Equitable Colourings of cycle systems

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Outline

1. A scheduling problem
2. Graph decompositions
3. Colourings
4. Equitable colourings

A scheduling problem

- A group of v people, representing c countries, attend a summit.

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- They will take part in meetings of k people at a time.
- Can we devise a meeting schedule so that:
  - Each person attends a meeting with each other person the same number, λ, of times.
  - Each meeting has, as much as possible, equal representation from every country. (I.e. at each meeting, the number of delegates from different countries differ by at most 1.)

Example: $v = 6$, $k = 5$, $\lambda = 4$, $c = 3$

Suppose we have the following delegates:

Canada	France	Italy
1, 4	2, 5	3, 6

Here is a possible schedule:

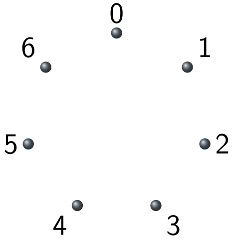
Meeting 1:	1	2	3	4	5
Meeting 2:	1	2	3	4	6
Meeting 3:	1	2	3	5	6
Meeting 4:	1	2	4	5	6
Meeting 5:	1	3	4	5	6
Meeting 6:	2	3	4	5	6

Graph Decompositions

Definition

- A collection $\{H_1, \ldots, H_t\}$ of subgraphs of Γ is a decomposition of Γ if the edge sets of $H_1, H_2, \ldots, H_t$ partition the edges of Γ.
- If $H_1 \cong \cdots \cong H_t \cong H$, then we speak of an H-decomposition of Γ.
- We call the subgraphs $H_1, \ldots, H_t$ blocks of the decomposition.

Example: A K_3-decomposition of K_7



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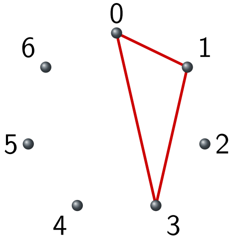

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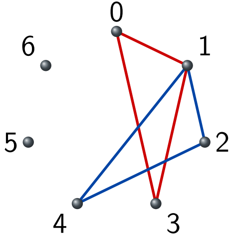

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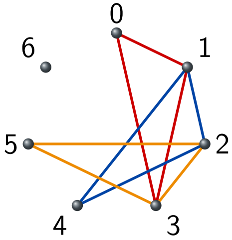

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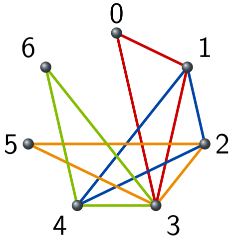

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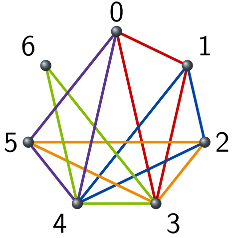

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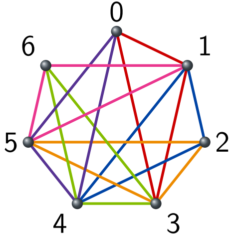

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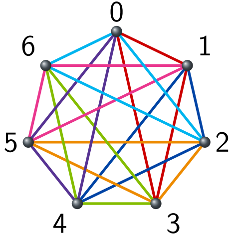

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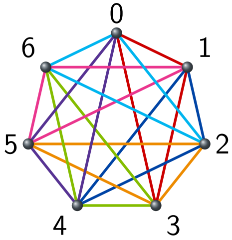

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Remark

This decomposition is cyclic, formed from the base block $\{0, 1, 3\}$ by adding elements of $\mathbb{Z}_7$.

Balanced incomplete block designs

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- Given k and λ, any $v \geq k$ satisfying the necessary conditions above will be called admissible for existence of a BIBD.
- For any k and λ, there exists a BIBD(v, k, λ) for any sufficiently large admissible v. (Wilson, 1975)
- A BIBD(v, k, λ) gives a meeting schedule with v people meeting in groups of k, with each pair of people attending λ meetings.

A BIBD(6, 5, 4)

1	2	3	4	5
1	2	3	4	6
1	2	3	5	6
1	2	4	5	6
1	3	4	5	6
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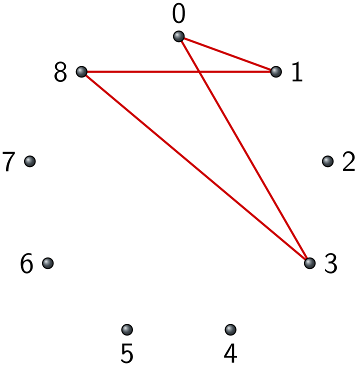
  - The meetings will take place at round tables.
  - Each person must sit next to each other person the same number of times.
- In this case, we would look for a cycle decomposition.

Example: A 4-cycle decomposition of K_9



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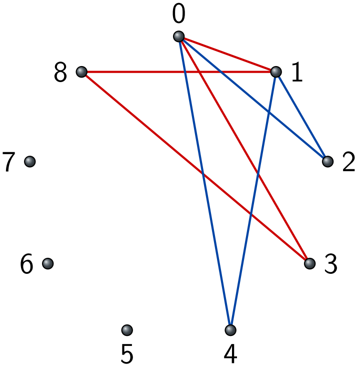

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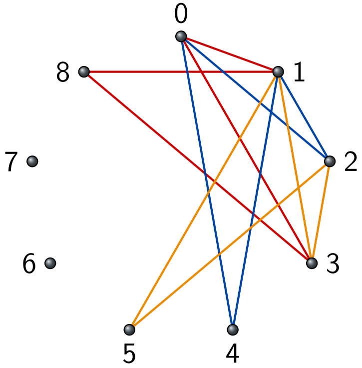

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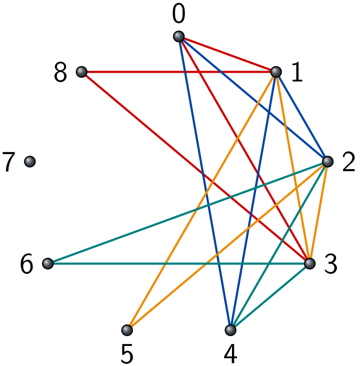

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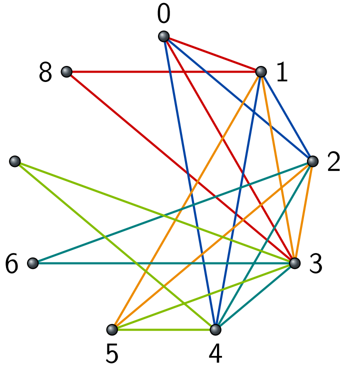

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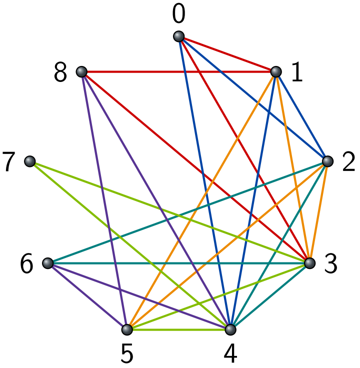

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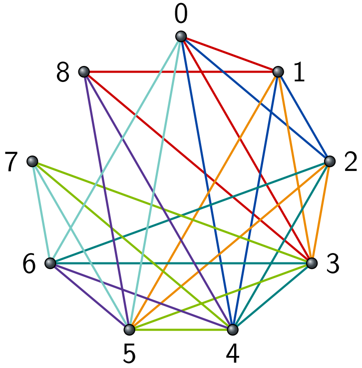

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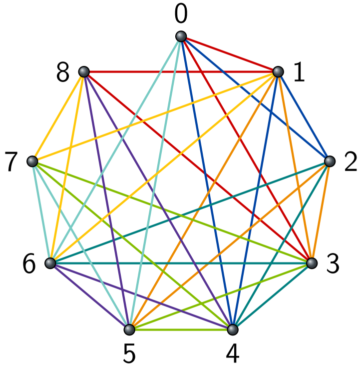

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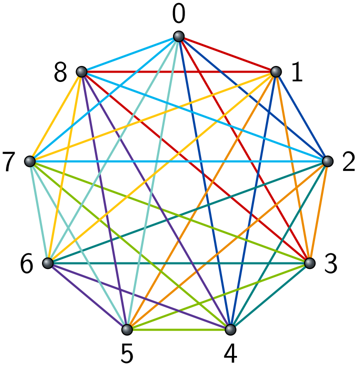

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Cycle decompositions of complete graphs

A k-cycle decomposition of K_v is also called a k-cycle system of order v.

Theorem (Alspach, Gavlas (2001); Šajna (2002))

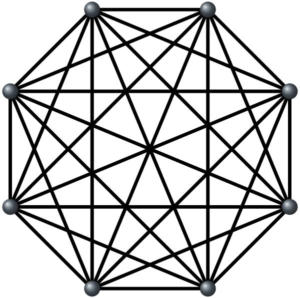
There exists a k-cycle decomposition of K_v if and only if:

- $3 \leq k \leq v$;
- v *is odd; and*
- $k \mid \binom{v}{2}$.

What if v is even?

The "trick" is to remove a 1-factor before decomposing into cycles.

So we look for a k-cycle decomposition of the cocktail party graph $K_v - I$.



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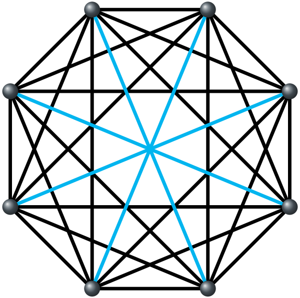

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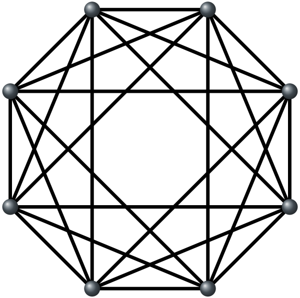

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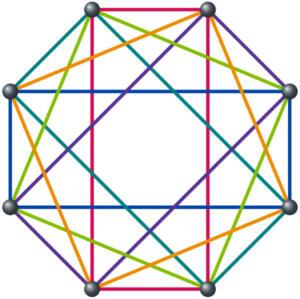

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We will refer to a value of v that satisfies the criteria for existence of a k-cycle decomposition of K_v or $K_v - I$ as k-admissible (or simply admissible) for existence of a k-cycle decomposition of K_v or $K_v - I$.

Cycle decompositions of complete multigraphs

Theorem (Bryant, Horsley, Maenhaut and Smith (2011))

There exists a k-cycle decomposition of λK_v if and only if:

- $2 \leq k \leq v$;
- *if* $k = 2$, *then* λ *is even;*
- $\lambda(v-1)$ *is even; and*
- $k \mid \lambda\binom{v}{2}$.

Theorem (Bryant, Horsley, Maenhaut and Smith (2011))

Let $v \geq 3$. There exists a k-cycle decomposition of $\lambda K_v - I$ if and only if:

- $3 \leq k \leq v$;
- $\lambda(v-1)$ *is odd; and*
- $k \mid \lambda\binom{v}{2} - \frac{v}{2}$.

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A c-colouring of $\mathcal{D}$ is an assignment of c colours to the vertices of Γ.

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- Strong if no block contains two vertices of the same colour.
- Equitable if for any two colours i and j, the number of vertices coloured i and j on any block differ by at most 1.

Weak colourings: Examples

A weak 3-colouring of a 4-cycle decomposition of K_9:

$$(0, 1, 8, 3)$$
$$(1, 2, 0, 4)$$
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Colour classes:

$$\{0, 2, 7\}, \{1, 4, 5\}, \{3, 6, 8\}$$

Weak colourings: Examples

The chromatic number $\chi(\mathcal{D})$ is the minimum number c of colours needed to weakly c-colour the decomposition $\mathcal{D}$.

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This decomposition has chromatic number 2.

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- There is a 2-chromatic BIBD$(v, 4, \lambda)$ for each admissible v. (Hoffman, Lindner and Phelps (1990); Hoffman, Lindner and Phelps (1991); Rosa and Colbourn (1992); Franek, Griggs, Lindner and Rosa (2002))

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- There is a 2-chromatic BIBD$(v, 4, \lambda)$ for each admissible v. (Hoffman, Lindner and Phelps (1990); Hoffman, Lindner and Phelps (1991); Rosa and Colbourn (1992); Franek, Griggs, Lindner and Rosa (2002))
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Weak colourings of cycle systems

- Every 3-cycle system of order $v \geq 7$ has chromatic number at least 3. (Rosa and Pelikán (1970))
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Equitable colourings: Examples

An equitably 3-colourable BIBD$(6, 5, 4)$:

1	2	3	4	5
1	2	3	4	6
1	2	3	5	6
1	2	4	5	6
1	3	4	5	6
2	3	4	5	6

Equitable colourings: Examples

An equitable 3-colouring of a 4-cycle decomposition of K_9:

$$(0, 1, 8, 3)$$
$$(1, 2, 0, 4)$$
$$(2, 3, 1, 5)$$
$$(3, 4, 2, 6)$$
$$(4, 5, 3, 7)$$
$$(5, 6, 4, 8)$$
$$(6, 7, 5, 0)$$
$$(7, 8, 6, 1)$$
$$(8, 0, 7, 2)$$

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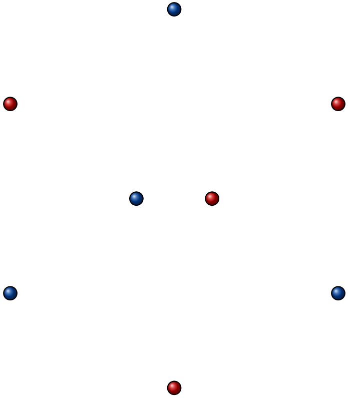
This decomposition cannot be equitably 2-coloured.

Equitable colourings: Examples

An equitable 2-colouring of a 6-cycle decomposition of $K_8 - I$:



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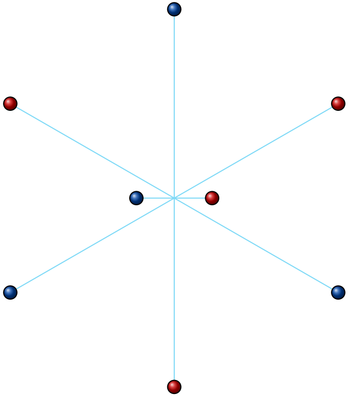

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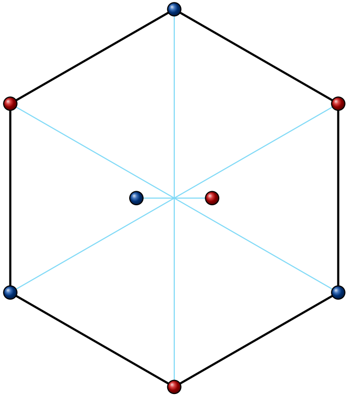

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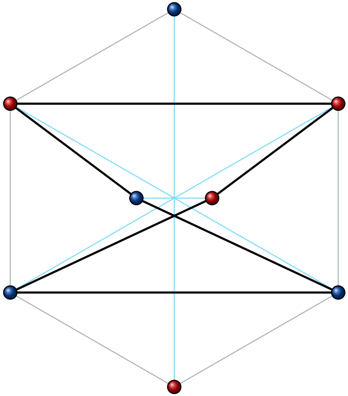

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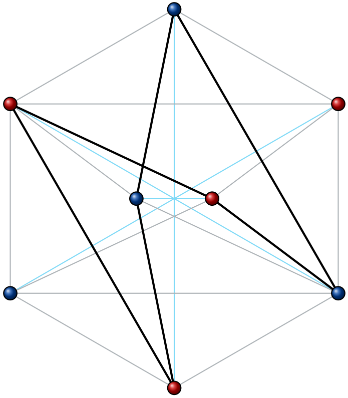

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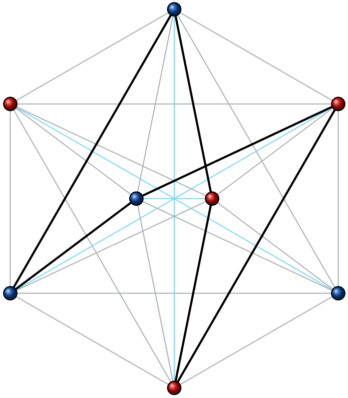

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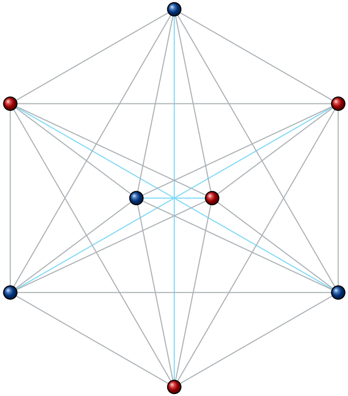

Equitable colourings of cycle decompositions

Lemma

Suppose there is an equitable c-colouring of a k-cycle decomposition of K_v or $K_v - I$, where $c \mid k$. Then:

- *Each cycle contains k/c vertices of every colour.*
- *$c \mid v$, and each colour class has size $\frac{v}{c}$.*

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Theorem (Adams, Bryant, Lefevre and Waterhouse (2004))

If there is an equitably c-colourable $(c+1)$-cycle decomposition of K_v, then $v \leq c^2$.

If there is an equitably c-colourable $(c+1)$-cycle decomposition of $K_v - I$, then $v \leq 2c^2$.

Equitable 2-colourings of cycle decompositions

Lemma (Adams, Bryant and Waterhouse (2007))

If k is even, then there is no equitably 2-colourable k-cycle decomposition of K_v.

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Theorem (Adams, Bryant, Lefevre and Waterhouse (2004))

There is an equitably 3-colourable 4-cycle decomposition of K_v (resp. $K_v - I$) if and only if $v = 9$ (resp. $v \in \{4, 6, 8, 10, 12, 18\}$).

Equitable 3-colourings of cycle decompositions

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There is an equitably 3-colourable 5-cycle decomposition of K_v *or* $K_v - I$ *for every admissible* v*. (I.e. iff* $v \equiv 1, 5 \pmod{10}$ *for decomposition of* K_v *and* $v \equiv 0$ *or* $2 \pmod{10}$ *for decomposition of* $K_v - I$*).*

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Theorem (Adams, Bryant, Lefevre and Waterhouse (2004))

There is an equitably 3-colourable 6-cycle decomposition of K_v if and only if $v \equiv 9 \pmod{12}$, and an equitably 3-colourable 6-cycle decomposition of $K_v - I$ if and only if $v \equiv 0 \pmod{6}$.

Existence of equitably coloured BIBDs

Theorem (Luther and Pike, 2016)

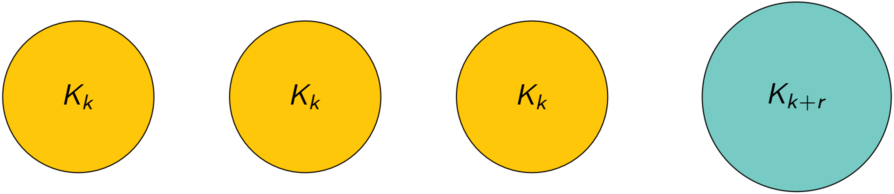
There is an equitably c-colourable BIBD(v, k, λ) with $k < v$ if and only if

- $c = v$, *or*
- $v = k + 1$, $\lambda \equiv 0 \pmod{k-1}$ *and* $k + 1 \equiv 0 \pmod{c}$.

Reduction step for equitably 2-colourable even cycle decomposition of $K_v - I$

Lemma (Burgess and Merola (2020+))

Let r and k be even, $0 \leq r < k$. If $K_{k+r} - I$ admits an equitably 2-colourable k-cycle decomposition, then so does $K_v - I$ for any $v \equiv r \pmod{k}$ with $v \geq k$.



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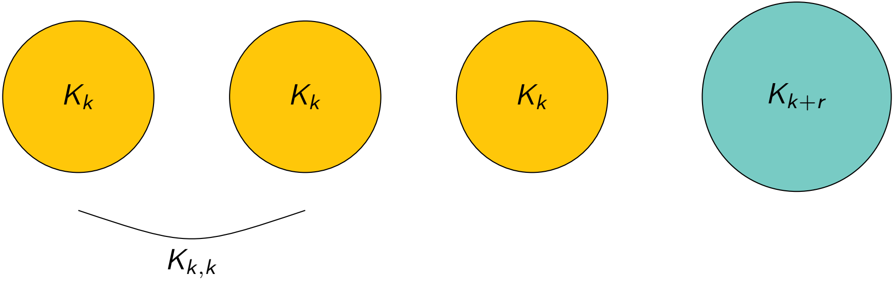

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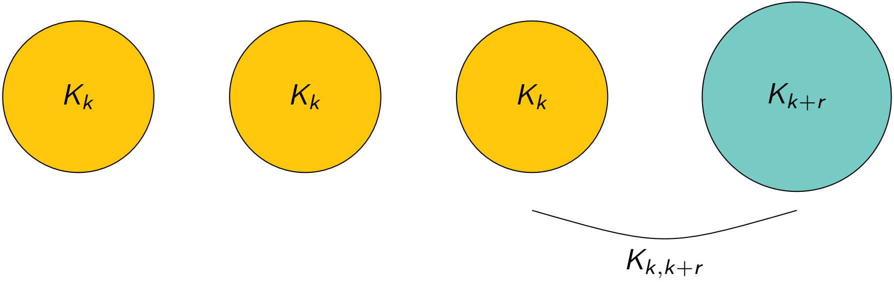
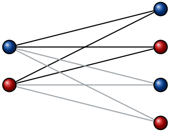


Doubly equitable decompositions of the complete bipartite graph

We say a cycle decomposition of $K_{m,n}$ is doubly equitably c-colourable if it admits a c-colouring ϕ such that:

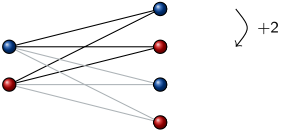
- ϕ is an equitable colouring
- ϕ equitably colours the parts



Theorem (Burgess and Merola (2020+))

Let k be even and $0 \leq r < k$. There exists a doubly equitably 2-colourable k-cycle decomposition of $K_{k,k+r}$.

- When $k \equiv 0 \pmod 4$, we split the part of size k into two sub-parts of size $k/2$, and decompose $K_{k/2,k+r}$.



- When $k \equiv 2 \pmod 4$, we use a variant of a decomposition due to Sotteau (1981).

Reduction step for equitably 2-colourable even cycle decomposition of $K_v - I$

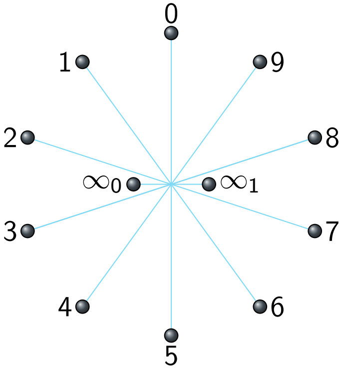
Theorem (Burgess and Merola (2020+))

Let $k \geq 4$ be even. If $K_v - I$ admits an equitably 2-colourable k-cycle decomposition for any k-admissible even v satisfying $k \leq v < 2k$, then $K_v - I$ admits an equitably 2-colourable k-cycle decomposition for any k-admissible even v.

$v \equiv 0$ or $2 \pmod{k}$

Theorem (Burgess and Merola (2020+))

Let k be even. There exist equitably 2-colourable k-cycle decompositions of $K_k - I$ and $K_{k+2} - I$. Hence there is an equitably 2-colourable k-cycle decomposition of $K_v - I$ whenever $v \equiv 0$ or $2 \pmod{k}$.



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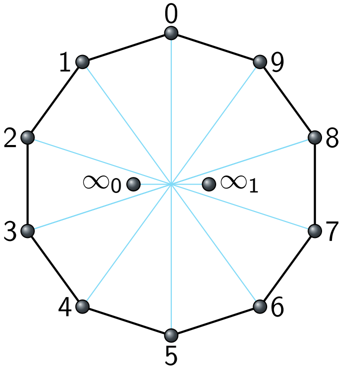

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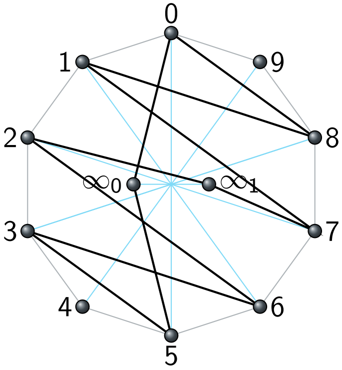

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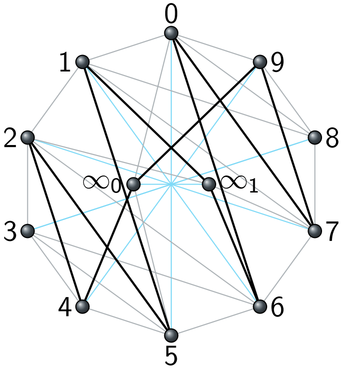

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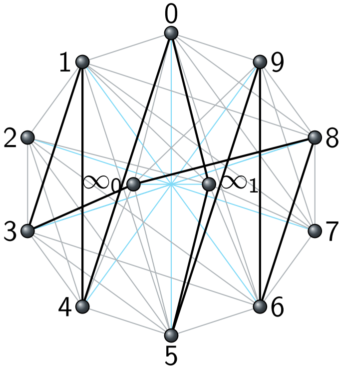

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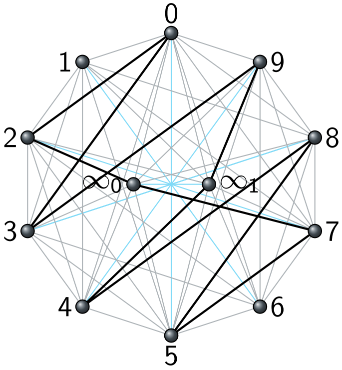

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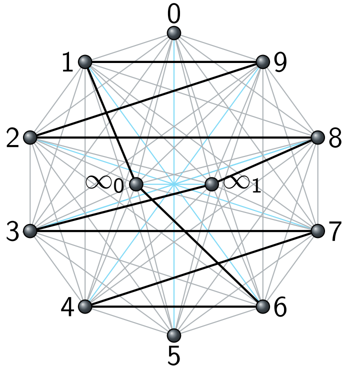

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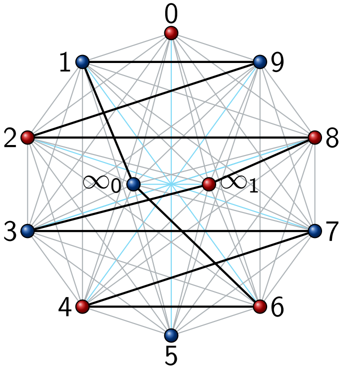

$k = 2q$ or $4q$, q a prime power

Corollary (Burgess and Merola (2020+))

Let q be an odd prime power. There is an equitably 2-colourable $2q$-cycle decomposition of $K_v - I$ if and only if v is $2q$-admissible.

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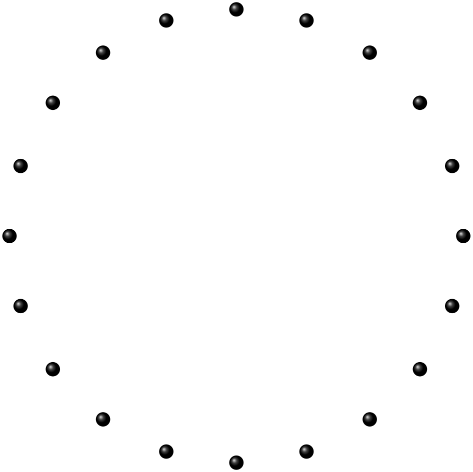
Theorem (Burgess and Merola (2020+))

Let q be an odd prime power. There is an equitably 2-colourable $4q$-cycle decomposition of $K_v - I$ if and only if v is $4q$-admissible.

Proof.

The $4q$-admissible orders $v \in [4q, 8q)$ are $v = 4q, 4q+2, 6q, 6q+2$. For $v \in \{6q, 6q+2\}$ we directly construct a equitably 2-colourable decomposition. □

An equitably 2-colourable 12-cycle decomposition of $K_{20} - I$



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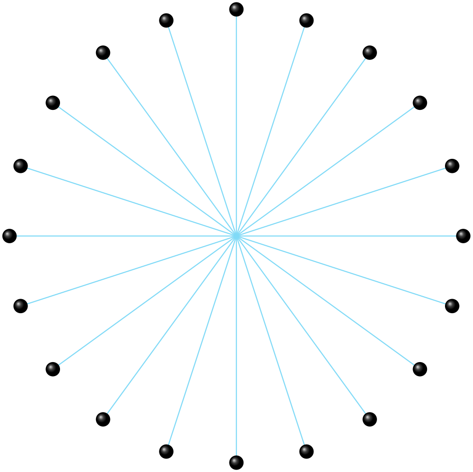

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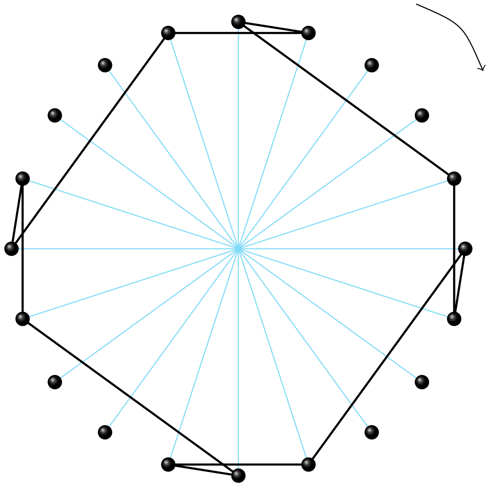

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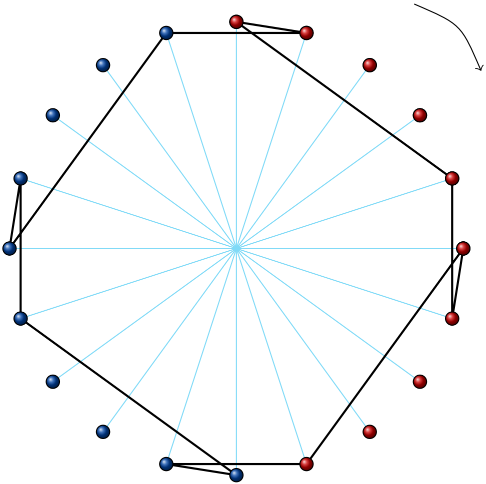

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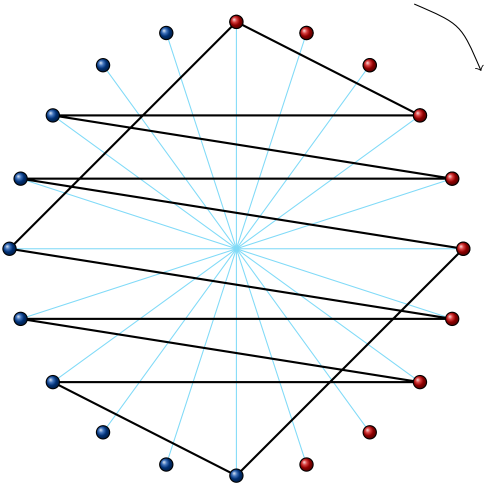

Cycle length $k \leq 30$

Theorem (Burgess and Merola (2020+))

If $4 \leq k \leq 30$ is even, then there is an equitably 2-colourable k-cycle decomposition of $K_v - I$ if and only if v is k-admissible.

Proof.

- The previous results cover all k-values except 24 and 30.
- For $k = 24$, we only need to check orders 32 and 42.
- For $k = 30$, we only need to check orders 42 and 50.
- We construct an equitably 2-colourable decomposition in each case.

□

Future directions

- Find equitably 2-colourable odd cycle decompositions of K_v or $K_v - I$.
- Find equitably c-colourable k-cycle decompositions of K_v or $K_v - I$.
- Complete the spectrum of equitably 2-colourable even cycle decompositions of $K_v - I$.
- Relax "equitable" condition
  - The number of vertices on a block with colours i and j may differ by at most d.
  - Not every colour need appear on every block, but those that do appear equitably.

Thanks!