The Tutte Polynomial
A Mathematical Catalyst

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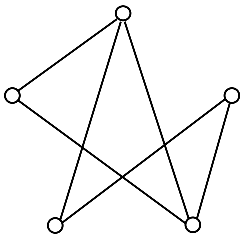
What is a graph polynomial?

- A graph polynomial is an algebraic object, namely a polynomial in one or more variables, associated to a graph.
- Typically, a graph polynomial is also a graph invariant, that is, two isomorphic graphs will have the same polynomial associated to them.
- One goal is to extract graph-theoretical information using algebraic tools.
- Another goal is to determine properties of the polynomials themselves.
- The graph polynomials have close connections to applications such as network reliability, scheduling, chip firing, knots, statistical mechanics, nanoscale self-assembly, and DNA sequencing.

Why are graph polynomials powerful?

Degree sequence polynomial $V(G;x) = \sum_i f_i(G) x^i$

Where $f_i(G)$ is the number of vertices of *G* with degree *i.*



$$V(G;x) = 2x^3 + 3x^2$$

(This is an example of a generating function formulation.)

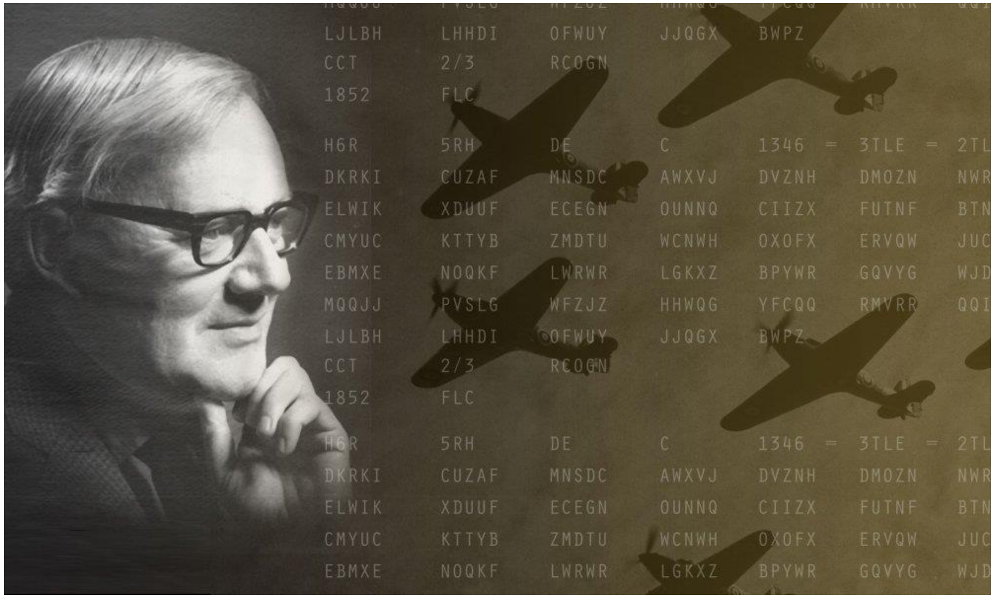
Can now use algebraic tools to extract combinatorial information.

$$\frac{1}{2}V'(G;1) = |E(G)|$$

Reason: $V'(G;1) = \sum_v \deg(v) = 2|E(G)|$

William Tutte, 14 May 1917 – 2 May 2002

Huge impact on combinatorics, but also Bletchley Park– much more challenging and critical cipher than Turing, but not declassified until decades later (mid-90’s), story just being told.



https://uwaterloo.ca/magazine/spring-2015/features/keeping-secrets


Applications
Interrelations
Generalizations
Specializations
Evaluations
Characterizations
Nucleation:
Original Tutte-
Whitney poly

The original catalyst

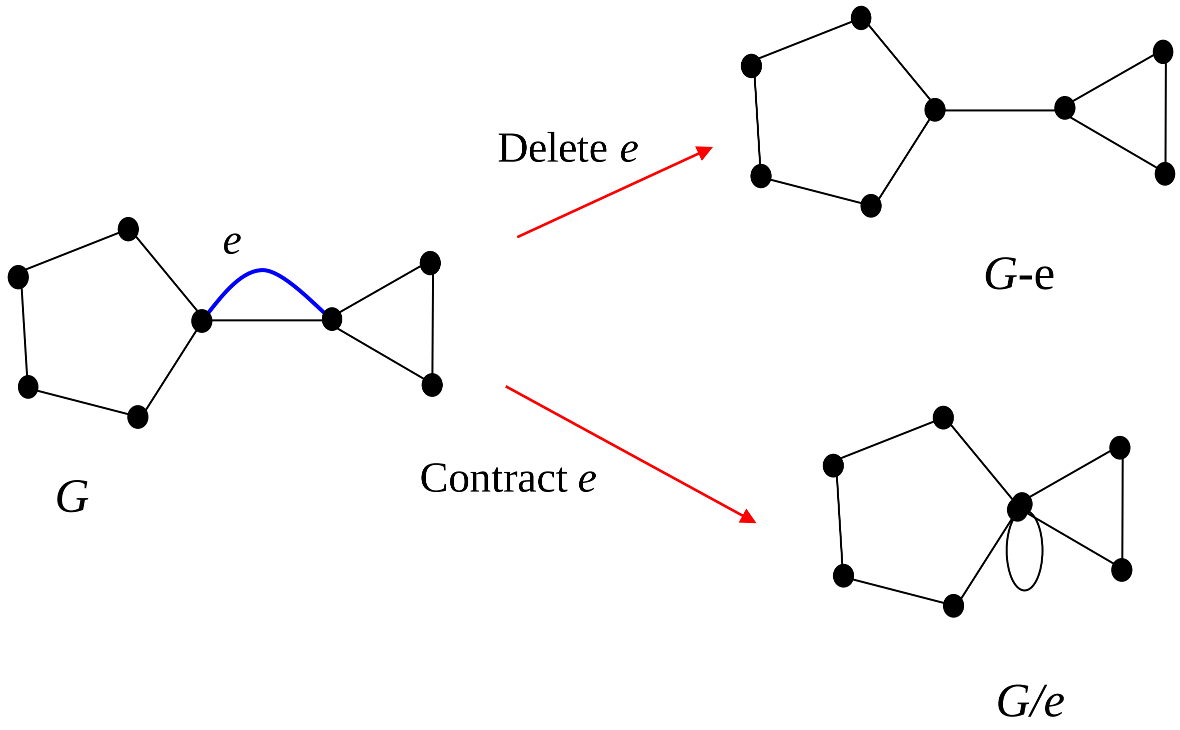
W. T. Tutte, A ring in graph theory, Proc. Camb. Phil. Soc. 43 (1947) 26-40

- Graph coloring = major impetus for graph theory at the beginning of the 20th century -> Birkoff's chromatic polynomial in 1912. Whitney (Birkoff's PhD student) in the 1930's -> coefficients. (A logical expansion in mathematics (1932) ; The coloring of graphs (1932), A set of topological invariants for graphs (1933))
- "A ring in graph theory" Tutte introduced explicit polynomials and established their fundamental properties , most notably deletion/contraction relations and universality.
- He recovers counting colorings, flows, spanning forests, and Whitney's numbers as the coefficients of a rank generating function, establishes universality, and provides a 'recipe theorem'.
- He further builds upon this work throughout his career, e.g.
  - An Algebraic Theory of Graphs, PhD thesis, (1948.).
  - A contribution to the theory of chromatic polynomials, (1954) .
  - On dichromatic polynomials, (1967) .
  - Some polynomials associated with graphs (1974).
  - The dichromatic polynomial (1976).
  - 1-Factors and polynomials (1980).
- And also, developed in parallel from a physics perspective, the Potts model partition function, introduced by Renfrey Potts in 1952 (we will return to this).

Broad influence

- Tutte’s approach has been so influential that now the attributes of the Tutte polynomial have shaped the field (large field –MSC 05C31).
- If a new graph/matroid polynomial arises, investigators often seek to establish:
  - Its relation to the Tutte polynomial (as a specialization, generalization, or through some transformation)
  - Similar attributes (having both state-sum and recursive expressions, having universality properties, etc.)

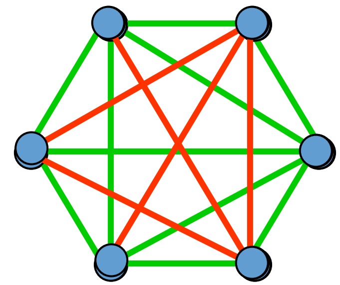
Deletion and contraction



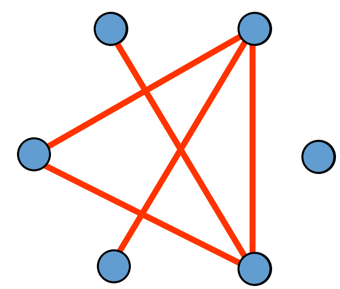
Some parameters of a graph *G*

- Components: $k(G)$
- Rank: $r(G) = v(G) - k(G)$
- Nullity: $n(G) = e(G) - r(G)$
- If *A* is a subset of *E*(*G*), then $k(A), r(A), n(A)$ are, respectively, the components, rank, and nullity of the spanning subgraph on *A*.

G, with *A* = the red edges



k(*G*)=1, *r*(*G*)=5, *n*(*G*)=10



k(*A*)=2, *r*(*A*)=4, *n*(*A*)=1

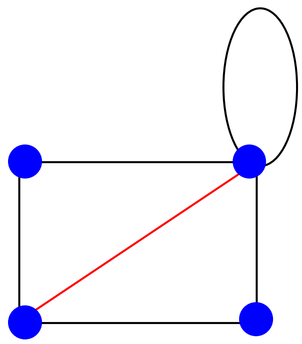
(These are the rank and nullity of the incidence matrix over $\mathbf{Z}_2$, but that is a story for another day)

Tutte Polynomial for graphs

Recursive definition:

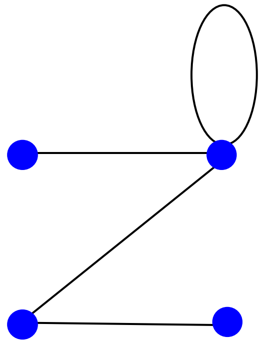
Let *e* be an edge of *G* that is neither a bridge nor a loop. Then,

$$T(G;x,y) = T(G-e;x,y) + T(G/e;x,y)$$



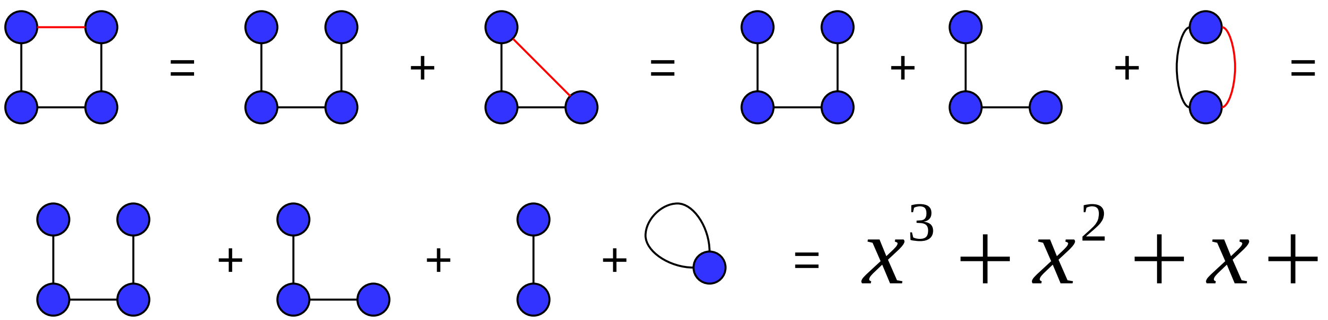
And if *G* consists of *i* bridges and *j* loops, then

$$T(G;x,y) = x^i y^j$$



Example

The Tutte polynomial of a cycle on 4 vertices…



$$= x^3 + x^2 + x + y$$

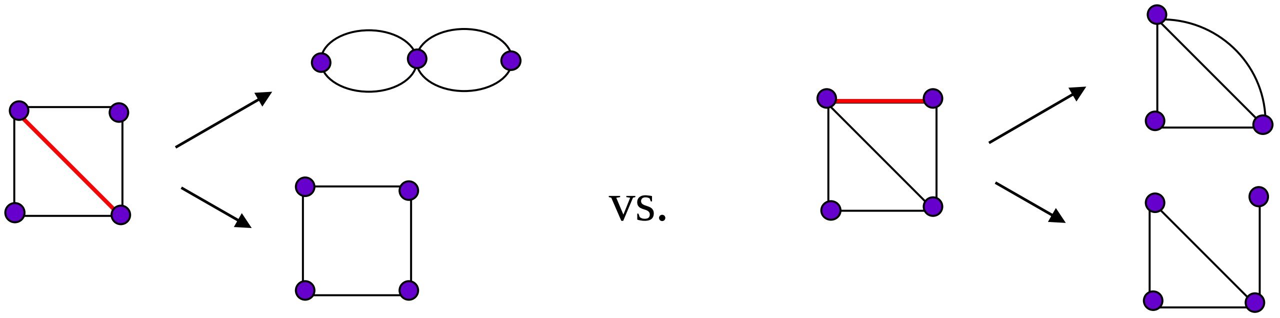
Notice that we choose an *order* in which to delete and contract the edges….

Does order matter?

This recursive definition means choosing an order of the edges, and deleting/contracting them in some order.

Is this well-defined?

How do we know we will get the same polynomial if we use a different order, especially since deleting and contracting different edges give different minors?



Some formulations

Rank-Nullity

$$R(G;x,y) = \sum_{A \subseteq E(G)} (x-1)^{r(G)-r(A)}(y-1)^{n(A)}$$

Deletion-contraction

$$T(G;x,y) = x^i y^j \text{, if } G \text{ is } i \text{ edges and } j \text{ loops}$$
$$= T(G-e;x,y) + T(G/e;x,y) \text{ else}$$

(with a transform)

$$Z(G;u,v) = \sum_{A \subseteq E(G)} u^{k(A)} v^{|A|}$$

Activities expansion

$$T(G;x,y) = \sum_{i,j} t_{ij} x^i y^j$$

where t_{ij} is the number of spanning trees with internal activity *i* and external activity *j*

Side note: Chromatic connection

$$C(G;x) = \sum_{A \subseteq E(G)} (-1)^{|A|} x^{k(A)} = Z(G;x,-1)$$

(use edge order)

(no edge order)

Conversion

$$R(G;x,y)=\sum_{A\subseteq E(G)}(x-1)^{r(G)-r(A)}(y-1)^{n(A)}$$

$r(A)=v(A)-k(A)$
$=v(G)-k(A)$

$r(G)-r(A)=k(A)-k(G)$

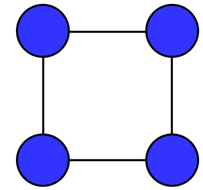
$n(A)=e(A)-r(A)$
$=|A|+k(A)-v(G)$

$$R(G;x,y)=(x-1)^{k(G)}(y-1)^{n(G)}\sum_{A\subseteq E(G)}(x-1)^{k(A)}(y-1)^{k(A)+|A|}$$

$$Z(G;u,v)=\sum_{A\subseteq E(G)}u^{k(A)}v^{|A|}$$

→ Let $u=(x-1)(y-1)$ and $v=(y-1)$

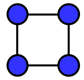
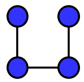
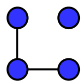
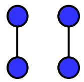
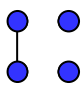
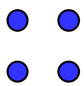
Dichromatic example



$$Z(G;u,v)=\sum_{A\subseteq E(G)} u^{k(A)}v^{|A|}$$

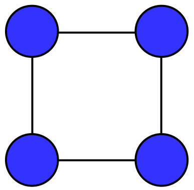
$$T(G;x,y)=(x-1)^{-k(G)}(y-1)^{-v(G)}Z\left(G;(x-1)(y-1),(y-1)\right)$$

$$Z\left(G;u,v\right)=u^{k(G)}v^{r(G)}T\left(G;\frac{u+v}{v},v+1\right)$$

	A	$k(A)$	$\|A\|$	term
1		1	4	uv^4
4		1	3	uv^3
4		2	2	u^2v^2
2		2	2	u^2v^2
4		3	1	u^3v
1		4	0	u^4

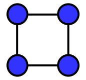
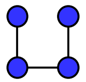
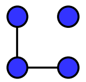
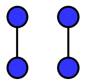
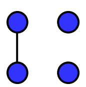
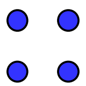
$$(x-1)^{-1}(y-1)^{-4}\left((x-1)(y-1)^5+4(x-1)(y-1)^4+6(x-1)^2(y-1)^4+4(x-1)^3(y-1)^4+(x-1)^4(y-1)^4\right)$$
$$=(y-1)+4+6(x-1)^1+4(x-1)^2+(x-1)^3=x^3+x^2+x+y$$

Rank-nullity example



$$R(G;x,y)=\sum_{A\subseteq E(G)}(x-1)^{r(G)-r(A)}(y-1)^{n(A)}$$

$$r(G)-r(A)=k(A)-k(G)$$

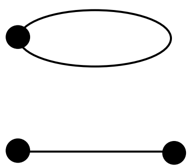
	A	$r(A)=v(A)-k(A)$	$n(A)=e(A)-r(A)$	term
1		4-1=3	4-3=1	$1(x-1)^{3-3}(y-1)^{1}$ = y-1
4		4-1=3	3-3=0	$4(x-1)^{3-3}(y-1)^{0}$ = 4
4		4-2=2	2-2=0	$4(x-1)^{3-2}(y-1)^{0} = 4(x-1)^{1}$
2		4-2=2	2-2=0	$2(x-1)^{3-2}(y-1)^{0} = 2(x-1)^{1}$
4		4-3=1	1-1=0	$4(x-1)^{3-1}(y-1)^{0} = 4(x-1)^{2}$
1		4-4=0	0-0=0	$1(x-1)^{3-0}(y-1)^{0} = 1(x-1)^{3}$

$$x^3+x^2+x+y$$

Conversion-by induction on number of edges

$T(G; x, y)\ via\ \ deletion - contraction$

Base case– easy to show that *T* and *R* are equal when *G* has just one edge.



$$R(G; x, y) = \sum_{A \subseteq E(G)} (x-1)^{r(G)-r(A)} (y-1)^{n(A)}$$

$$= \sum_{\substack{A \subseteq E(G) \\ e \notin A}} (x-1)^{r(G)-r(A)} (y-1)^{n(A)} + \sum_{\substack{A \subseteq E(G) \\ e \in A}} (x-1)^{r(G)-r(A)} (y-1)^{n(A)}$$

$$= R(G-e; x, y) + R(G/e; x, y)$$

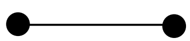
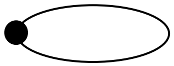
$$= T(G-e; x, y) + T(G/e; x, y)$$
$$= T(G; x, y)$$

Universality (Recipe Theorem)

THEOREM: (various forms—Tutte,Brylawski, Welsh, Oxley, Bollobas, etc.)
If f is a graph invariant such that

a) $f(G) = a\, f(G\text{-}e) + b\, f(G/e)$ whenever e is ordinary,

b) $f(GH) = f(G)f(H)$ where GH is either the disjoint union of G and H, or one point join of G and H.

c) and f () = x_0 , and f () = y_0.

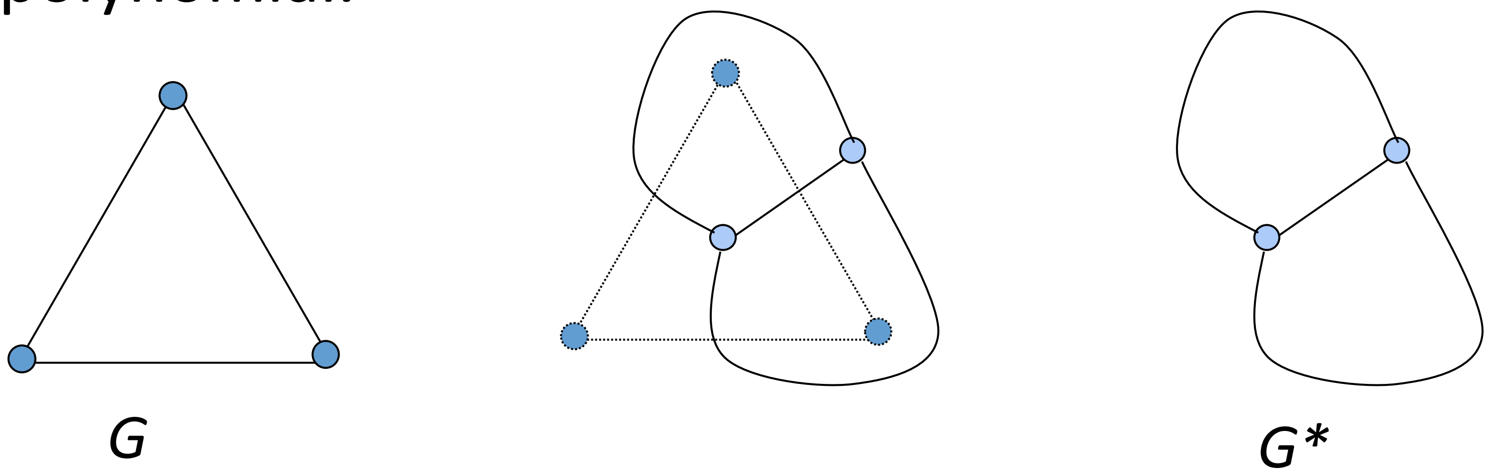
Then,

$$f(G) = a^{n(G)} b^{r(G)}\ T\left(G; \frac{x_0}{b}, \frac{y_0}{a}\right)$$

Thus, the Tutte polynomial is universal for multiplicative deletion-contraction invariants.

Duality

- We will thread this theme as a representative example through the expansion of the Tutte polynomial.



- If G is a plane graph, then

$$T(G;x,y) = T(G^*;y,x)$$

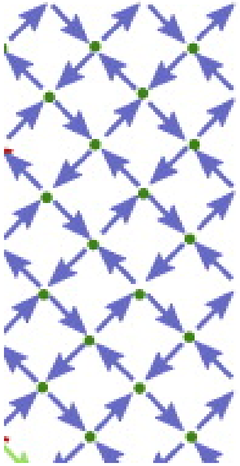
Combinatorial Evaluations

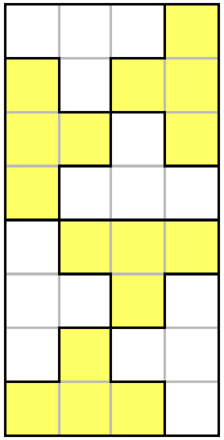
- If G is a connected graph, then
  - $T(G; 1,1)$ = the number of spanning trees/score vectors of G,
  - $T(G; 2,1)$ = the number of spanning forests of G
  - $T(G; 1,2)$ = the number of spanning connected subgraphs of G
  - $T(G; 2,2) = 2^{|E(G)|}$
  - $T(G; -1,-1) = (-1)^{|E(G)} (-2)^{\dim(C\cap C\perp)}$, where C is the space of the incidence matrix of G over $\mathbf{Z}_2$
  - $T(G; 2,0)$ = the number of acyclic orientations of G, representable matroids, hyperplane arrangements
  - $T(G; 0,2)$ = totally cyclic orientations of G
  - $T(G; 1,0)$ = the number of acyclic orientations of G with a single specified source
  - T(G;0,1)=special score vectors of G (out degree sequence of an orientation of G).

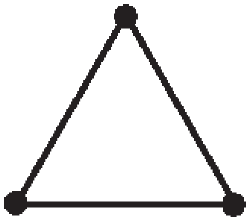
Brylawski, Gioan, Green, Las Vergnas, Lucas, Read, Rosenstiehl, Stanley, Winder, Zaslavsky, etc., etc.…

Evaluations in diverse settings

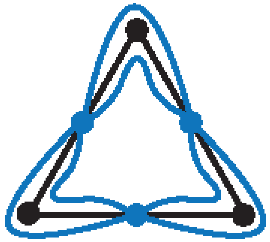
- $T(G;\, 0,-2)$ counts ice configurations if G is 4-regular
- $T(G;\, 3,3)$ counts claw coverings if G is plane, T-tetrominoes when G is a grid graph.
- $T(G;-1,-1) = (-1)^{|E(G)|}(-2)^{a(\vec{G}_m)-1}$, here a = anticircuits
- $T(G;\, 1+n,\, 1+n)$ counts monochromatic vertices in cycle n-colorings of $\vec{G}_m$



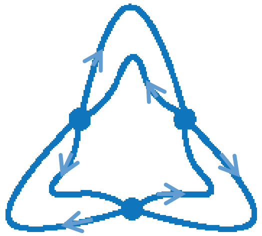




G



constructing G_m



$\vec{G}_m$

Korn & Pak, Las Vergnas, Lieb, Pauling, E-M

Proof techniques

- The universality theorem
- Induction and deletion-contraction
- Manipulation and interpretation of the rank-nullity formulation
- Connections between the Tutte polynomial and other polynomials
- Correspondences between objects—e.g. between ice models and flows for 4 regular graphs.

Even 'easy' is hard....

Conjecture (Welsh and Merino) :

$$T(G;1,1) \leq \max\{T(G;2,0), T(G;0,2)\}$$

|spanning trees| $\leq$ max{ |acyclic orientations|, |totally cyclic orientations|}.

- Thomassen proved that (roughly) if G has few edges, then $T(G;2,0) \geq T(G;1,1)$

(acyclic) and if lots of edges then $T(G;0,2) \geq T(G;1,1)$, (totally cyclic)

- Known for various classes of graphs, e.g. series parallel, and cubic (see Chavez-Lomelí, Merino, Noble, Ramírez-Ibañez, Royle, Thomassen , etc.), but not in general.

In general, when is it true that

$$T(G;x,y)T(G;y,x) \geq T(G;z,z)^2 \quad ?$$

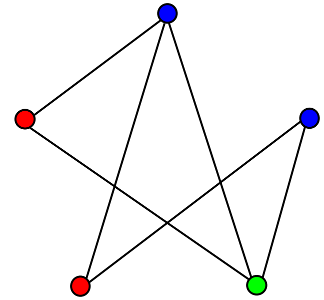
- Jackson proved this for $y = 0$ if $x > z^2 + 2z - 1$ (hence the original conjecture is true if 2 is replaced by 3).

Many Specializations-- Chromatic and Flow polynomials

- The Chromatic Polynomial-counts proper colorings

$$C(G;x) = C(G-e;x) - C(G/e;x), \;\; C(E_n;x) = x^n$$

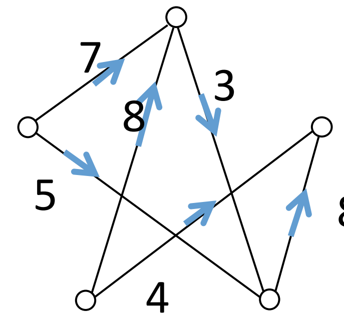
$$C(G;x) = (-1)^{r(G)} x^{k(G)} T(G;1-x,0)$$



- The Flow Polynomial-counts nowhere zero *H*-flows

$$F(G;x) = F(G/e;x) - F(G-e;x), \;\; F(E_n;x) = x^n$$

$$F(G;x) = (-1)^{|E(G)|-r(G)} T(G;0,1-x)$$



- Duality: if *G* is a connected plane graph, then

$$C(G;x) = xF(G^*;x)$$

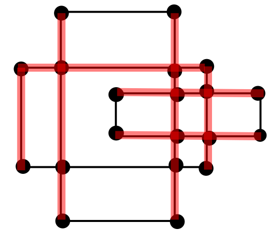
- Convolution:

$$T(G;x,y) = \sum_{A \subseteq E(G)} T(G/A;x,0)T(G|A;0,y)$$

Kook, Reiner, Stanton

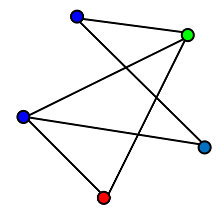
Pervasive applications

- Reliability - p = probability an edge functions



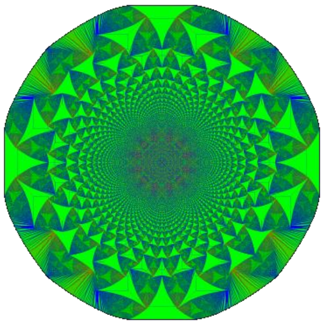
$$R(G;p) = (1-p)R(G-e;p) + p\,R(G/e;p)$$

- Bad coloring – x colors, j monochromatic edges



$$B(G;x,t) = \sum_j b_j(G;x)t^j = t^{r(G)}x^{k(G)}T(G;\frac{x+t}{t},1+t)$$

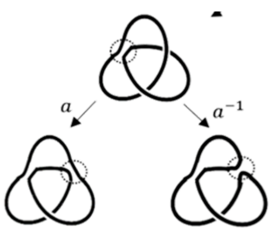
- Sand pile model- c_i stable configurations of level *i.*



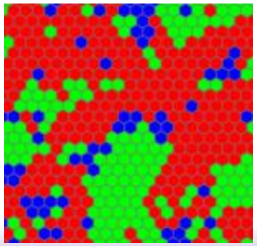
$$P(G;y) = \sum_i c_i t^i = T(G;1,y)$$

http://www.natureincode.com/code/various/sandpile.html

- Kauffman Bracket – a knot invariant



- Weight enumerator of a linear code
- Characteristic Poly of hyperplane arrangements
- The Potts Model– a statistical mechanics model



Generalizations in all directions

- The preceding concerned the 'classical' Tutte polynomial for graphs.
- But then the power of its fundamental properties expanded outward, as the flexibility and broader applicability of these ideas became apparent
- Both the parameter space and the domain have seen fundamental growth

(Movie Trailers/Highlights Tour up next…..)

Expansion of variables—edge weights

Originally:

$$Z(G;u,v)=\sum_{A\subseteq E(G)} u^{k(A)}v^{|A|}$$

Now add edge weights:

$$Z(G;q,\mathbf{v})=\sum_{A\subseteq E(G)} q^{k(A)}\prod_{e\in A} v_e$$

$$Z(G;u,\mathbf{v})=Z(G-e;u,\mathbf{v})+v_e Z(G/e;u,\mathbf{v})$$

Replaces $v^{|A|}$ by a product of the weights on the edges in *A*.

Traldi '89

From physics:

$$Z(G;q,\mathbf{v},\mathbf{w})=\sum_{A\subseteq E(G)} q^{k(A)}\prod_{e\in A} v_e\prod_{e\in A^c} w_e$$

$$Z(G;u,\mathbf{v},\mathbf{w})=w_e Z(G-e;u,\mathbf{v},\mathbf{w})+v_e Z(G/e;u,\mathbf{v},\mathbf{w})$$

Doubly weighted, but requires $w_e + v_e = 1$, so equivalent to above .

Fortuin & Kastelyn '72

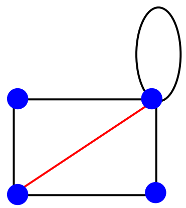
More edge possibilities

Note that there are four things that can happen to an edge as the Tutte polynomial is computed:

deleted, contracted, evaluated as a bridge, evaluated as a loop.

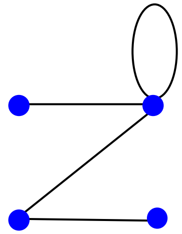
Let *e* be an edge of *G* that is neither a bridge nor a loop. Then,

$$T(G;x,y) = T(G-e;x,y) + T(G/e;x,y)$$



And if *G* consists of *i* bridges and *j* loops, then

$$T(G;x,y) = x^i y^j$$



Fully parameterized Tutte polynomial

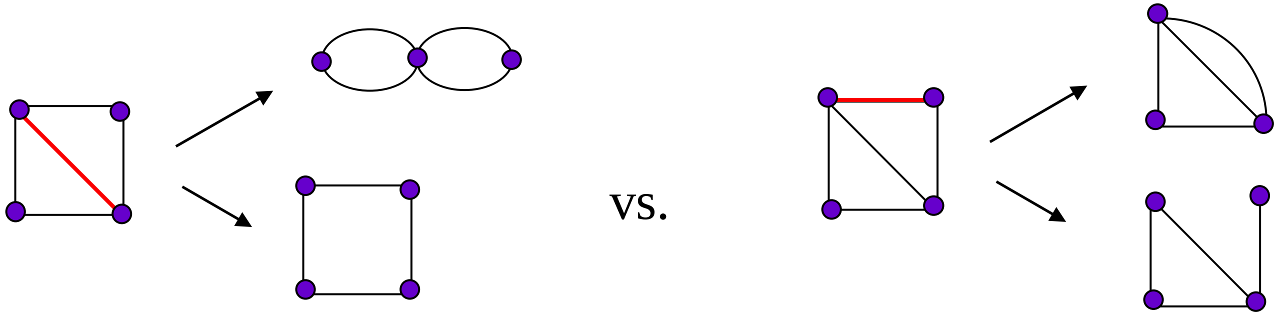
Zaslavsky ’92, Bollobás & Riordan ’99, (E-M &Traldi ’06)

Let $W(E_n, c) = \alpha_n$, where E_n is the edgeless graph on *n* vertices.

$$W(G,c) = \begin{cases} X_e W(G/e,c) & \text{if } e \text{ is a bridge} \\ Y_e W(G-e,c) & \text{if } e \text{ is a loop} \\ x_e W(G/e,c) + y_e W(G-e,c) & \text{else} \end{cases}$$

Each edge has four variables associated with it: one for contracting, one for deleting, a loop value, and a bridge value.

Need to be VERY careful about order…

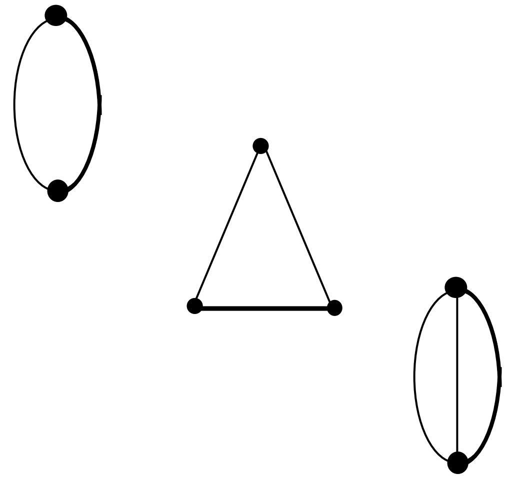


Need to have:

$$X_\lambda y_\mu - y_\lambda X_\mu - x_\lambda Y_\mu + Y_\lambda x_\mu = 0$$

$$Y_\nu \left(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu \right) = 0$$

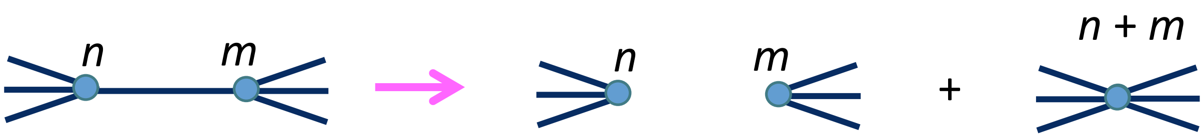
$$X_\nu \left(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu \right) = 0$$



Necessary and sufficient to assure the function is well-defined, i.e. independent of the order of deletion and contraction.

Expansion of variables—vertex weights

- Noble and Welsh, 1999, The *U*- and *W*- polynomials
- Take vertex weights in **Z⁺**, indeterminates $x_1, x_2 \ldots$
- Compute as follows:
  - If *e* is not a loop, then $W(G) = W(G-e) + W(G/e)$, where deletion is as usual, and contraction adds weights:



  - If *e* is a loop, then

$$W(G) = yW(G-e)$$

  - If E_m consists of *m* isolated vertices, with weights $n_1, n_2 \ldots$ then $$W(E_m) = \prod_{i=1}^{m} x_{n_i}$$

(knot theory Vassiliev invariants)

U initialized weights to 1, giving a graph invariant.

The V-polynomial—putting it together

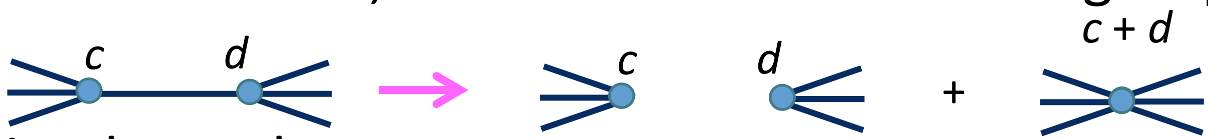
- Edge weights/indeterminates indexed by the edges---(**γ**).
- Vertex weights in a semigroup S--- (ω)
- Indeterminates indexed by S--- (***x***)

$$V(G) = V(G, \omega; \mathbf{x}, \gamma) \in \left[\{\gamma_e\}_{e \in E(G)}, \{x_k\}_{k \in S} \right]$$

E-M, Moffatt, 2011

Recursive and state model definitions

- Recursive:
  - If e is not a loop, then $V(G) = V(G-e) + \gamma_e V(G/e)$, where deletion is as usual, and contraction adds semigroup weights:



  - If e is a loop, then

$$V(G) = (\gamma_e + 1)V(G-e)$$

  - If E_m consists of m isolated vertices, with weights $c_1, c_2 \dots$ then $V(E_m) = \prod_{i=1}^{m} x_{c_i}$

State Model: $V(G) = \sum_{A \subseteq E(G)} \prod_{i=1}^{k(A)} x_{c_i} \prod_{e \in A} \gamma_e$ where c_i sums weights on the i^{th} component.

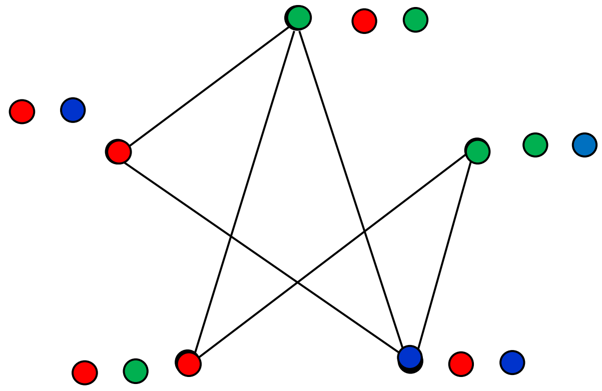
Also a Spanning Tree Expansion, —McDonald & Moffatt 2012

The List Chromatic Polynomial

- Let G be a graph with lists l_i from some set L at the vertices.
- Let S be the semigroup 2^L under intersection.
- Assign edge weights of -1 to each edge

$$C(G,\{l_i\}) := \mathbf{V}(G,\{l_i\};\mathbf{x},\mathbf{-1})$$

This gives the number of ways to properly color G from the given lists of colors at the vertices when evaluated at $x_s = |s|$.



Can be properly colored from this set of lists

E-M, Moffatt, 2015

Zeros

- The zeros of the Tutte polynomial are intimately connected with major driving questions in graph theory, e.g.:
  - The Four Color Theorem -- Planar graphs are 4-colorable, i.e. $C(G;4) > 0$ if G is planar.
    - (Also– zero temperature phase transitions in statistical mechanics. Here they hope to clear regions of the plane of zeros.)
  - Tutte's Five Flow Conjecture-- every bridgeless graph has a nowhere-zero 5-flow, i.e., $F(G;5) > 0$.
    - Known for 6 (Seymour, 1981), and hence (Tutte), all higher. But still open.

Multivariable breakthroughs

- The multivariable versions of the Tutte polynomial allow manipulations such as merging series or parallel edges by combining weights (Sokal, 2005)
- This was used to very good effect to clear regions of the Tutte plane of zeros (c.f. Jackson & Sokal, 2009)

$$Z(G;u,v) = \sum_{A \subseteq E(G)} u^{k(A)} v^{|A|}$$

$$Z(G;q,\mathbf{v}) = \sum_{A \subseteq E(G)} q^{k(A)} \prod_{e \in A} v_e$$

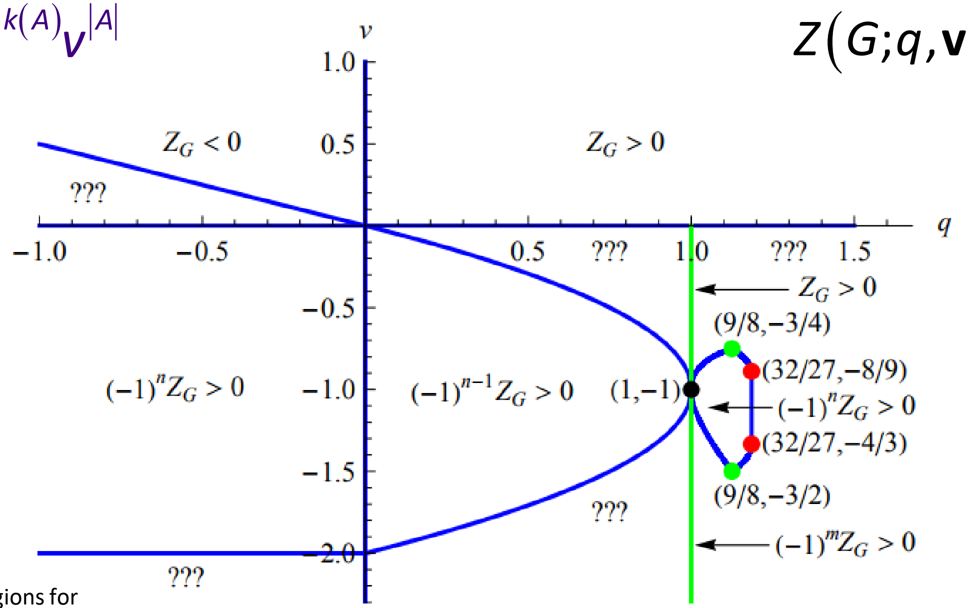
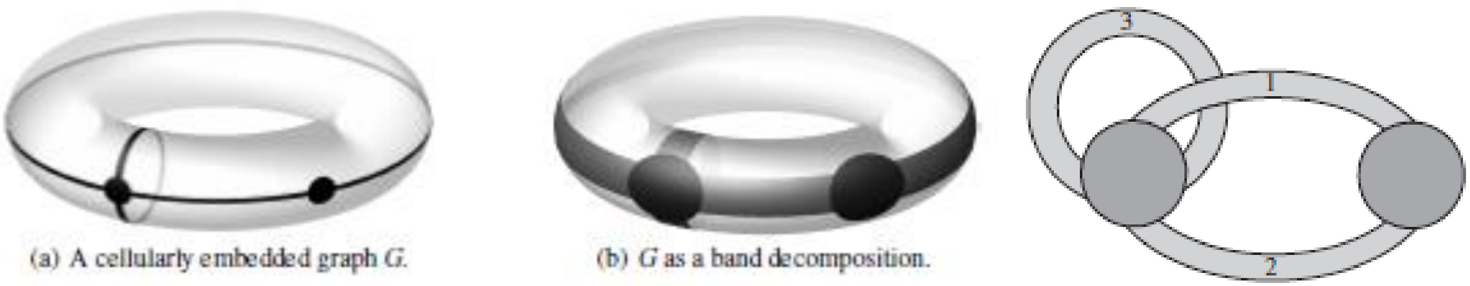


Figure from Jackson & Sokal "Zero-free regions for multivariate Tutte polynomials (alias Potts-model partition functions) of graphs and matroids"

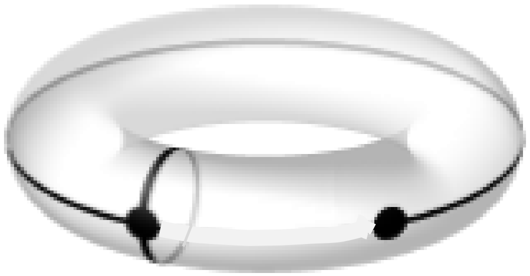
Expanding the domain

- Cellularly embedded graphs-some representations



(a) A cellularly embedded graph G. (b) G as a band decomposition. (c) G as a ribbon graph.

- A non-cellularly embedded graph



Ribbon graph parameters

- Rank: $r(G) = v(G) - k(G)$
- Nullity: $n(G) = e(G) - r(G)$
- Boundary components (same as faces): $bc(G)$
- Orientability index: $t(G)$
- Genus: sum (Euler) genus over components—
  - Recall $v\text{-}e\text{+}f = 2k - \gamma$

The Ribbon Graph Polynomial of Bollobás and Riordan (2001, 2002)

Let G be a ribbon graph and $w=w^2$.

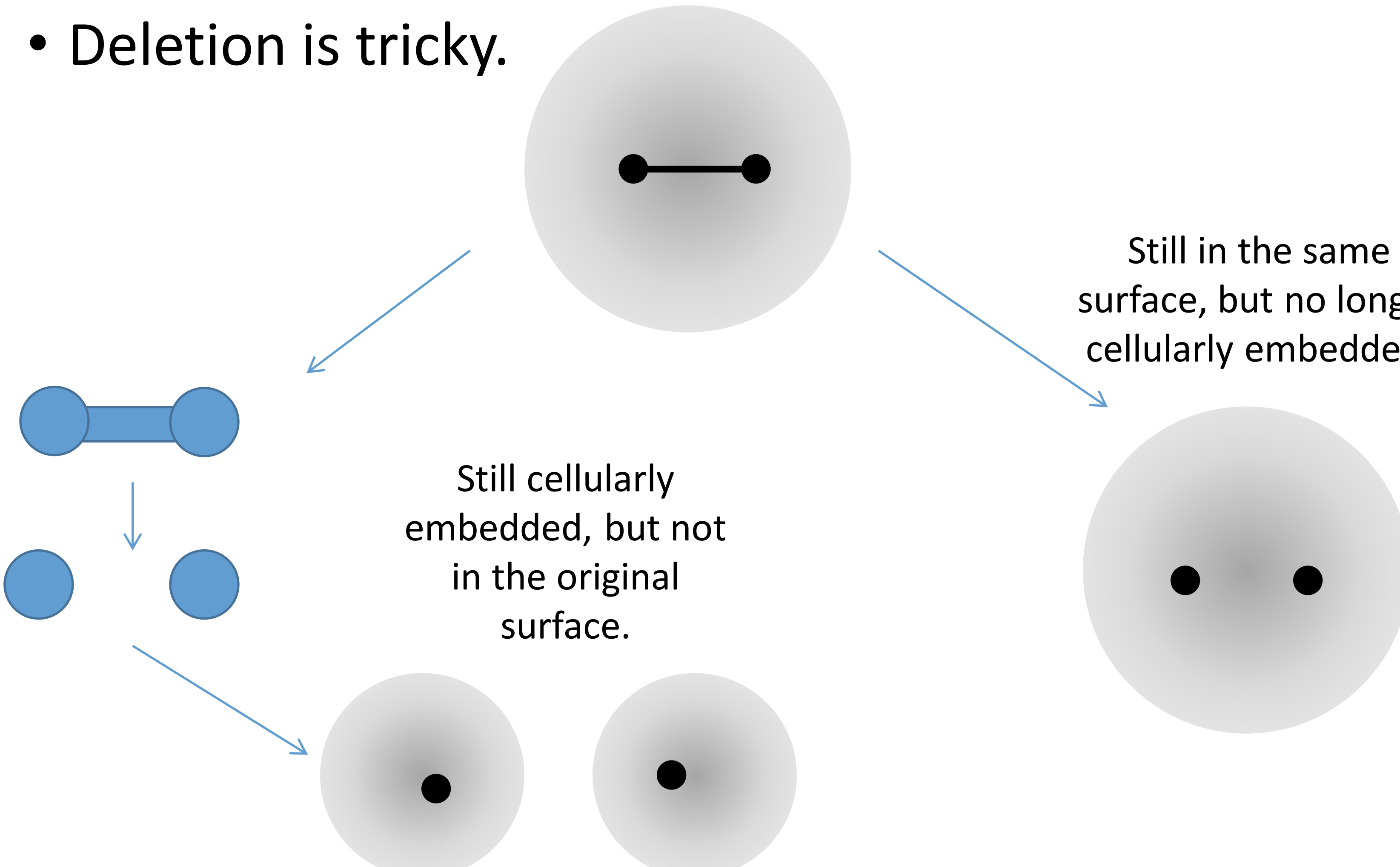
$$R(G;x,y,z,w) =$$

$$\sum_{A\subseteq E(G)} \overbrace{(x-1)^{r(G)-r(A)}\, y^{n(A)}}^{\text{Classical Tutte}}\, z^{\overbrace{k(A)-bc(A)+n(A)}^{\text{Basically genus}}}\, w^{\overbrace{t(A)}^{\text{Records orientability}}}$$

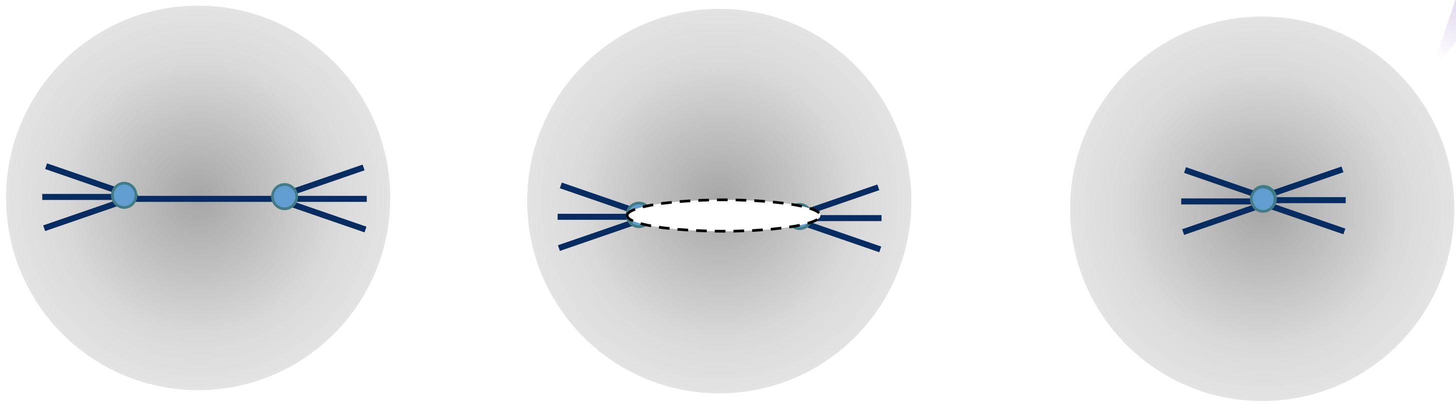
This is where things really took off in this direction.

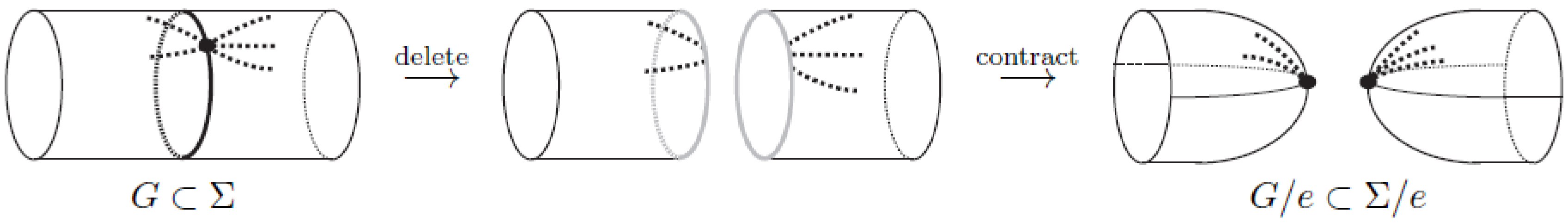
Deletion

- Deletion is tricky.



Surface contraction





Delete the interior of a regular neighborhood of the edge which creates a new boundary component(s), then contract this boundary component(s) to a point (or two), carrying the drawing of *G* along with the surface, and then placing a new vertex on the resulting point(s).

Graphs → Matroids

- Matroids $$M=(E,B),\ B\subseteq 2^{E},\ B\neq\varnothing$$

$$\text{if } X,Y\in B,\ x\in X-Y,\ \text{then } \exists\, y\in Y \text{ with } X-x\cup y\in B$$

E.g, *E* = edges, *B* = spanning trees; *E* = vectors, *B* = bases

  - These show up in Tutte's thesis, but the extension of the Tutte polynomial to matroids really takes off with Crapo's 1969 *The Tutte polynomial*, which extends many of the fundamental properties of the Tutte polynomial to matroids, and establishes them as perhaps the 'natural' domain of the Tutte polynomial.
  - Then Brylawski 1972, *A decomposition for combinatorial geometries* & *The Tutte-Grothendieck ring* establishes the matroid decompositions necessary to extend deletion-contraction to matroids. C.f. Brylawski and Oxley *The Tutte polynomial and its applications.*
  - TONS of combinatorial structures can be expressed as matroids, and this means that the Tutte polynomial encodes all of them
  - Also…delta matroids, multimatroids, etc. Versions of the Tutte polynomial for all of these too.
- Duality again—but now unrestricted:

$$T(M;x,y)=T(M^*;y,x)$$

Delta Matroids

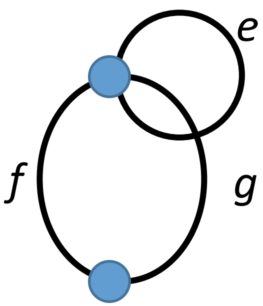
- Matroids $M=(E,B),\ B\subseteq 2^{E},\ B\neq\varnothing$

  if $X,Y\in B,\ x\in X-Y$, then $\exists\ y\in Y$ with $X-x\cup y\in B$

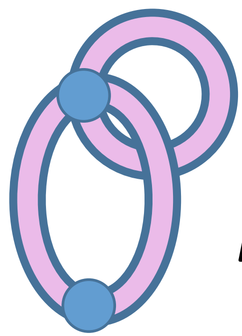
- Delta Matroids $D=(E,F),\ F\in 2^{E},\ F\neq\varnothing$

  if $X,Y\in F,\ e\in X\triangle Y$, then $\exists\ f\in X\triangle Y$ with $X\triangle\{e,f\}\in F$

Indept'ly, 80's, Bouchet, Chandrasekaran&Kabadi, Dresss&Havel



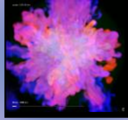
$M=(\{efg\},\{\{e\},\{f\}\})$



$D=(\{efg\},\{\{efg\},\{e\},\{f\}\})$

As matroids are to abstract graphs, so delta matroids are to embedded graphs.

Recent work lifts topological Tutte polynomials (and many others) to this setting. Several recent papers by Chun, (Chun), Moffatt, Noble, and Rueckriemen



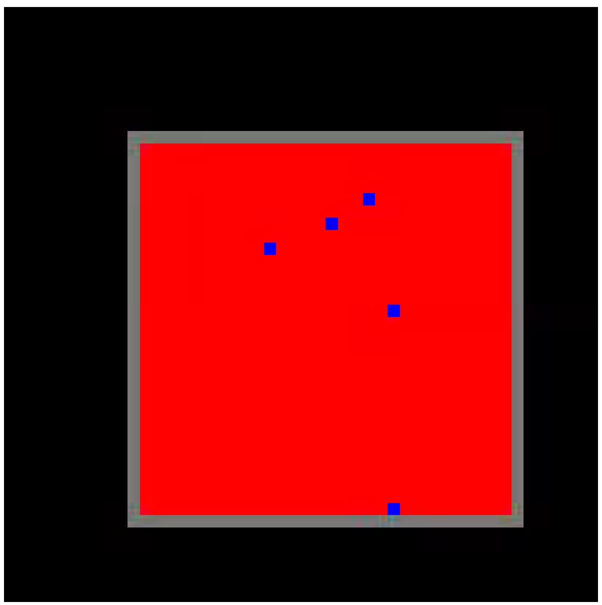
And now a little physics…

- The Ising Model (1925) and Potts Model (1952) are important models of nearest neighbor complex systems where local interactions determine global behaviors.
- They are Bolzmann distributions, with thermodynamic properties computed from the normalization factor (partition function).

This is an important example that touches on many of the ideas discussed here.

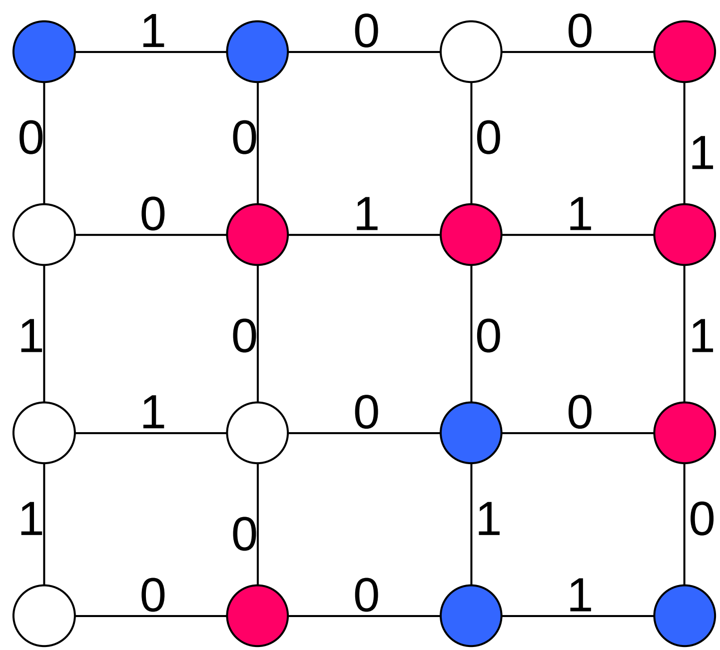
The Hamiltonian

The **Hamiltonian** measures the overall energy of the a state of a system.



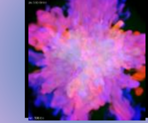
$$H(S) = \sum_{edges} -J\delta_{a,b}$$

The Hamiltonian of a state of a 4X4 lattice with 3 choices of spins (colors) for each element.



$$H = -10J$$

The probability of a state



The probability of a particular state *S* occurring depends on the *temperature, T*

(or other measure of activity level in the application)

--Boltzmann probability distribution--

$$P(S) = \frac{\exp(-\beta H(S))}{\sum_{\text{all states } \mathbf{S}} \exp(-\beta H(\mathbf{S}))}$$

$\beta = \frac{1}{kT}$ where $k = 1.38 \times 10^{-23}$ joules/Kelvin and T is the temperature of the system.

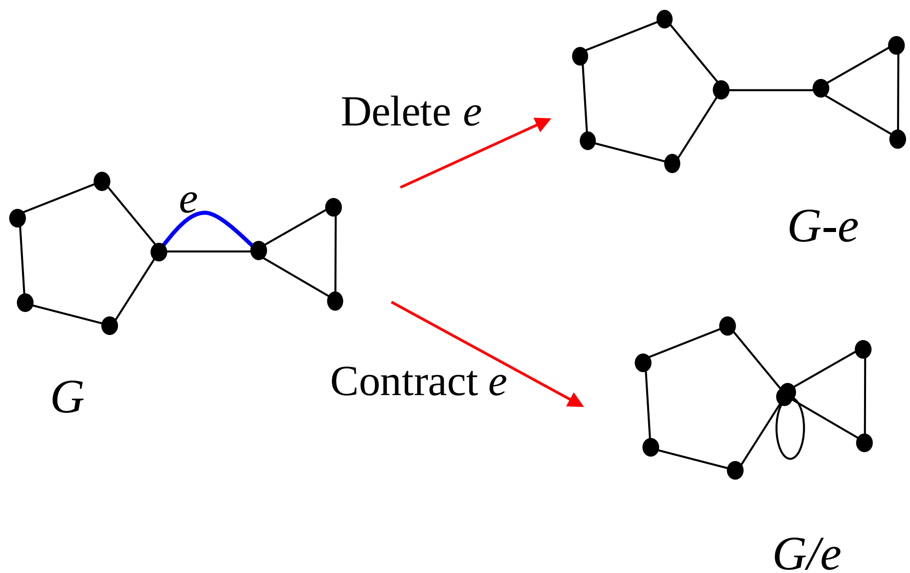
The numerator is easy. The denominator, $Z = \sum_{\text{all states } \mathbf{S}} \exp(-\beta H(\mathbf{S}))$

called the *Potts Model Partition Function,*

is the interesting (hard) piece.

Fundamental Observation

- If two vertices have different spins, they don't interact, so there might as well not be an edge between them (so delete it).
- If two adjacent vertices have the same spin, they interact with their neighbors in exactly the same way, so they might as well be the same vertex (so contract the edge)*.

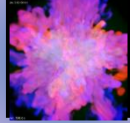


*with a weight for the interaction energy

$$Z(G;q,v)=q^{k(G)}\left(v\right)^{|V(G)|-k(G)}T\left(G;\frac{q+v}{v},1+v\right)$$

This is the connection to the Tutte polynomial--Fortuin and Kasteleyn, 1972. This means that results for the Tutte polynomial carry over to the Potts model and vice versa.

Phase Transitions and zeros of the Chromatic polynomial

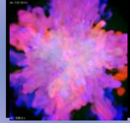


In the infinite volume limit, the ground state entropy (temperature -> 0) per vertex of the Potts antiferromagnetic model becomes:

$$S = \kappa \lim_{n \to \infty} \frac{1}{|V(G_n)|} \ln\left(C(G_n; q)\right)$$

- Thus, phase transitions correspond to the accumulation points of roots of the chromatic polynomial in the infinite volume limit.

- This is another reason for focusing on zeros



Limitations of the classical connection

Many applications

- Liquid-gas transitions
- Foam behaviors
- Magnetism
- Biological membranes
- Ghetto formation
- Separation in binary alloys
- Cell migration
- Spin glasses
- Neural networks
- Flocking birds

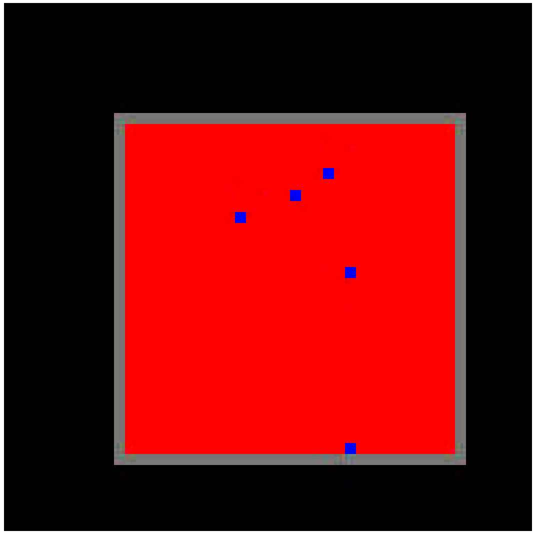
However…
Most applications include additional terms in the Hamiltonian, and the classical theory of the Tutte-Potts connection does not encompass this.

A Simple External Field

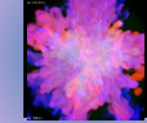
The first spin is favored, and *M* is the strength of the favoritism

$$H(w) = \sum_{edges} -J\delta_{a,b} \quad \longrightarrow \quad H(w) = \sum_{edges} -J\delta_{a,b} + \sum_{vertices} -M\delta_{1,a}$$

- In the first sum, *a* and *b* are the spins on endpoints of the edge
- In the second sum, *a* is the spin on the vertex.



http://pages.physics.cornell.edu/sss/ising/ising.html



Need more sophisticated models for these applications

- Allow edge-dependent interaction energies--- (**γ**).

$$H(S) = -\sum_{edges} J_{ij}\delta(s_i s_j)$$

- Also allow *q*-dimensional magnetic field vectors via a vector ($M_{i,1}$ … $M_{i,q}$) associated to each vertex v_i --**(M)**

$$H(S) = -\sum_{edges} J_{ij}\delta(s_i s_j) - \sum_{vertices}\sum_{a=1}^{q} M_{i,a}\delta(a, s_i$$

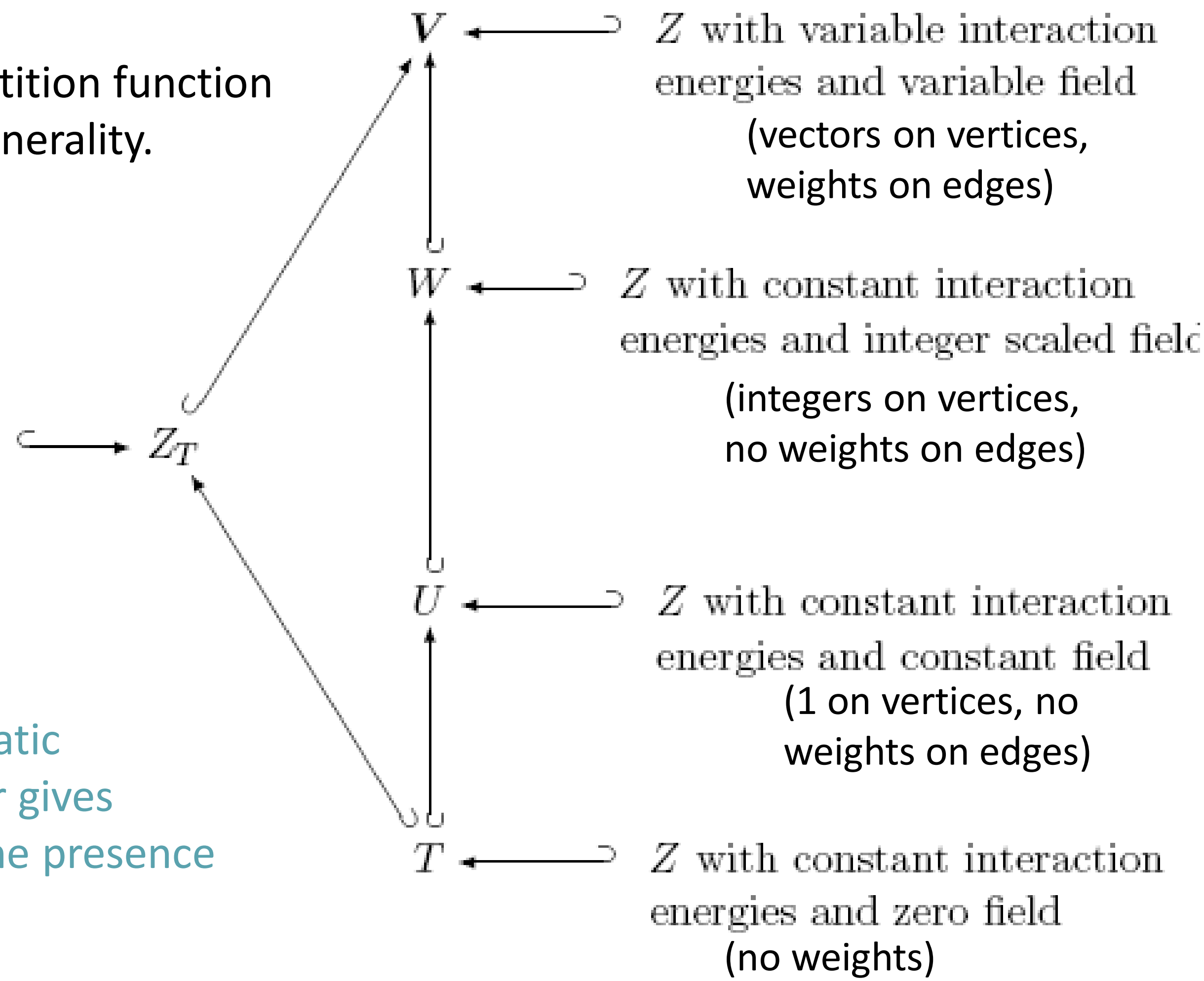
Variable (edge-dependent) energies and a variable (vertex-dependent) external field.

Appropriate choices of **M** and **γ** yield familiar models:
Preferred Spin, Spin Glass, Random Field Ising Model, etc.

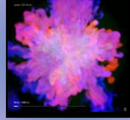
E-M&Moffatt 2011

The *V*-polynomial captures these external field models

Z is the Potts model partition function in varying degrees of generality.



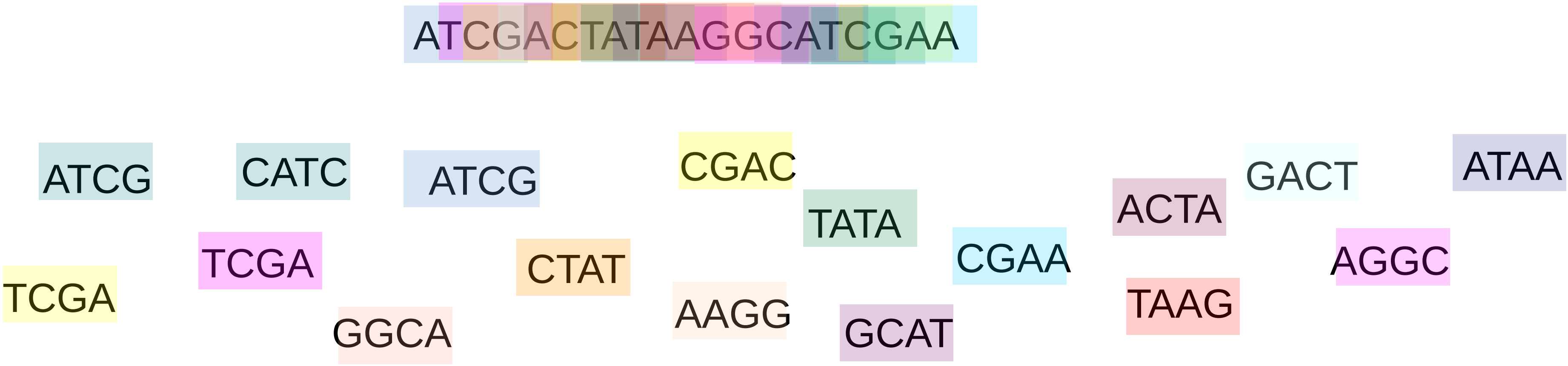
Side note: The list chromatic polynomial we saw earlier gives ground state entropy in the presence of external fields.



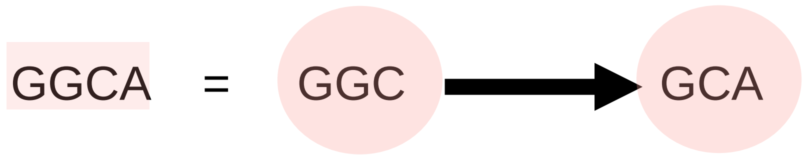
A wide range of applications

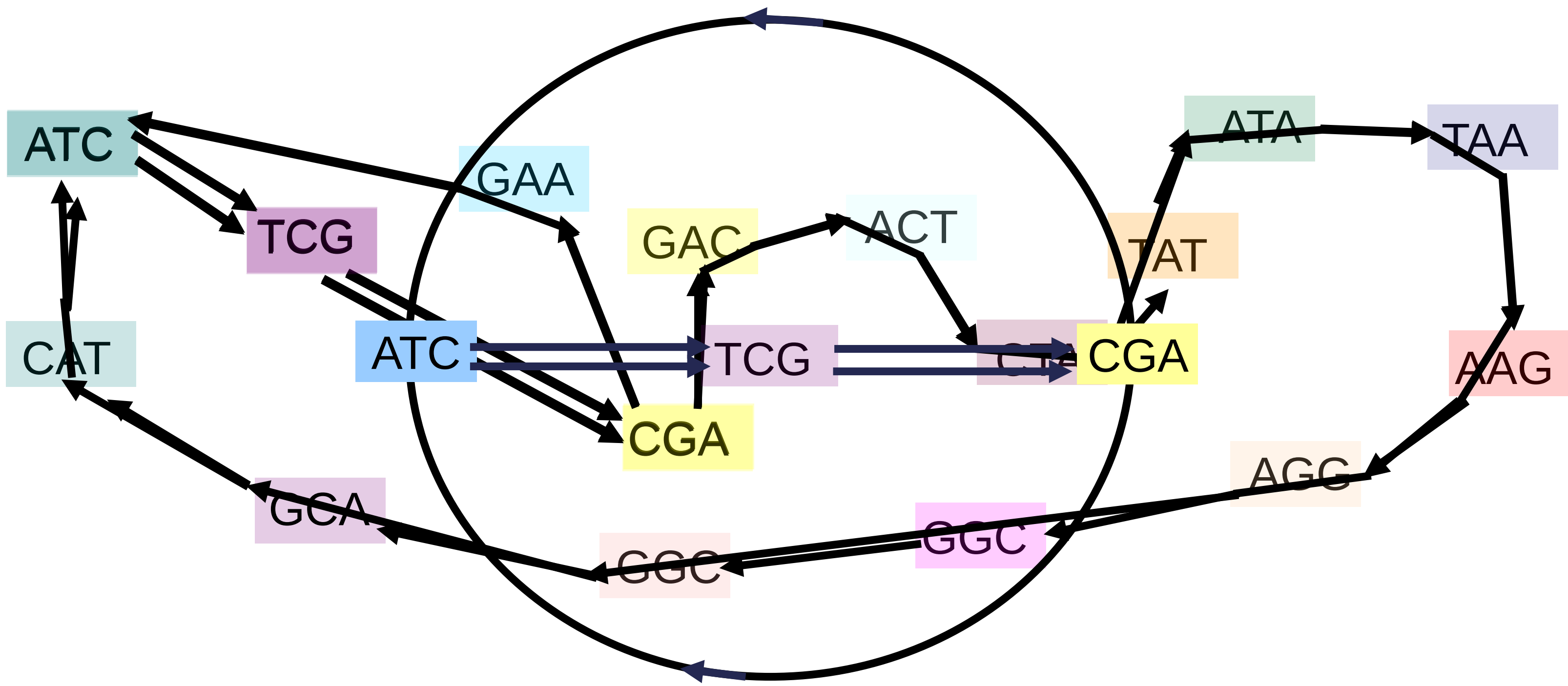
DNA sequencing (String reconstruction in general)

- Say we want to read this piece of single-stranded DNA
- We can't read it all in one piece, so we break it up into *l*-length "fragments," and then piece it back together to reconstruct the DNA sequence.
- Here, we'll use fragments of 4 nucleotides.
- For the sequence above, we would get fragments of:



The DeBruijn graph, a 2-in 2-out digraph



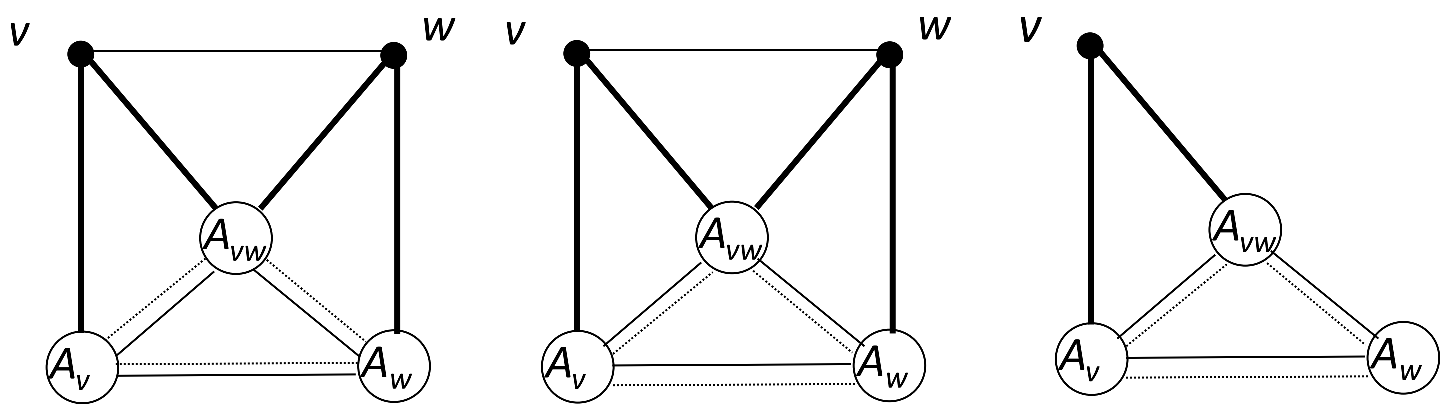


Interlacements through vertices of degree 4 confound reassembly. One one Euler circuit corresponds to the correct sequencing. But how to model the interlacing?

The Interlace Polynomial

- Arratia, Bollobás, Coppersmith, Sorkin, papers 2000 -2004.

$$\mathbf{q}(G,x)=\begin{cases} x^n & \text{if } G=E_n\text{, the edgeless graph on } n \text{ vertices} \\ \mathbf{q}(G-v,x)+\mathbf{q}\left(G^{vw}-w,x\right) & \text{if } vw\in E(G) \end{cases}$$



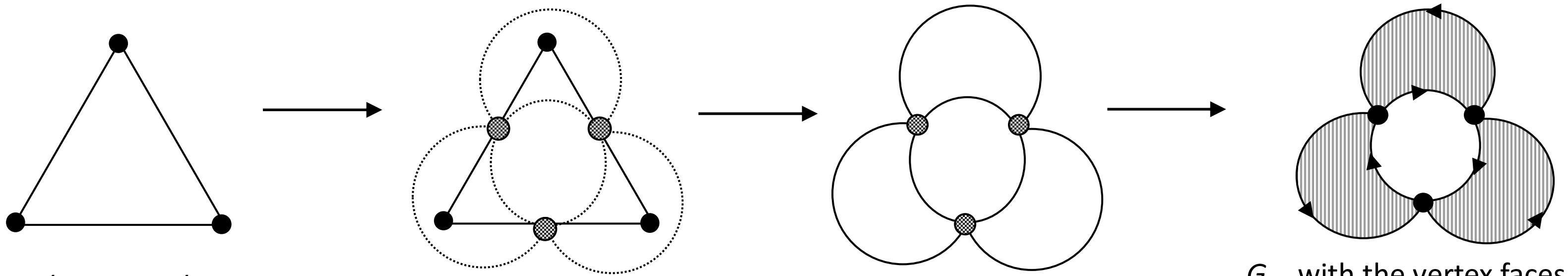
G $\quad$ G^{vw} $\quad$ G^{vw}-w

(note interchange of edges and non-edges among A_v, A_w and A_{vw})

(cf. Bouchet *local complementation* and *Tutte-Martin poly of isotropic systems*)

From Interlace to Tutte

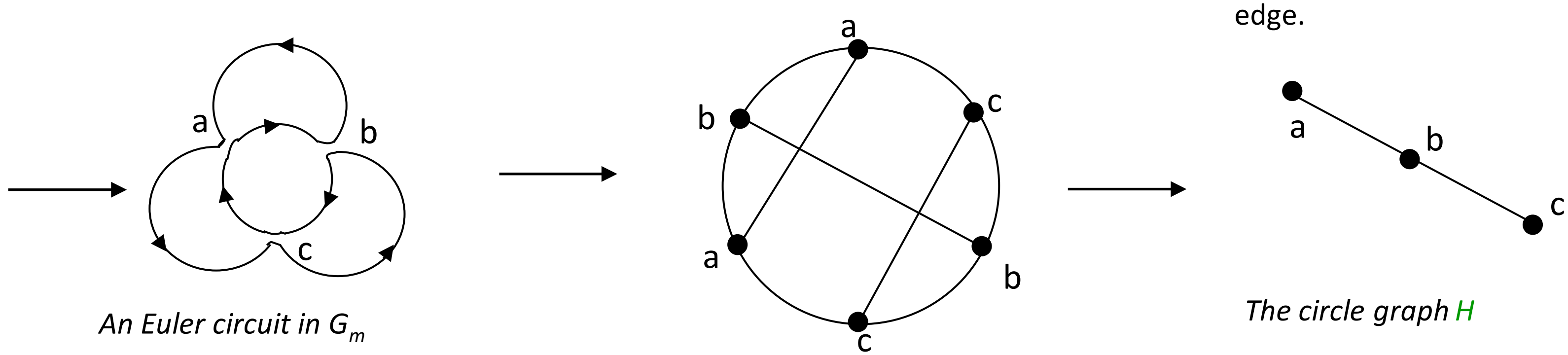
- **Theorem:** If G is a planar graph, and H is the circle graph of some Eulerian circuit of $\vec{G}_m$, then $t(G; x, x) = q_N(H; x)$.



A planar graph G

The medial graph G_m

G_m with the vertex faces colored black, oriented so that black faces are to the left of each edge.



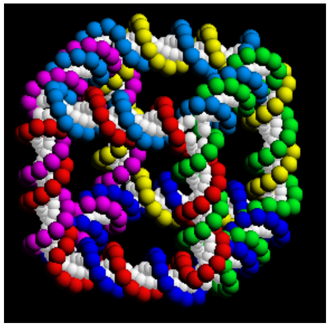
An Euler circuit in G_m

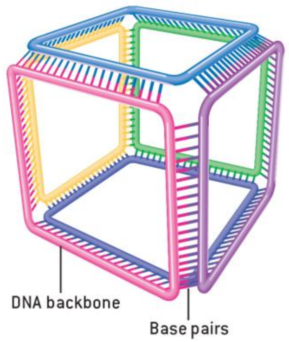
The chord diagram

The circle graph H

NOTE: The interlace polynomial is also now being lifted to delta matroids– see Brijder, Hoogeboom, Traldi, etc.

DNA self-assembly and Ciliates





http://seemanlab4.chem.nyu.edu/

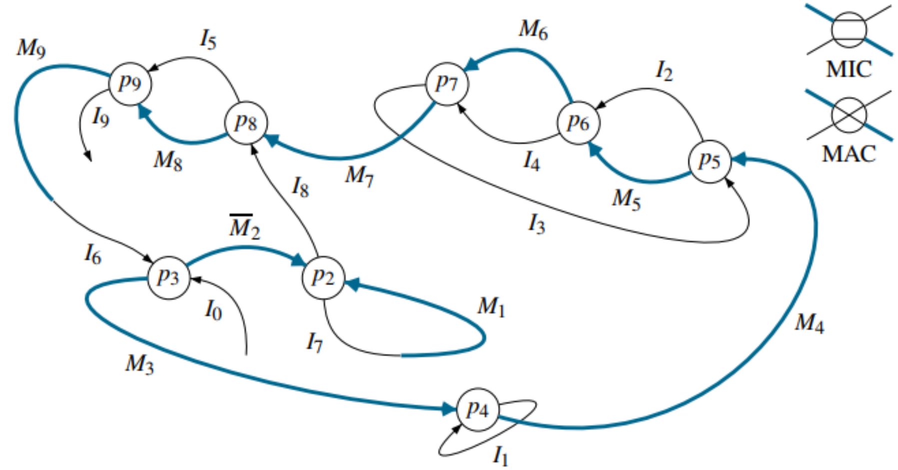
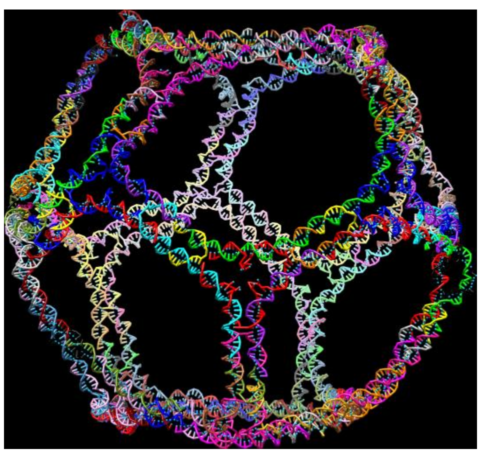


Fig. 1. Actin I gene of Sterkiella nova. Schematic diagram, based on [18]



https://dna.physics.ox.ac.uk/index.php/Main_Page

Ribbon graphs are an ideal model, and graph polynomials a target for biomolecular computing.

The behavior of DNA in ciliates is captured by the circuits and anticircuits of 4-regular graphs whose properties are encoded by the Tutte polynomial.
Figure from “Graph polynomials motivated by gene rearrangements in ciliates”, Brijder&Hoogeboom, 2014.
CF others by these authors.

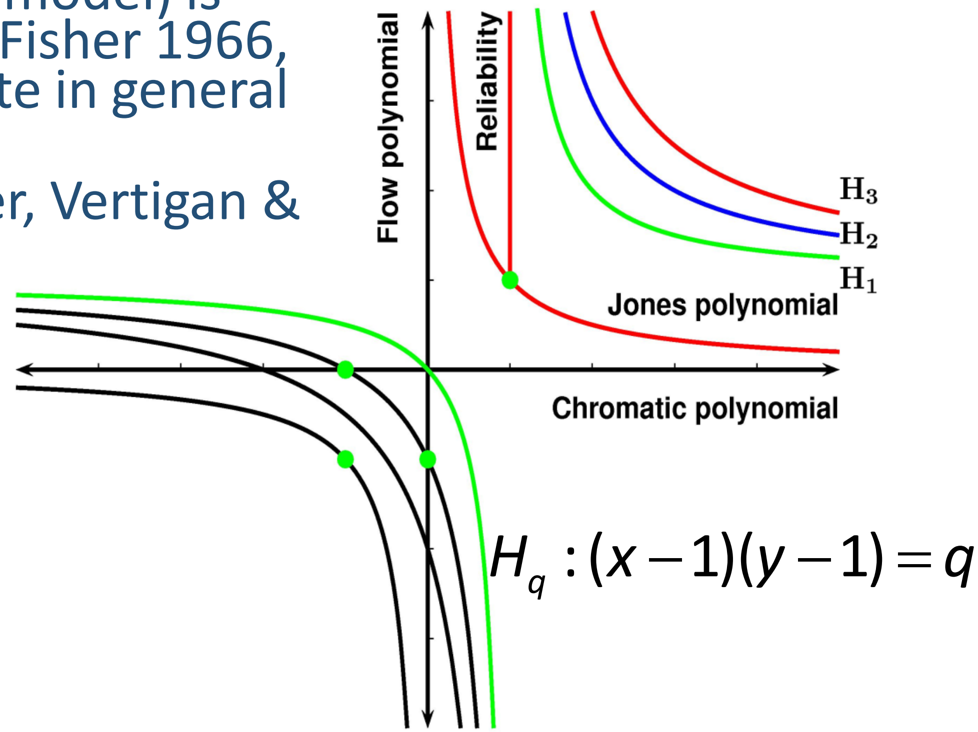
Computational Complexity

- Tutte-Potts crossovers of computational complexity results.
  - The Ising Model (q=2 Potts model) is tractable for plane graphs (Fisher 1966, Kastelyn 1967), #P-Complete in general (Jerrum, 1987)
  - Foundational paper—Jaeger, Vertigan & Welsh, 1990:

The Tutte polynomial is #P-Complete for general graphs, except when

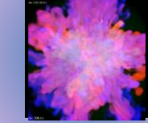
- q = 1 (trivial) ,
- when q = 2 as above,

or 9 special points, {(1, 1), (-1,-1), (0,-1), (-1, 0), (i,-i), (-i, i), (j, j^2), (j^2, j)}, where j = exp(2i/3)



And then another explosion of results, for the classical Tutte, and all these outgrowths….

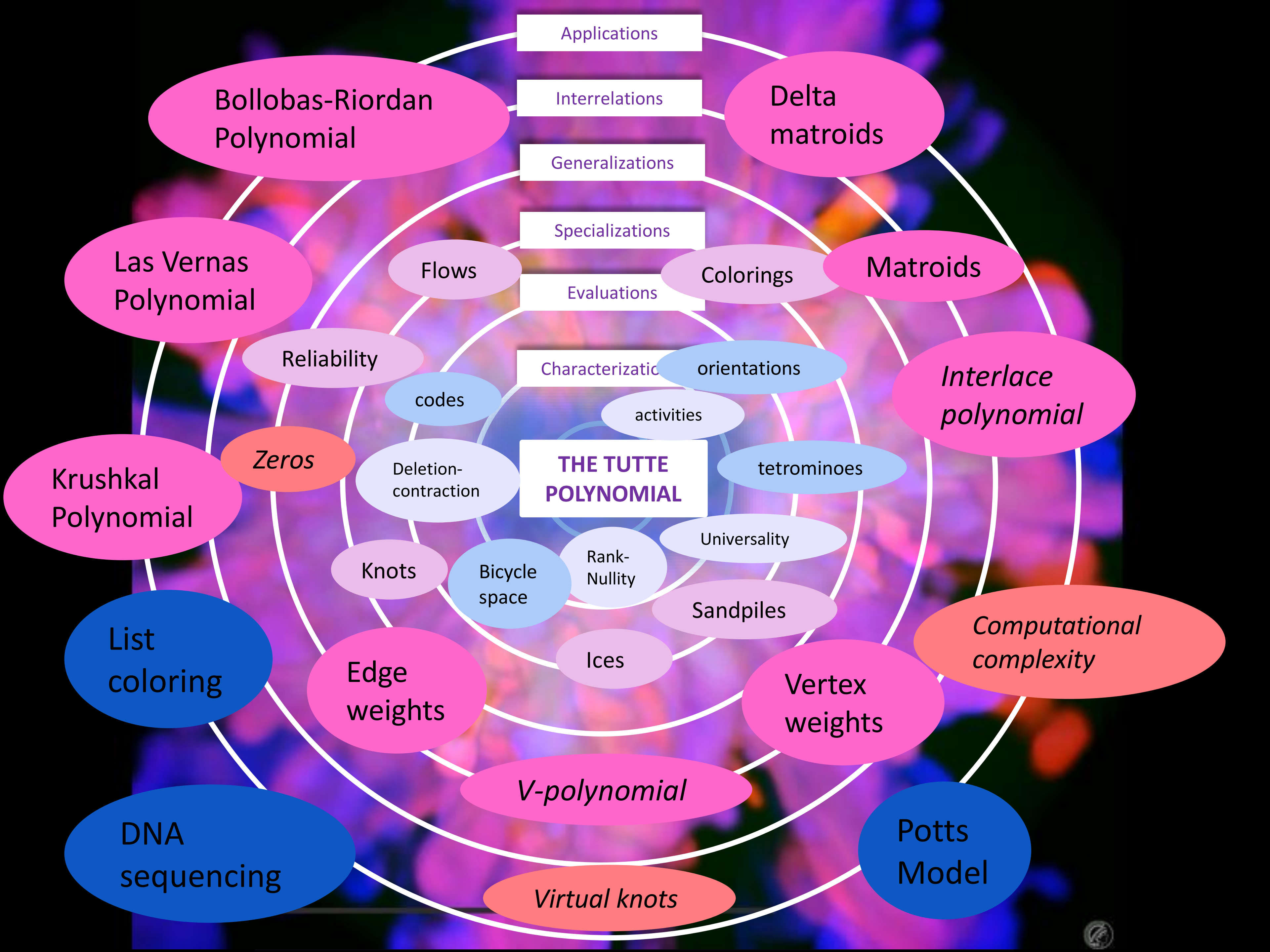
(Very nice figure of the Tutte plane from Steve Nobel's webpage)



Open

- Merino-Welsh: $T(G;1,1) \leq \max\{T(G;2,0), T(G;0,2)\}$
- Which polynomials are T(G) for some G?
- Tutte uniqueness– for what graphs is T(G) unique?
- Flow/coloring of course
- Prove or disprove that a graph with a palindromic Tutte polynomial is 4-colorable (open since 1932…)

  - *D* is diagonal with entries all the Hamiltonians
  - *A* is the adjacency matrix of the 2^n-cube


Applications
Interrelations
Generalizations
Specializations
Evaluations
Characterizatio
THE TUTTE POLYNOMIAL
Bollobas-Riordan Polynomial
Delta matroids
Las Vernas Polynomial
Flows
Colorings
Matroids
Reliability
orientations
Interlace polynomial
codes
activities
Zeros
Krushkal Polynomial
Deletion-contraction
tetrominoes
Universality
Knots
Bicycle space
Rank-Nullity
Sandpiles
List coloring
Ices
Computational complexity
Edge weights
Vertex weights
V-polynomial
DNA sequencing
Potts Model
Virtual knots

Duality –and coloring –again

- The following are equivalent

1 *the Four Colour Theorem is true;*
2 *for every connected, loopless plane graph* G, $(-1)^{v(G)}R(G^{\times};-2,-3,1/3)<0$;
3 *for every connected, loopless plane graph* G, $(-1)^{v(G)}R(G^{\times};-3,-4,1/4)<0$.
4 *for every connected, loopless plane graph* G, $R(G^{\times};3,2,-1/2)\neq 0$.

G^x is the *Petrie Dual* – i.e. give every edge a half-twist: