Optimizing Student Course Preferences in School Timetabling

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Atlantic Graph Theory Seminar

Brief Bio

- M.Sc. and Ph.D., supervised by Jason Brown (2001-2007)
- Canada Border Services Agency (2006-2010)
- Post-Doctoral Fellowship in Japan (2010-2013)
- Quest University Canada (2013-2020)
- Northeastern University (2020-present)

Outline of the Talk

- Presentation and Discussion of Fun Puzzle
- Motivation for Optimal School Timetabling
- Two-stage Timetabling Algorithm
  1. **Bundle One-Section Courses (Chromatic Number)**
  2. **Solve an Integer Linear Program**
- Deployment at a High School in British Columbia
- Conclusion

Motivation for Optimal School Timetabling

- Schools need to produce a Master Timetable:

  For each section of each course: time slot, teacher, classroom

  For each teacher and student: time slot, course section, classroom

- 10% Math and 90% Politics

- Numerous hard constraints (e.g., teachers, classrooms)

- School administrators don't know Discrete Optimization

St. Margaret’s School

Canada’s first all-girls STEM high school, located in Victoria, BC

Weekly Schedule of 9 Blocks

Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
Block 1	Block 2	Assembly	Block 1	Block 2
Block 7	Block 8	Block 8	Block 9	Block 3
Block 4	Block 3	Block 6	Block 5	Block 7
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St. Margaret's Optimal Timetable

Block 1	Block 2	Block 3
Culinary Arts 11A (DR)	Culinary Arts 11B (DR)	Comp. Prog. 11 (WF)
Core French Intro 11 (AS)	Life Education 11A (DH)	Life Education 11B (DH)
Philosophy 12 (JP)	Entrepreneurship 11 (CJ)	Spoken Language 11 (MC)
Economics 12 (SW)	Drama 11/12 (NC)	Japanese Intro 11 (MH)
Life Connections 12A (KD)	Life Connections 12B (KD)	Composition 12 (NP)
EarthSci 11/12 Combo (CJ)	Law Studies 12 (SW)	Spanish 12 (BP)
Pre-Calc 11/12 Combo (CT)	Social Justice 12 (JP)	Comp. Cultures 12 (SW)
Block 4	**Block 5**	**Block 6**
Chemistry 11A (SB)	Art Studio 11 (LH)	Physics 11 (CT)
Active Living 11 (JS)	Fitness 11 (JS)	Life Sciences 11 (DR)
AP Studio Art 12 (LH)	Japanese 12 (MH)	Pre-Calculus 11A (WF)
English Studies 12A (NP)	Pre-Calculus 12A (CT)	English Studies 12B (NP)
AP Calculus 12 (CT)	Chemistry 12A (SB)	
Block 7	**Block 8**	**Block 9**
New Media 11 (NP)	Pre-Calculus 11B (WF)	Chemistry 11B (SB)
Creative Writing 11A (CN)	Creative Writing 11B (NP)	Core French 12 (AS)
Anatomy/Physio 12 (SB)	Chemistry 12B (SB)	Human Geog 12 (LZ)
Pre-Calculus 12B (CT)	EarthSci 11/12 Combo (CJ)	Univ/Grad Prep 12 (KD)
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Maximize 4x+3y

Given the following constraints:

$$x + y \leq 11$$

$$5x + 2y \leq 40$$

$$x \geq 0$$

$$y \geq 0$$

Integer Linear Program

Binary Variables

$$X_{t,c,b} = \begin{cases} 1 & \text{if teacher } t \text{ is assigned} \\ & \text{to course } c \text{ in block } b. \\ 0 & \text{otherwise.} \end{cases}$$

t = teacher (e.g. 1 ... 19)

c = course (e.g. 1 ... 39)

b = block (e.g. 1 ... 9)

Integer Linear Program

Binary Variables

$$Y_{s,c,b} = \begin{cases} 1 & \text{if student } s \text{ takes} \\ & \text{course } c \text{ in block } b. \\ 0 & \text{otherwise.} \end{cases}$$

s = student (e.g. $1 \ldots 58$)

c = course (e.g. $1 \ldots 39$)

b = block (e.g. $1 \ldots 9$)

For each teacher-course-block triplet and
for each student-course-block triplet consider

$$X_{t,c,b} \qquad\qquad Y_{s,c,b}$$

Integer Linear Program

Desirability and Preferences Coefficients

For each teacher-course-block triplet and
for each student-course-block triplet consider

$$\cdot X_{t,c,b} \qquad\qquad \cdot Y_{s,c,b}$$

Integer Linear Program

Desirability and Preferences Coefficients

For each teacher-course-block triplet and
for each student-course-block triplet consider

$$\underbrace{D_{t,c,b}}_{\text{Desirability coefficient}} \cdot X_{t,c,b} \qquad \underbrace{P_{s,c,b}}_{\text{Preference coefficient}} \cdot Y_{s,c,b}$$

Integer Linear Program

Desirability and Preferences Coefficients

For each teacher-course-block triplet and
for each student-course-block triplet consider

$$\underbrace{D_{t,c,b}}_{\text{Desirability coefficient}} \cdot X_{t,c,b} \qquad \underbrace{P_{s,c,b}}_{\text{Preference coefficient}} \cdot Y_{s,c,b}$$

The larger the $D_{t,c,b}$, the stronger the willingness to teach.
The larger the $P_{s,c,b}$, the stronger the preference to take a course.

Integer Linear Program

Objective Function

Then our ILP has the following objective function, where T, S, C, B are the set of teachers, students, courses, and blocks:

$$\max \sum_{t \in T} \sum_{c \in C} \sum_{b \in B} D_{t,c,b} \cdot X_{t,c,b} + \sum_{s \in S} \sum_{c \in C} \sum_{b \in B} P_{s,c,b} \cdot Y_{s,c,b}$$

Integer Linear Program

Constraints: In any block, a teacher can teach at most one course

$$\sum_{c=1}^{39} X_{t,c,b} \leq 1 \qquad \forall\ t \in T, b \in B \quad (1)$$

Integer Linear Program

Constraints: Ensure a maximum class size for each course

Let M_c be the maximum class size for course c.

$$\sum_{s=1}^{58} Y_{s,c,b} \leq M_c \qquad \forall c \in C, b \in B \quad (2)$$

Integer Linear Program

Constraints: A student can only enroll only in a course that is scheduled

$$Y_{s,c,b} \leq \sum_{t=1}^{19} X_{t,c,b} \quad \forall \ s \in S, c \in C, b \in B \tag{3}$$

Integer Linear Program

Optimal Solution

St. Margaret's objective function is:

$$\max \sum_{t=1}^{19} \sum_{c=1}^{39} \sum_{b=1}^{9} D_{t,c,b} \cdot X_{t,c,b} + \sum_{s=1}^{58} \sum_{c=1}^{39} \sum_{b=1}^{9} P_{s,c,b} \cdot Y_{s,c,b}$$

Our ILP has $(|T| + |S|) \cdot |C||B| = 27,027$ variables.

Integer Linear Program

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Results

- Our ILP is solved in Python, using the Google OR-Tools optimization package.
- Students get into 167 out of 167 core courses (100%). Students get into 262 out of 280 elective courses (94%).
- Overall success rate: 429 out of 447 total courses (96%).
- Total run time: 202.6 seconds.

(1) Construct Weighted Conflict Graph

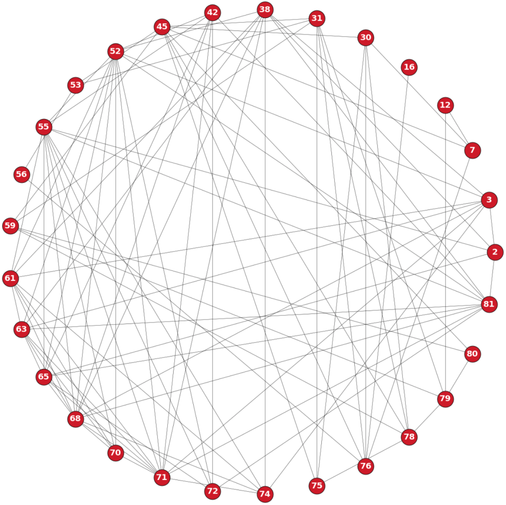

Figure: A conflict graph.

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(1) Construct Weighted Conflict Graph

1. Define the set of courses $C = C_M \cup C_O$,
where C_M is the set of multiple-section courses,
and C_O is the set of one-section courses

2. Let C_O be the set of vertices of the *weighted conflict graph* G

3. For each pair $x, y \in C_O$, let the edge weight $w(x, y)$ of G be:
- w_t if teacher t is assigned to both x and y
- w_r if room r is assigned to both x and y
- w_s if student s is assigned to both x and y

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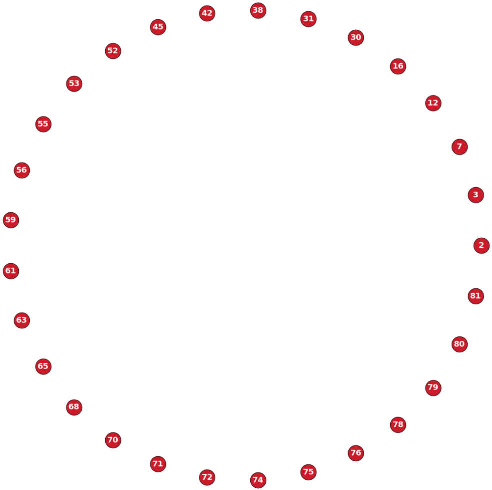

Figure: Conflict graph G.

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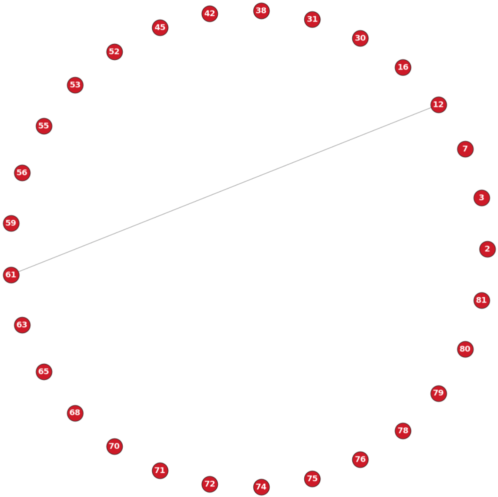

Figure: Graph G with one conflict,
i.e. one edge.

(1) Construct Weighted Conflict Graph

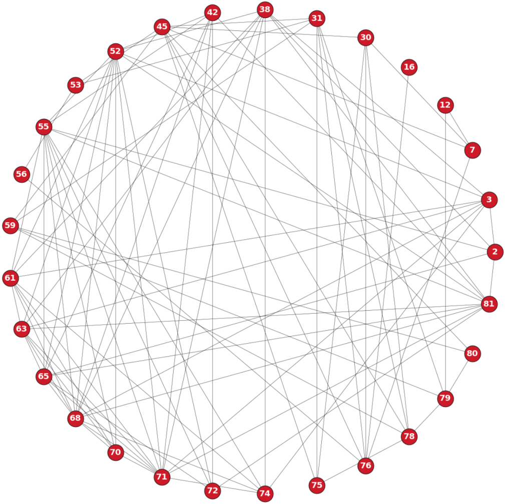

Figure: Graph G_0 with 30 vertices and 94 edges.

(2) Find a Graph Colouring

1. $\forall\, i \geq 0$, let G_i be the graph whose vertices are the set C_O and whose edges have weight greater than i

Insight: if we find a graph colouring for G_i so that the *chromatic number*, $\chi(G_i) \leq |B|$, then we can bundle together courses in the same colour class and assign them to the same block

2. Starting with $i = 0$, calculate $\chi(G_i)$.
If $\chi(G_i) \leq |B|$, then stop.
Otherwise, increment i by 1 until
we find some index $i = t$
for which $\chi(G_t)_{\leq}|B|$

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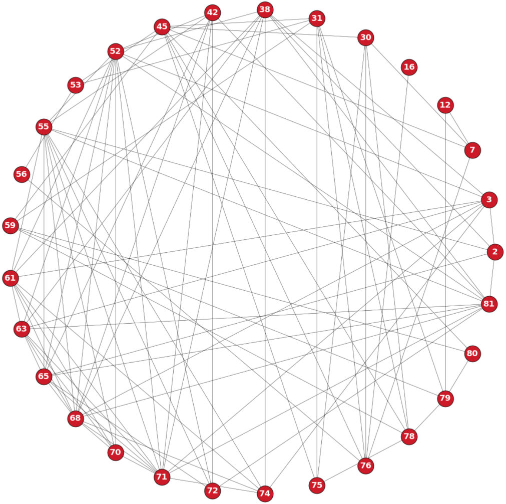

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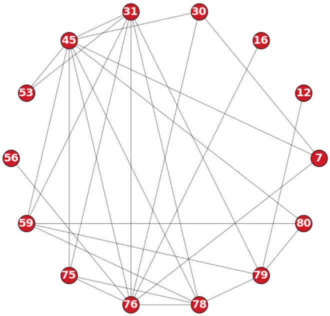

Figure: $G_0\,'$

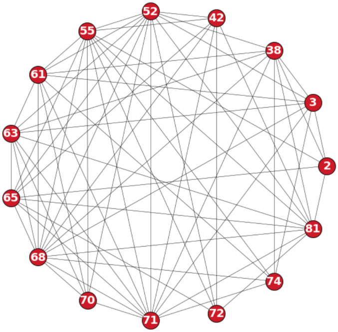

Figure: $G_0\,''$

Graph G_0 is a two-component graph.

(2) Find a Graph Colouring

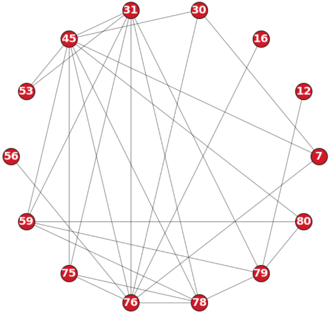

Figure: $\chi({G_0}')=5$

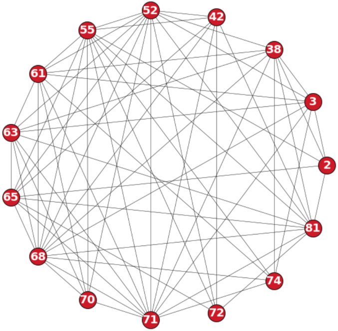

Figure: $\chi({G_0}'')=7$

$$\chi(G_0)=5+7=12>|B|$$

(2) Find a Graph Colouring

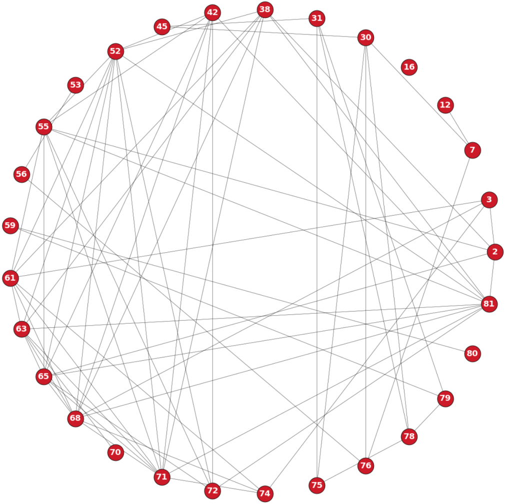

Figure: Graph G_1 with 30 vertices and 69 edges.

(2) Find a Graph Colouring

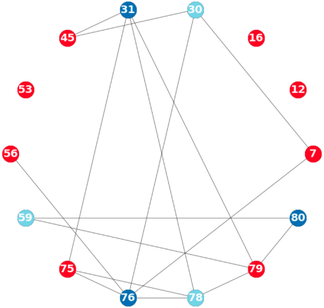

Figure: $\chi(G_1') = 3$

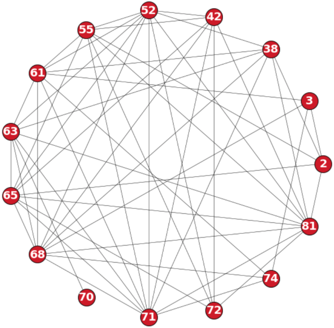

Figure: $\chi(G_1'') = 6$

$$\chi(G_1) = 3$$

(2) Find a Graph Colouring

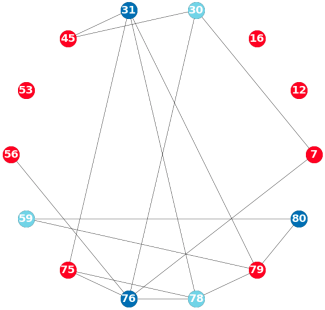

Figure: $\chi(G_1') = 3$

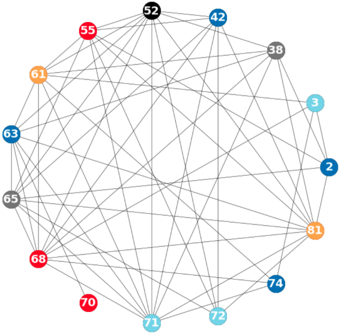

Figure: $\chi(G_1'') = 6$

$$\chi(G_1) = 3 + 6$$

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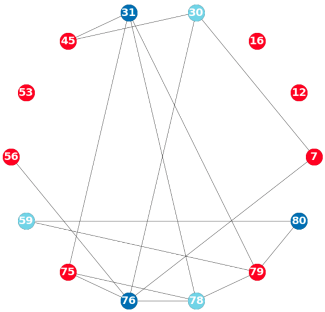

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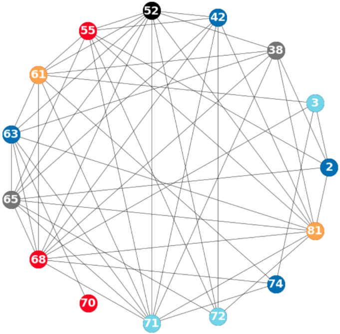

Figure: $\chi(G_1'') = 6$

$$\chi(G_1) = 3 + 6 \leq |B| = \mathbf{9}$$

(3) Course Bundles Become ILP Input

1. Consider the colouring of G_t we found and let X_j be that subset of C_O assigned to color j

2. Redefine $C = C_M \cup X_1 \cup X_2 \cup ... \cup X_{|B|}$, where $X_1,\ X_2,\ ..., X_{|B|}$ are bundles of one-section "super courses" taught by multiple teachers.

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Results

	enrolled core courses	enrolled elective courses	enrolled total courses	total runtime (in seconds)
ILP	167/167	262/280	429/447	202.6
ILP & bundling	167/167	250/280	417/447	4.17

- enrolled total courses reduced from 96% to 93%
- total run-time reduced from 202 seconds to 4 seconds

St. Margaret’s 2019-2020 Timetable

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Conclusion

- This work led to a publication at the CPAIOR 2020 conference.
- I created the 2020-2021 Master Timetable for four high schools, and the 2021-2022 Master Timetable for five high schools.
- I have currently signed seven high schools for 2022-2023, all in British Columbia.
- Combining Graph Theory with Linear Programming creates a triple-win: for school administrators, teachers, and students.

Contact Information

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