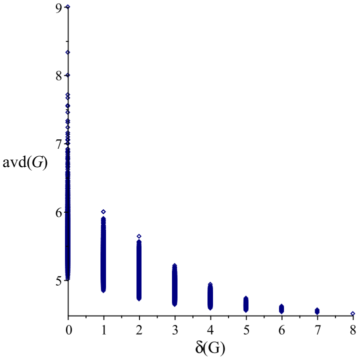


Figure 1: $\mathrm{avd}(G)$ over the minimum degree $\delta(G)$ for graphs of order 9.

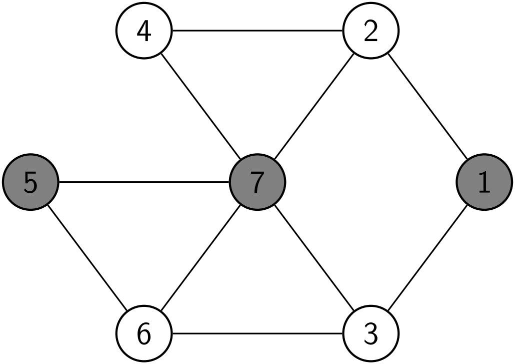
Conjecture

Let G be a graph with $n \geq 2$ vertices. If G has no isolated vertices (so, in particular, if G is connected) then $\mathrm{avd}(G) \leq \frac{2n}{3}$.

Lets take a little journey...

For a dominating set S, let $a(S)$ denote the **domination critical vertices** of S. That is

$$a(S) = \{v \in S : S - v \text{ is not a dominating set}\}.$$



If $S = \{1, 5, 7\}$ then $a(S) = \{1, 7\}$

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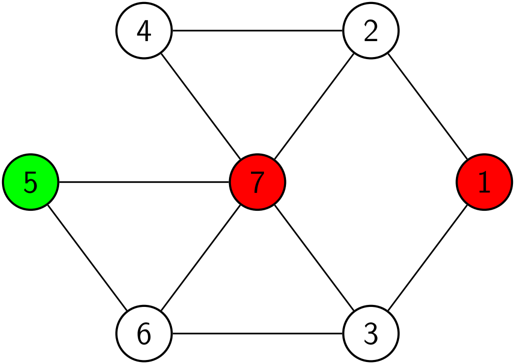


If $S = \{1, 5, 7\}$ then $a(S) = \{1, 7\}$

For a graph G, let $\mathcal{D}_k(G)$ denote the collection of dominating sets of order k.

$$\sum_{S \in \mathcal{D}_k(G)} |a(S)| = k \cdot d_k - (n-k+1) \cdot d_{k-1}.$$

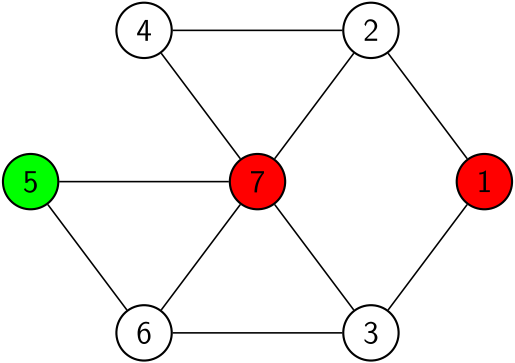
Lemma (B., Brown, 2020+)

For a graph G with n vertices.

$$\sum_{S \in \mathcal{D}(G)} |a(S)| = 2D'(G,1) - nD(G,1).$$

Where $\mathcal{D}(G)$ denotes the collection of all dominating sets of G

For a dominating set S and $v \in a(S)$ there is a non-empty set of vertices which are not dominated by $S - v$. We call these the **private neighbours** of v in S and denote them by $\text{Priv}_S(v)$.

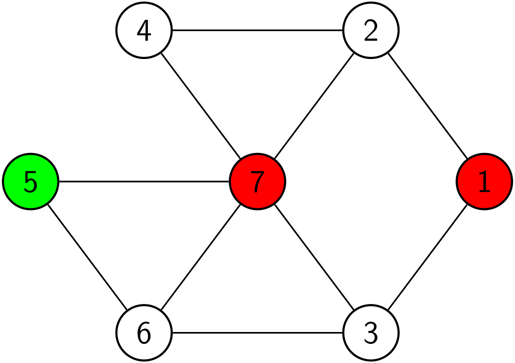


$S = \{1, 5, 7\}$
$a(S) = \{1, 7\}$

$\text{Priv}_S(7) = \{4\}$
$\text{Priv}_S(1) = \{1\}$

We can then partition $a(S)$ into two parts:

$$a_1(S) = \{v \in a(S) : \text{Priv}_S(v) \cap (V - S) \neq \emptyset\}$$
$$a_2(S) = \{v \in a(S) : \text{Priv}_S(v) = \{v\}\}.$$

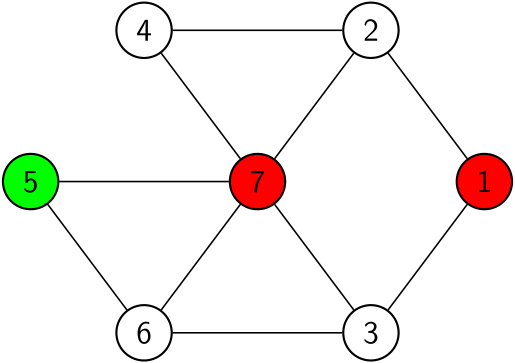


$S = \{1, 5, 7\}$
$a(S) = \{1, 7\}$

$a_1(S) = \{7\}$
$a_2(S) = \{1\}$

Furthermore, we can then partition $V - S$ into two parts:

$$N_1(S) = \{v \in V - S : |N[v] \cap S| = 1\}$$
$$N_2(S) = \{v \in V - S : |N[v] \cap S| \geq 2\}.$$



$S = \{1, 5, 7\}$ $a_1(S) = \{7\}$ $N_1(S) = \{4\}$
$a(S) = \{1, 7\}$ $a_2(S) = \{1\}$ $N_2(S) = \{2, 3, 6\}$

Intuition for $a_1(S)$ and $N_1(S)$

For a dominating set S:

- $a_1(S)$: Vertices in S which have private neighbours outside S.
- $N_1(S)$: Vertices not in S with one neighbours in S.
  - $N_1(S)$: Vertices which the are private neighbours outside of S.

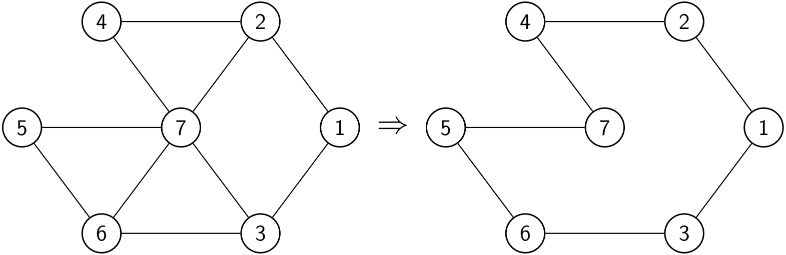
Lemma (B., Brown, 2020+)

Let G be a graph. For any dominating set, $|a_1(S)| \leq |N_1(S)|$.

Lemma (B., Brown, 2020+)

Let G be a graph with is quasi-regularizable. For any dominating set, $|a_2(S)| \leq |N_2(S)|$.

A graph G is called **quasi-regularizable** if one can replace each edge of G with a non-negative number of parallel copies, so as to obtain a regular multigraph of minimum degree at least one.



If G is quasi-regularizable then got every dominating set S:

$$|a(S)| = |a_1(S)| + |a_2(S)| \leq |N_1(S)| + |N_2(S)| = n - |S|.$$

So if we sum this over all dominating sets be get

$$\sum_{S \in \mathcal{D}(G)} |a(S)| \leq \sum_{S \in \mathcal{D}(G)} (n - |S|) = \sum_{k=0}^{n} (n-k) \cdot d_k = nD(G,1) - D'(G,1).$$

Then if we combine this with the previous result:

$$\sum_{S \in \mathcal{D}(G)} |a(S)| = 2D'(G,1) - nD(G,1).$$

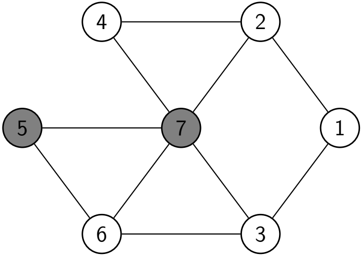
We obtain:

Theorem (B., Brown, 2020+)

If G is a quasi-regularizable graph then $\mathrm{avd}(G) \leq \frac{2n}{3}$.

Bounding $a_2(S)$ for general graphs

Let $p_v(G)$ denote the collection of dominating sets of $G - v$ which are not dominating sets of G. Below is an example of $p_1(G)$.



For each dominating set S and $v \in a_2(S)$, $S - v$ is set which dominates $G - v$ but not G.

Lemma (B., Brown, 2020+)

For any graph G be a graph, $\sum_{S \in \mathcal{D}(G)} |a_2(S)| = \sum_{v \in V(G)} |p_v(G)|$

Note that for each vertex v, $(2^{\deg(v)+1} - 1)|p_v(G)| \leq D(G,1)$

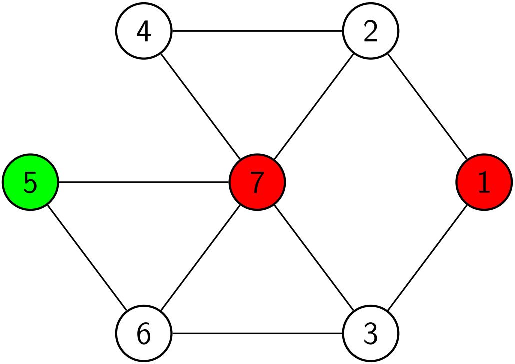
Theorem (B., Brown, 2020+)

Let G be a graph with $n \geq 2$ vertices and minimum degree $\delta \geq 1$. Then

$$\mathrm{avd}(G) \leq \frac{2n(2^\delta - 1) + n}{3(2^\delta - 1) + 1},$$

and so $\mathrm{avd}(G) \leq \frac{3n}{4}$.

We can do better...



$S = \{1, 5, 7\}$
$a(S) = \{1, 7\}$

$a_1(S) = \{7\}$
$a_2(S) = \{1\}$

$N_1(S) = \{4\}$
$N_2(S) = \{2, 3, 6\}$

Lemma (B., Brown, 2020+)

For any graph G, $\sum_{S \in \mathcal{D}(G)} |N_1(S)| = \sum_{e \in E(G)} |\mathcal{D}(G) - \mathcal{D}(G-e)|$.

Lemma (Kotek, Preen, Simon, Tittmann, Trinks, 2012)

Let G *be a graph. For every edge* $e = \{u, v\}$ *of* G,

$|\mathcal{D}(G) - \mathcal{D}(G-e)| = |p_u(G-e)| + |p_v(G-e)| - |p_u(G)| - |p_v(G)|$.

Theorem (B., Brown, 2020+)

For any graph G *with no isolated vertices,*

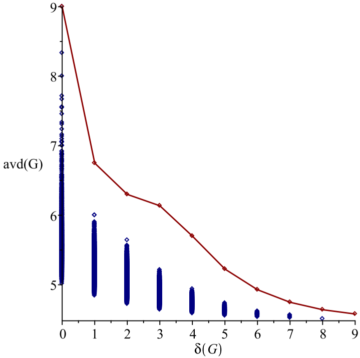
$$\mathrm{avd}(G) \leq \frac{n}{2} + \sum_{v \in V(G)} \frac{\deg(v)}{2^{\deg(v)+1} - 2}.$$

Corollary (B., Brown, 2020+)

For a graph G with minimum degree $\delta \geq 1$.

$$\operatorname{avd}(G) \leq \frac{n}{2}\left(1 + \frac{\delta}{2^{\delta} - 1}\right).$$

In particular if $\delta \geq 2\log_2(n)$ then $\operatorname{avd}(G) \leq \frac{n+1}{2}$.



Distribution of $\mathrm{avd}(G)$

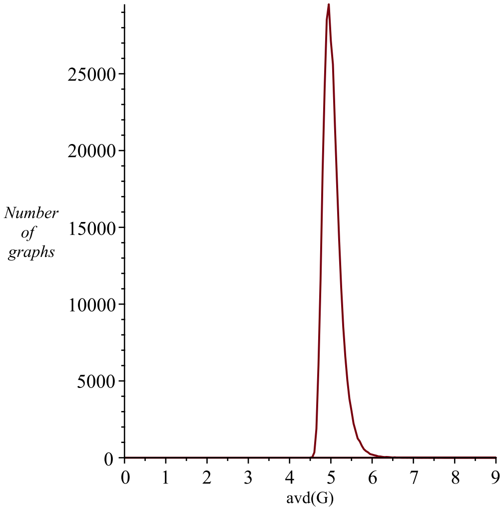


Figure 2: Distribution of $\mathrm{avd}(G)$ for all graphs of order 9

Let $\mathcal{G}(n,p)$ denote the sample space of random graphs on n vertices (each edge exists is independent present with probability p)

Theorem (B., Brown, 2020+)

For a fixed $p \in (0,1]$ let $G_n \in \mathcal{G}(n,p)$. Then

$$\lim_{n\to\infty} \widehat{\mathrm{avd}}(G_n) = \frac{1}{2}.$$

Proposition (B., Brown, 2020+)

The set $\left\{\widehat{\mathrm{avd}}(G) : G \text{ is a graph}\right\}$ is dense in $[\frac{1}{2}, 1]$.

Future Work and Open Problems

- Can we show for all graphs with $\delta(G) \geq 0$, $\mathrm{avd}(G) \leq \frac{2n}{3}$?
- Can we extend the work to the unimodality conjecture of the Domination Polynomial?

Conjecture (Alikhani, Peng, 2009)

The domination polynomial of any graph is unimodal.

$$\sum_{S \in \mathcal{D}_k(G)} |a(S)| = k \cdot d_k - (n-k+1) \cdot d_{k-1}.$$

- For a non-empty graph G, does there exists a vertex v and edge e such that

$$\mathrm{avd}(G-v) < \mathrm{avd}(G) < \mathrm{avd}(G-e).$$

THANK YOU!