Orthogonal Colourings of Graphs

Kyle MacKeigan

Dalhousie University

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Vertex Colouring

- A **proper vertex colouring** of a graph G is an assignment of colours (labels) to the vertices of G so that no two adjacent vertices receive the same colour.
- The **chromatic number** of a graph G, denoted $\chi(G)$, is the minimum number of colours required for a proper vertex colouring.

Example of a Vertex Colouring

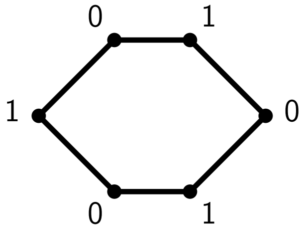


Figure: Vertex Colouring of C_6

Orthogonal Colourings

- Two colourings of a graph G are **orthogonal** if they have the property that when two vertices are coloured with the same colour in one of the colourings, then those vertices must receive distinct colours in the other colouring.
- A k-**orthogonal colouring** of a graph G is a collection of k mutually orthogonal vertex colourings of G.
- The k-**orthogonal chromatic number** of a graph G, denoted $O\chi_k(G)$, is the minimum number of colours required for a proper k-orthogonal colouring of G.

Example of an Orthogonal Colouring

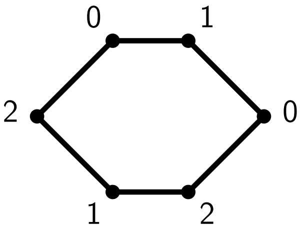


Figure: Colouring 1 of C_6

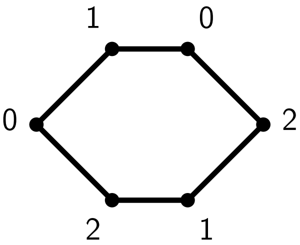


Figure: Colouring 2 of C_6

Example of an Orthogonal Colouring

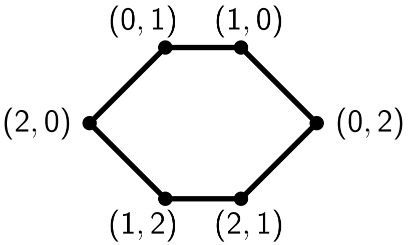


Figure: Orthogonal Colouring C_6

\	0	1	2
0		x	x
1	x		x
2	x	x	

Figure: Orthogonality Grid

Optimal Orthogonal Colourings

- The orthogonal chromatic number is bounded by the following chain of inequalities:
- $\max\{\chi(G), \lceil\sqrt{n}\rceil\} \leq O\chi(G) \leq \cdots \leq O\chi_k(G) \leq n$.
- If $O\chi_k(G) = \lceil\sqrt{n}\rceil$, we say that G has a *k*-**optimal orthogonal colouring** (*k*-OOC).
- Optimal orthogonal colourings are of particular interest, in part because of their application to independent coverings.

Applications

- An **independent transversal** of a graph, with respect to a vertex partition P, is an independent set that contains exactly one vertex from each vertex class.
- An **independent covering** of a graph, with respect to a vertex partition P, is a collection of disjoint independent transversals with respect to P that spans all of the vertices.
- A graph is **[n,k,r]-partite** if the vertices can be partitioned into n independent sets of size k, with exactly r independent edges between every pair of independent sets.

Example of an Independent Covering

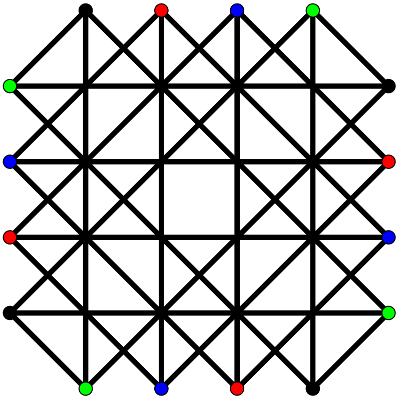


Figure: $[4, 4, 4]$-Partite Graph

Applications

- In general, an independent covering with respect to a partition of the vertices into independent sets gives an orthogonal colouring.
- On the other hand, an orthogonal colouring only gives an independent covering if the sizes of the colour classes in the first colouring are the same and the sizes of the colour classes in the second colouring are the number of colours used in the first colouring.
- In an optimal orthogonal colouring of a graph with n^2 vertices, this occurs.

Applications

- Let $c(k, r)$ denote the maximal n such that all $[n, k, r]$-partite graphs have an independent covering with respect to the given $[n, k, r]$-partition.

Theorem (Yuster, 1997) [4]

$k \geq c(k, r) \geq \min\{k, k - r + 2\}$.

Conjecture (Yuster, 1997) [4]

For all $r \leq k$, $c(k, r) = k$.

Theorem (M., 2020) [3]

For all $r \leq k$, $c(k, r) \geq \lceil \frac{k}{2} \rceil$

Counter Example

- Yuster's [4] conjecture was shown false with the following graph [3].

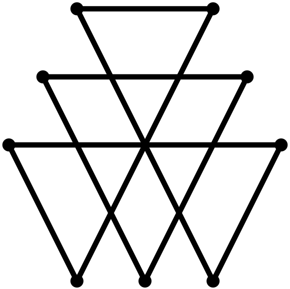


Figure: Counter Example Graph

Counter Example

- However, an orthogonal colouring using 3 colours does exist, as shown in the following figure.

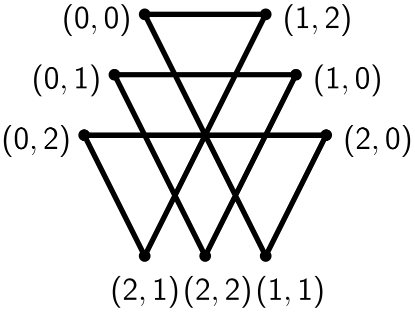


Figure: Counter Example Graph

Optimal Orthogonal Colourings

- Optimal orthogonal colourings give a partition that has an independent covering.
- Graphs having optimal k-orthogonal colourings are also applied to time scheduling and transversal design problems.
- Graphs having optimal k-orthogonal colourings can be categorized by the following graphs.
- Take the complete graph K_{n^2} and delete k edge-disjoint K_n factors.
- In the case $k = 2$, the graph obtained by this process is independent of the K_n factors chosen.

Optimal Orthogonal Colourings

Theorem (M. and Janssen, 2020) [2]

A graph G with $|V(G)| = n$ has an optimal orthogonal colouring if and only if $G \subseteq K_N \times K_N$, where $N = \lceil \sqrt{n} \rceil$.

$c_1 \backslash c_2$	0	1	2
0	$v_{0,0}$	$v_{0,1}$	$v_{0,2}$
1	$v_{1,0}$	$v_{1,1}$	$v_{1,2}$
1	$v_{2,0}$	$v_{2,1}$	$v_{2,2}$

Figure: Orthogonality Grid

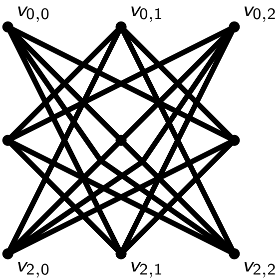


Figure: $K_3 \times K_3$

Orthogonal Colourings of Tree Graphs

- A **tree graph** is a connected graph containing no cycles.
- A *d*-**degenerate graph** is a graph such that there exists an ordering of the vertices, in which each vertex has *d* or fewer neighbours that are earlier in the ordering.
- We call such an ordering a *d*-**degenerate ordering**.

Degenerate Graphs

- For example, the tree graph has the 1-degenerate ordering of: $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$.

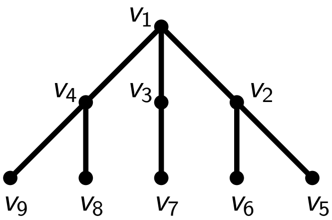


Figure: Tree Graph

Orthogonal Colourings of Degenerate Graphs

- Using the degenerate ordering, an upper bound on the orthogonal chromatic number of degenerate graphs is obtained.

Theorem (Caro and Yuster, 1999) [1]

If G is a d-degenerate graph with n vertices, then $O\chi(G) \leq d + \lceil \sqrt{n-d} \rceil$.

- This bound is tight by considering the join of an independent set I_{n-d} and a clique K_d.
- The orthogonal chromatic number of a tree graph T_n is one a two values, $\lceil \sqrt{n} \rceil$, or $\lceil \sqrt{n} \rceil + 1$.

Classifying the Orthogonal Chromatic Number of Tree Graphs

- Let $T_{n,k}$ denote the class of all trees on n vertices having maximum degree k. Then we are interested in bounding the following parameter:

$$\Delta_1 = \max\{k : \forall n \forall T \in T_{n,k}, O\chi(T) = \lceil \sqrt{n} \rceil\}$$

- Let D_n denote the graph obtained by joining two $K_{1,\frac{n}{2}-1}$ graphs at the roots.

Orthogonal Colourings of Double Star Graphs

Theorem (M., 2020) [3]

$O\chi(D_n) = \lceil \sqrt{n} \rceil$ if and only if n is not a perfect square.

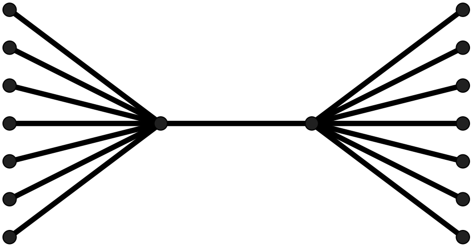


Figure: Double Star Graph D_{16}

Orthogonal Colourings of D_{14}

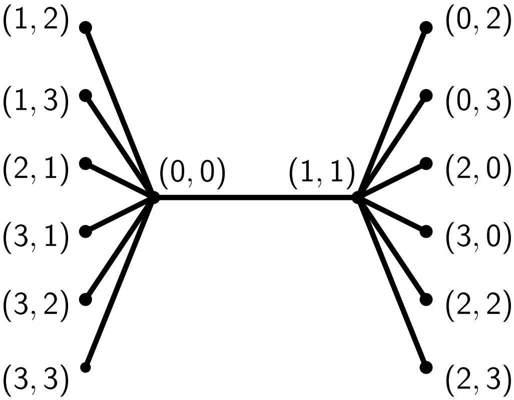


Figure: Orthogonal Colouring of D_{14}

Orthogonal Colourings of Tree Graphs Conjecture

- Firstly, this corrected the result in [1] published by Caro and Yuster, that stated that $O\chi(D_n) = \lceil \sqrt{n} \rceil + 1$ whenever n is even and $\lceil \sqrt{n} \rceil \lceil \sqrt{n} - 1 \rceil < n$.
- Secondly, this theorem gives us that $\Delta_1 < \frac{n}{2}$.

Conjecture (Caro and Yuster, 1999) [1]

$\Delta_1 = \frac{n}{2} - 1$.

Counter Example Graph

- This conjecture is false, as shown with the following graph.

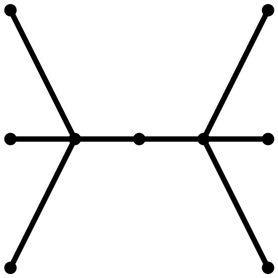


Figure: Counter Example Graph

Optimal Orthogonal Colouring using Degree

- On the other hand, the following result gives a lower bound for Δ_1.

Theorem (Caro and Yuster, 1999) [1]

If G is an n-vertex graph and $\Delta(G) \leq \frac{\sqrt{n}-1}{4}$, then $O\chi(G) = \lceil \sqrt{n} \rceil$.

- Therefore, $\Delta_1 \geq \frac{\sqrt{n}-1}{4}$

Optimal Orthogonal Colouring using Degeneracy

Theorem (M., 2020) [3]

If G is d-degenerate with $\Delta(G) < \frac{\sqrt{n}-2d-1}{2}$, then $O\chi(G) = \lceil \sqrt{n} \rceil$.

Idea of Proof

- Consider a d-degenerate ordering $\{v_1, v_2, \ldots, v_n\}$.
- Let G_t be the graph where all the edges to the vertices $v_{t+1}, \ldots, v_n$ are removed.
- The goal is to inductively colour G_t with $\lceil \sqrt{n} \rceil$ colours. $(G_1 = I_n, G_n = G)$.
- Suppose that we have a proper orthogonal colouring of G_{t-1}. Assign the same colouring to G_t.

Optimal Orthogonal Colouring using Degeneracy

Idea of Proof

- Let $N_t(v_t)$ be the neighbourhood of v_t is G_t.
- Let W be the set of vertices having the property that for some vertex $v \in N_t(v_t)$, $c_1(v) = c_1(w)$ or $c_2(v) = c_2(w)$.
- Let Y_t denote the set of vertices having the property that $c_1(y) = c_1(v_t)$ or $c_2(y) = c_2(v_t)$.
- Let $N(Y_t)$ be the union of open neighbourhoods of these vertices in G.
- Let $X = V(G) \backslash (W \cup N(Y_t)$.

Cayley Graphs

- Let $(G, \circ)$ be a group and let S be a generating set of G.
- The associated **Cayley graph** is denoted $\Gamma(G, S)$.
- There is a vertex for each element of G.
- There is a directed edge between two elements u and v if and only if $u \circ v^{-1} \in S$.
- To get simple, undirected Cayley graphs, we assume that the generating set is self-inverse and that $1 \notin S$.

Circulant Graphs

- **Circulant graphs** are Cayley graphs of cyclic groups.
- For instance, the cycle graph C_n is a circulant graph.
- Consider $\mathbb{Z}_n$ with addition as its group operation.
- Then the Cayley graph $\Gamma(\mathbb{Z}_n, \{1, -1\}) \cong C_n$.

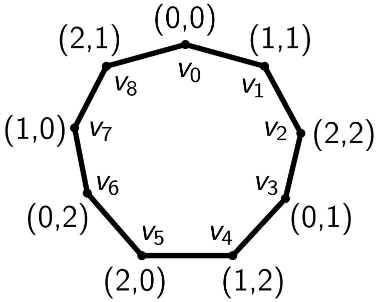
Orthogonal Chromatic Number of Cycles

Theorem (Janssen and M., 2020) [2]

$O\chi(C_n) = \lceil \sqrt{n} \rceil$ if and only if $n > 4$.

$c_1 \backslash c_2$	0	1	2
0	v_0	v_3	v_6
1	v_7	v_1	v_4
2	v_5	v_8	v_2

Figure: Orthogonality Grid



k-Orthogonal Chromatic Number of Cycles

Theorem (Janssen and M., 2020) [2]

If $\lceil \sqrt{n} \rceil = p$ is a prime number, then $O\chi_{p-2}(C_n) = p$.

- $|0, 1, 2, 3, 4|0, 1, 2, 3, 4|0, 1, 2, 3, 4|0, 1, 2, 3, 4|0, 1, 2, 3, 4|$
- $|0, 1, 2, 3, 4|1, 2, 3, 4, 0|2, 3, 4, 0, 1|3, 4, 0, 1, 2|4, 0, 1, 2, 3|$
- $|0, 1, 2, 3, 4|2, 3, 4, 0, 1|4, 0, 1, 2, 3|1, 2, 3, 4, 0|3, 4, 0, 1, 2|$
- $|0, 1, 2, 3, 4|3, 4, 0, 1, 2|1, 2, 3, 4, 0|4, 0, 1, 2, 3|2, 3, 4, 0, 1|$

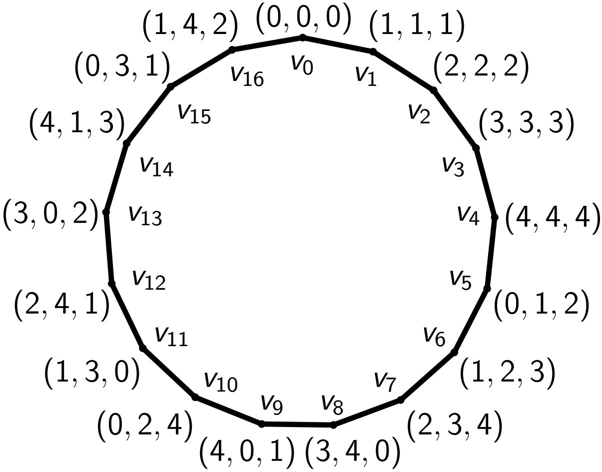


Figure: 3-Orthogonal Colouring of C_{17}

Orthogonal Colourings of Circulant Graphs

Lemma (Janssen and M., 2020) [2]

For every $\alpha \in \mathbb{Z}_p \backslash \{0\}$, $\hat{F}_{\alpha,p}(i,j) : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_{n^2}$ defined by $F_{\alpha,n}(i,j) = ((\alpha(j-i) \pmod{p} + p(2i-j)) \pmod{p^2})$ is a bijection.

Theorem (Janssen and M., 2020) [2]

For p a prime, if $|S| < \frac{p-1}{2}$, then $O\chi(\Gamma(\mathbb{Z}_{p^2}, S)) = p$.

Idea of Proof

- Suppose that two vertices k and l receive the same colour in the first colouring.
- That is, $\hat{F}_{\alpha,p}^{-1}(k) = (i, (j+x) \pmod p)$ and $\hat{F}_{\alpha,p}^{-1}(l) = (i, j)$ for some $i, j \in \mathbb{Z}_p$ and $x \in \mathbb{Z}_p \setminus \{0\}$.
- Then $k - l = ((\alpha x) \pmod p) - px) \pmod{p^2}$.
- Let $A_\alpha = \{((\alpha x) \pmod p) - px) \pmod{p^2} | x \in \mathbb{Z}_p \setminus \{0\}\}$.
- Differences in the second colouring are of the form $((-\alpha x) \pmod p)) + 2px) \pmod{p^2}$.
- $B_\alpha = \{((-\alpha x) \pmod p)) + 2px) \pmod{p^2} | x \in \mathbb{Z}_p \setminus \{0\}\}$.
- Therefore, there is a colour conflict in the first and second colouring if and only if $S \cap A_\alpha \neq 0$ and $S \cap B_\alpha \neq 0$ respectively.

Idea of Proof

- $A_\alpha = B_{2\alpha \pmod p}$
- The $A'_\alpha s$ together with $\{m|m \in \mathbb{Z}_p\}$ and $\{mp|m \in \mathbb{Z}_p\}$ are disjoint.
- Let $c \in S$ and suppose that $c \in \{m|m \in \mathbb{Z}_p\}$ or $c \in \{mp|m \in \mathbb{Z}_p\}$. Since these are disjoint with the A_α's, $c \notin A_\alpha$ for any $\alpha \in \mathbb{Z}_p \backslash \{0\}$.
- Let $c \in S$ and suppose that $c \in A_\alpha$ for some α. Then $c \in B_{2\alpha}$. But since the A_α's and B_α's are disjoint, $c \notin A_{\alpha'}$ and $c \notin B_{2\alpha'}$ for any $\alpha' \neq \alpha$.
- Therefore, each c removes two options for α. There are less than $\frac{p-1}{2}$ choices for c, so less than $p-1$ restrictions on α. But there are $p-1$ options for α, so at least one of these will work.

Open Problems

Theorem (Walter and Torsten, 2016)

If $S = \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$, then $\Gamma(\mathbb{Z}_{n^2}, S)$ is uniquely n-colourable.

- Therefore, we need $|S| \leq 2n - 4$.
- However, there are still circulant graphs with $|S| > 2n - 4$ that have optimal orthogonal colourings.

Lemma (Janssen and M., 2020) [2]

If $n \nmid \alpha$, then $F_{\alpha,n} : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_{n^2}$ defined by $F_{\alpha,n}(i,j) = in + j\alpha$ is a bijection.

- Note that for $\alpha = 1$, $F_{1,n}^{-1}$ is the orthogonal colouring used on C_n.

Theorem (Janssen and M., 2020) [2]

If $|S| < n$, then $O\chi(\Gamma(\mathbb{Z}_{n^2}, S)) = n$.

Payley Graphs

- The Payley graph is denoted $QR(q)$.
- Let $\mathbb{F}_q$ be the finite field of order q.
- Let S be the set of quadratic residues.
- Then $\Gamma(\mathbb{F}_q, S) = QR(q)$.

Theorem (Janssen and M., 2020) [2]

For p a prime and an integer $r \geq 1$, $O\chi_{\frac{p^r+1}{2}}(QR(p^{2r})) = p^r$.

Orthogonal Colourings of Hamming Graphs

- **Hamming graphs**, denoted $H(d, q)$, are a special class of Cayley graphs.
- Consider $\mathbb{Z}_q^d$ with operation $+$ coordinate wise.
- Let $S = \{\mathbb{Z}_q \backslash \{0\}\}$.
- Then $H(d, q) \cong \Gamma(\mathbb{Z}_q^d, S^d)$.
- Alternatively, $H(d, q)$ can be constructed as the Cartesian product of d complete graphs K_q.
- The Cartesian product of two graphs G and H, denoted by $G \square H$, has vertex set $V(g) \square V(H)$, and two vertices (u_1, v_1) and (u_2, v_2) in $G \square H$ are adjacent if and only if $u_1 u_2 \in E(G)$ and $v_1 = v_2$, or $v_1 v_2 \in E(H)$ and $u_1 = u_2$.

Orthogonal Colourings of Hamming Graphs

- Note that $H(d+2, q) \cong H(d, q) \square H(2, q)$.

Theorem (Janssen and M., 2020) [2]

If $|V(G)| = n^2$, $|V(H)| = m^2 \leq n^2$, and $O\chi(G) = n$, then $O\chi(G \square H) = nm$.

Corollary (Janssen and M., 2020) [2]

$O\chi(H(2d, q)) = q^d$ if $q \neq 2, 6$ and $d \geq 1$.

Optimal Orthogonal Colourings

- A **Latin square** is an $n \times n$ array, filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

0	1	2
1	2	0
2	0	1

Figure: 3x3 Latin square.

0	1	2
2	0	1
1	2	0

Figure: 3x3 Latin square.

- Two Latin squares L_1 and L_2 of order n are **orthogonal** if for any ordered pair (s, t) there is exactly one pair (i, j) for which $L_1(i, j) = s$ and $L_2(i, j) = t$.

Optimal Orthogonal Colourings

Lemma (M. and Janssen)

$O\chi(K_n \square K_n) = n$ for $n \neq 2, 6$.

$(0,0)$	$(1,1)$	$(2,2)$
$(1,2)$	$(2,0)$	$(0,1)$
$(2,1)$	$(0,2)$	$(1,0)$

Figure: Orthogonal Latin squares.

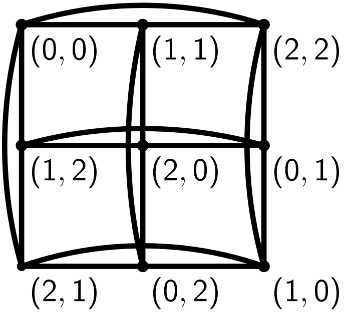
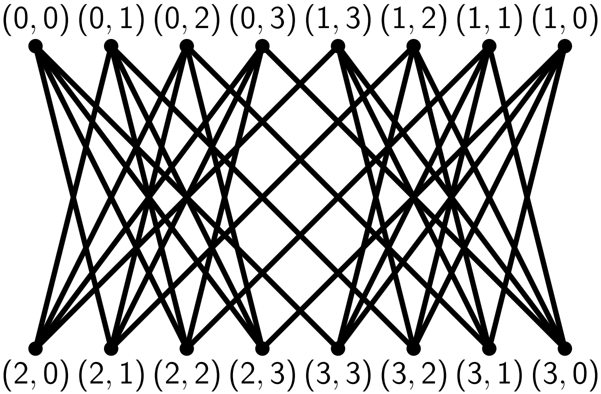


Figure: $K_3 \square K_3$

Orthogonal Colourings of Hamming Graphs

Theorem (M. and Janssen)

$O\chi(H(4d,2)) = q^d$ and $O\chi(H(4d,6)) = q^d$ if $d \geq 2$.



Tensor Graph Product

- The tensor product of two graphs G and H, denoted by $G \times H$, has vertex set $V(G) \times V(H)$, and two vertices (u_1, v_1) and (u_2, v_2) in $G \times H$ are adjacent if and only if $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$.

Theorem

If G has n^2 vertices, H has m^2 vertices, and $O\chi(G) = n$, then $O\chi(G \times H) = nm$.

Theorem

If G has n^2 vertices with $O\chi_k(G) = n$ and H has m^2 vertices with $O\chi_k(H) = m$, then $O\chi_k(G \times H) = nm$.

Idea of Proof

- For $0 \leq r < k$ and $0 \leq i < n$, let $G_{r,i}$ be the i-th colour class in the r-th colouring of G.
- For $0 \leq r < k$ and $0 \leq j < m$, let $H_{r,j}$ be the j-th colour class in the r-th colouring of H.
- Let $I_{r,i,j} = \{(u, v) | u \in G_{r,i}, v \in H_{r,j}\}$.
- For (u_1, v_1) and $(u_2, v_2) \in I_{r,i,j}$, $u_1 u_2 \notin G$ and $v_1 v_2 \notin H$. Thus $I_{r,i,j}$ is an independent set in $G \times H$.
- Note that it is also an independent set in $G \square H$, and $G \boxtimes H$.

Theorem

If G has n^2 vertices, H has p^2 vertices where p is a prime, and $O\chi_k(G) = n$ with $k \leq p$,then $O\chi_k(G \times H) = np$.

Idea of Proof

- Label $V(H) = \{(u_i, u_j) : 0 \leq i, j < p\}$. For $O \leq r < k$ and $0 \leq s < n$
- Let $I_{r,s}$ be the s-th colour class in the r-th colouring of G.
- For $0 \leq j < p$, let $\bar{I}_{r,s,j} = \{(v, (u_i, u_{(ir+j)(modp)}) | v \in I_{r,s}, 0 \leq i < p\}$.

Reference

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