Gracefully labelling windmills using Skolem-like sequences

Jared Howell - Grenfell Campus, MUN

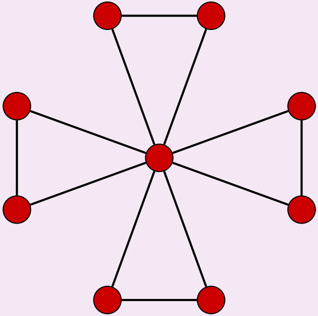
work with Danny Dyer & Ahmad Alkasasbeh

Windmills

- A windmill (with t vanes) is a graph created from a smaller graph G.
- First select a vertex $v \in G$ to be the central vertex.
- Next take t copies of G and identifying each copy of v so that they form one central vertex.
- We denote this windmill by G^t

Windmills

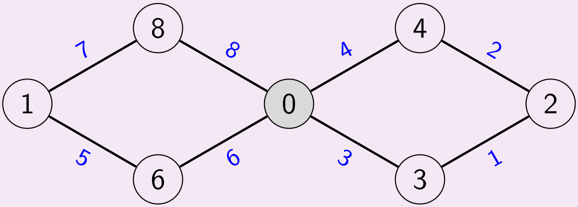
This is the windmill C_3^4.



Graceful Labellings

- Introduced by A. Rosa (as β-labellings).
- $G = (V, E)$ is a graph with m edges.
- Each $v \in V$ is given a distinct label $\ell(v)$ from $[0, m]$.
- Edge labels are induced by the vertex labels: $|\ell(u) - \ell(v)|, \forall uv \in E$.
- The labelling is a graceful labelling if and only if the set of edge labels is $[1, m]$.
- If G has a graceful labelling then we say G is graceful.

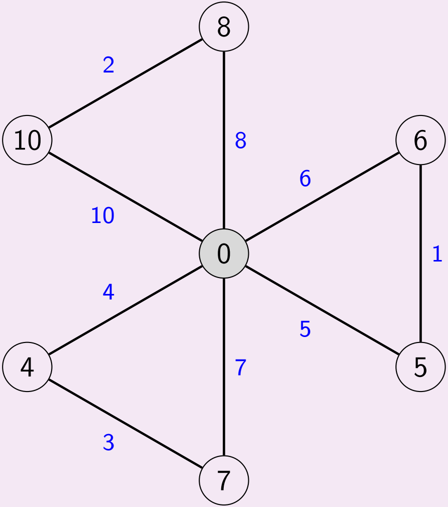
Graceful Labellings



Near Graceful Labellings

- Similar to graceful labellings
- $G = (V, E)$ is a graph with m edges.
- Each $v \in V$ is given a distinct label $\ell(v)$ from $[0, m+1]$.
- Edge labels are induced by the vertex labels: $|\ell(u) - \ell(v)|, \forall uv \in E$.
- The labelling is a near graceful labelling if and only if the set of edge labels is $[1, m]$ or $[1, m+1] \setminus \{m\}$.
- If G has a near graceful labelling then we say G is near graceful.

Near Graceful Labellings



Graceful and Near Graceful Labellings

Conjecture: [Koh, Rogers, Lee, Toh]

C_n^t is graceful if $nt \equiv 0, 3 \pmod 4$.

The "if and only if" version of this conjecture has been shown to hold for

- C_3^t [Bermond]
- C_4^t [Bermond, Brouwer, Germa]
- C_5^t [Yang,Lin,Yu]
- C_6^t [Ma]
- C_{4k}^t [Koh, Rogers, Lee, Toh]

Graceful and Near Graceful Labellings

Graceful labellings have been shown to exist for

- C_7^t [Yang, Xu, Xi, Li, Haque]
- C_9^t [Yang, Xu, Xi, Huijun]
- C_{11}^t [Xu, Yang, Li, Xi]
- C_{13}^t [Xu, Yang, Han, Li]

Skolem Sequences

- A Skolem sequence of order n is a sequence $K = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers.
- $\forall\ k \in \{1, 2, 3, ..., n\}$, $\exists\ s_i, s_j \in K$ such that $s_i = s_j = k$;
- if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $(1, 1, 3, 4, 2, 3, 2, 4)$ is a Skolem sequence of order 4.

λ-fold Skolem Sequences

- A λ-fold Skolem Sequence of order n is a sequence $K = (s_1, s_2, \ldots, s_{2\lambda n})$ of $2\lambda n$ positive integers.
- Each $k \in \{1, 2, 3, ..., n\}$ occurs 2λ times in K.
- These occurrences can be partitioned into λ disjoint pairs, (s_i, s_j), such that $s_i = s_j = k$ and $j - i = k$

For example, $(1, 1, 2, 3, 2, 3, 3, 2, 3, 2, 1, 1)$ is a two-fold Skolem sequence of order 3.

Skolem-like Sequences

- A Skolem-like sequence of order n is a sequence $K = (s_1, s_2, \dots, s_{2n})$ of $2n$ integers.
- $\forall\ k \in H$, where H is a set of n distinct positive integers, $\exists\ s_i, s_j \in K$ such that $s_i = s_j = k$;
- if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $(6, 4, 1, 1, 3, 4, 6, 3)$ is a Skolem-like sequence of order 4 with $H = \{1, 3, 4, 6\}$.

Other Skolem-like Sequences

- A hooked Skolem-like sequence of order n has $2n+1$ positions, but the penultimate position not used ($s_{2n}=0$).
- Example: $(3,1,1,3,2,0,2)$ is a hooked Skolem sequence of order 3.
- A near-Skolem sequence of order n and defect m is a Skolem-like sequence of order $n-1$ with $H=\{1,2,3,...,n\}\setminus\{m\}$.
- Example: $(1,1,6,3,7,5,3,2,6,2,5,7)$ is a 4-near-Skolem sequence of order 7.
- We can combine these ideas as well.
- Example: $(2,5,2,4,6,7,5,4,1,1,6,0,7)$ is a hooked 3-near-Skolem sequence of order 7.

Existence of Skolem-like Sequences

Sequence	Necessary and Sufficient conditions
Skolem sequence of order n	$n \equiv 0, 1 (\mathrm{mod}\ 4)$
hooked Skolem sequence of order n	$n \equiv 2, 3 (\mathrm{mod}\ 4)$
Langford sequence of order l and defect d	$l \geq 2d - 1$, $l \equiv 0, 1 \pmod 4$ and d is odd, or $l \equiv 0, 3 \pmod 4$ and d is even
hooked Langford sequence of order l and defect d	$l\,(l - 2d + 1) + 2 \geq 0$, $l \equiv 2, 3 \pmod 4$ and d is odd, or $l \equiv 1, 2 \pmod 4$ and d is even
m-near-Skolem sequence of order n	$n \equiv 0, 1 \pmod 4$ and m is odd, or $n \equiv 2, 3 \pmod 4$ and m is even
hooked m-near-Skolem sequence of order n	$n \equiv 0, 1 \pmod 4$ and m is even, or $n \equiv 2, 3 \pmod 4$ and m is odd
m-fold Skolem sequence of order n	$n \equiv 0, 1 \pmod 4$ and any m, or $n \equiv 2, 3 \pmod 4$ and m is even
hooked m-fold Skolem sequence of order n	$n \equiv 2, 3 \pmod 4$ and m is odd

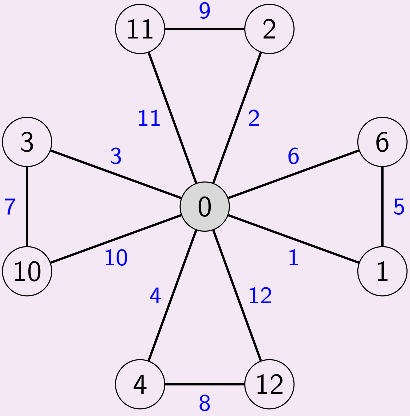
Why Skolem-like Sequences?

- Consider the Skolem sequence of order n.
- Now consider any number, say k, that occurs in the sequence. We often refer to these as differences.
- k occurs in two positions, a_k and b_k, where $a_k < b_k$.
- By definition $b_k - a_k = k$ and so we can form a triple $(0, k, b_k + n)$ which gives pairwise differences k, $a_k + n$, and $b_k + n$.
- Since $1 \leq k \leq n$, and the a_k's and b_k's are all distinct, we get unique labels and differences.
- Note that the triples of the form $(0, a_k + n, b_k + n)$ give the same differences.

C_3^t

- Example: Suppose we want to label C_3^4.
- Consider the Skolem sequence $(1, 1, 3, 4, 2, 3, 2, 4)$.
- For each $1 \leq k \leq 4$, get the triple $(0, k, b_k + 4)$.
- We get $(0, 1, 6)$, $(0, 2, 11)$, $(0, 3, 10)$, and $(0, 4, 12)$.
- These are the vertex labels we want.

C_3^t



C_3^t

We can do a similar construction with hooked Skolem sequences to get near-graceful labellings of C_3^t.

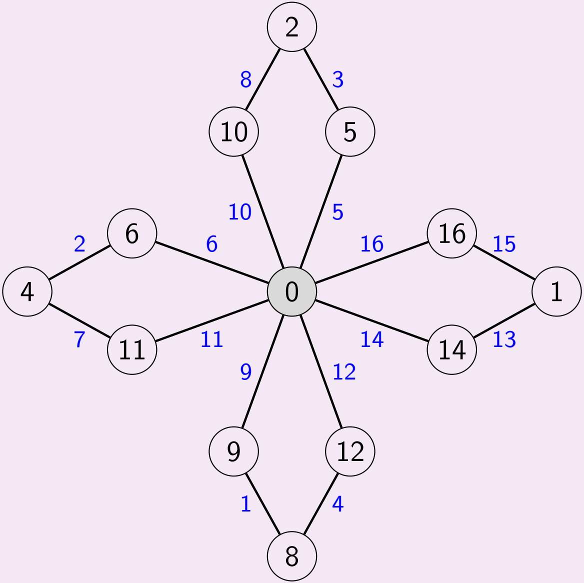
Theorem

C_3^t is graceful when $t \equiv 0, 1 \pmod 4$ and near-graceful when $t \equiv 2, 3 \pmod 4$.

C_4^s

- Example: Suppose we want to label C_4^4.
- Consider the two-fold Skolem-like sequence $(8, 4, 2, 8, 2, 4, 4, 2, 8, 2, 4, 8, 1, 1, 1, 1)$.
- Each difference k occurs in two pairs (a_k, b_k) and (c_k, d_k).
- For each $k \in \{1, 2, 4, 8\}$, get the quadruple $(0, b_k, k, d_k)$.
- We get $(0, 16, 1, 14)$, $(0, 10, 2, 5)$, $(0, 11, 4, 6)$, and $(0, 12, 8, 9)$.
- In the sequence, the positions $\{1, 2, 4, 8\}$ are all left endpoints. So these are the vertex labels we want.

C_4^s



C_4^s

We don't currently have a construction that gives sequences with the desired properties to label all C_4^s. We know the following theorem holds from other constructions.

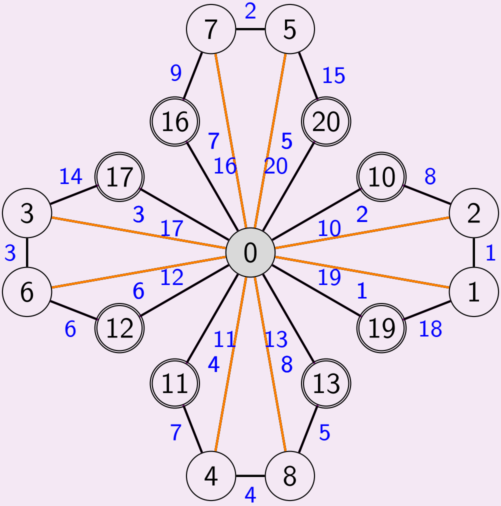
Theorem

C_4^s is graceful.

C_5^p

- Example: Suppose we want to label C_5^4.
- First, consider the Skolem sequence $(1, 1, 3, 4, 2, 3, 2, 4)$, which we used to label C_3^4.
- Next take the triples of the form $(0, a_i, b_i)$, we get $(0, 1, 2)$, $(0, 5, 7)$, $(0, 3, 6)$, and $(0, 4, 8)$.
- We will turn these triples into 5-tuples by replacing each edge incident with the central vertex (0) by two edges.
- Now consider the Skolem sequence $(8, 6, 4, 2, 7, 2, 4, 6, 8, 3, 5, 7, 3, 1, 1, 5)$.
- We replace each triple $(0, a_i, b_i)$ by $(0, b_{a_i} + 4, a_i, b_i, b_{b_i} + 4)$, from this sequence. We get $(0, 19, 1, 2, 10)$, $(0, 20, 5, 7, 16)$, $(0, 17, 3, 6, 12)$, and $(0, 11, 4, 8, 13)$.

C_5^p



7
5
2
9
15
16
20
7
5
16
20
17
10
14
8
3
2
3
17
10
2
0
3
1
6
1
12
19
6
1
6
12
19
18
11
13
11
13
4
8
7
5
4
8
4

C_5^p

This construction relies on the existence of the sequences and the interaction between them.

$p \pmod 4$	0	1	2	3
1st sequence	S,p	S,p	hS,p	hS,p
2nd sequence	S,2p	hS,2p	hnS,2$p+1$	nS,2$p+1$

Theorem

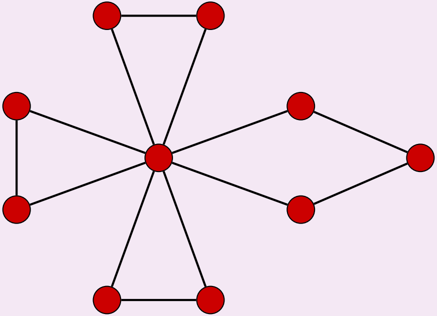
C_5^p is graceful when $p \equiv 0, 3 \pmod 4$ and near graceful when $p \equiv 1, 2 \pmod 4$.

Variable Windmills

- A variable windmill is a graph created from a set of windmills $\mathcal{W} = \{G_1^{t_1}, G_2^{t_2}, ..., G_k^{t_k}\}$.
- To create this variable windmill, identify the set of central vertices from the windmills in $\mathcal{W}$.
- We denote this variable windmill by $G_1^{t_1} G_2^{t_2} ... G_k^{t_k}$.

Variable Windmills

This is the variable windmill $C_3^3 C_4^1$.



This is the main theorem for $C_3^t C_4^s$ windmills. There are several different construction techniques we use to get this.

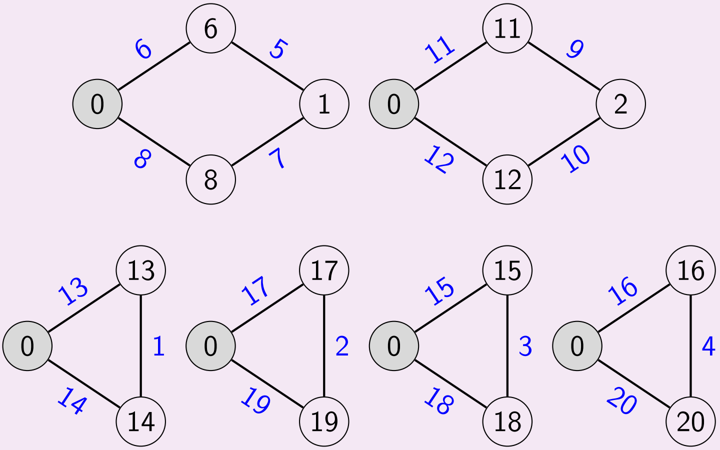
Theorem

$C_3^t C_4^s$ is graceful when $t \equiv 0, 1 \pmod 4$ and near graceful when $t \equiv 2, 3 \pmod 4$

$C_3^t C_4^s$

- Example for $t \geq s \geq 1$: Suppose we want to label $C_3^4 C_4^2$.
- First, consider the 2-fold Skolem sequence $(1, 1, 1, 1, 2, 2, 2, 2)$ and the Skolem sequence $(1, 1, 3, 4, 2, 3, 2, 4)$.
- Next take the quadruples of the form $(0, b_i + 4, i, d_i + 4)$ from the 2-fold Skolem sequence and the triples of the form $(0, a_i + 12, b_i + 12)$ from the Skolem sequence. We get $(0, 6, 1, 8)$, $(0, 11, 2, 12)$, $(0, 13, 14)$, $(0, 17, 19)$, $(0, 15, 18)$, and $(0, 16, 20)$.
- These are the vertex labels we want.

$C_3^t C_4^s$



0
6
1
8
6
5
8
7
0
11
2
12
11
9
12
10
0
13
14
13
1
14
0
17
19
17
2
19
0
15
18
15
3
18
0
16
20
16
4
20

$C_3^t C_4^s$

- Example for $s > 3t+1$ and $t \geq 4$: Suppose we want to label $C_3^4 C_4^{15}$.
- Consider the 2-fold Skolem-like sequence
  $(1, 1, 19, 17, 15, 13, 11, 9, 7, 18, 16, 14, 12, 10, 8, 7, 9, 11, 13, \to$
  $15, 17, 19, 8, 10, 12, 14, 16, 18, 19, 17, 15, 13, 11, 9, 7, 18, 16, \to$
  $14, 12, 10, 8, 7, 9, 11, 13, 15, 17, 19, 8, 10, 12, 14, 16, 18, \to$
  $1, 1, 2, 2, 2, 2)$
- Take the quadruples of the form $(0, b_i + 4, i, d_i + 4)$. We get

$(0, 6, 1, 60)$	$(0, 63, 2, 64)$	$(0, 20, 7, 46)$	$(0, 27, 8, 53)$
$(0, 21, 9, 47)$	$(0, 28, 10, 54)$	$(0, 22, 11, 48)$	$(0, 29, 12, 55)$
$(0, 23, 13, 49)$	$(0, 30, 14, 56)$	$(0, 24, 15, 50)$	$(0, 31, 16, 57)$
$(0, 25, 17, 51)$	$(0, 32, 18, 58)$	$(0, 26, 19, 52)$	

- Use the Skolem sequence $(1, 1, 3, 4, 2, 3, 2, 4)$ and get triples of the form $(0, a_i + 64, b_i + 64)$. We get $(0, 65, 66)$, $(0, 67, 69)$, $(0, 65, 68)$, and $(0, 66, 70)$.
- These are the vertex labels we want.

$C_3^t C_4^s$

- We can manipulate the 2-fold Skolem-like sequence in this example to give us graceful labellings for $C_3^4 C_4^{16}$, $C_3^4 C_4^{17}$, and $C_3^4 C_4^{18}$.
- We leave the red part as is, call it S.
- For $C_3^4 C_4^{16}$, use $(1, 1, S, 1, 1, 2, 3, 2, 3, 3, 2, 3, 2)$ and shift the triples by 68 (4 more).
- For $C_3^4 C_4^{17}$, use $(1, 1, S, 4, 4, 1, 1, 4, 4, 2, 3, 2, 3, 3, 2, 3, 2)$ and shift the triples by 72 (8 more).
- For $C_3^4 C_4^{18}$, use $(1, 1, S, 4, 4, 1, 1, 4, 4, 2, 2, 2, 2, 5, 3, 5, 3, 3, 5, 3, 5)$ and shift the triples by 76 (12 more).
- We can't include 6 as it is already used as a vertex label.

$C_3^t C_5^p$

We also have some partial results for $C_3^t C_5^p$. These all come from the same constructive technique.

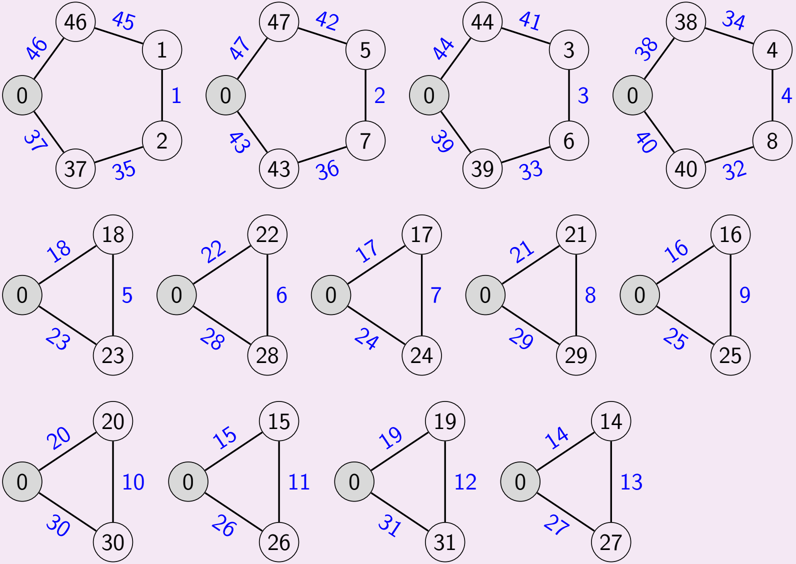
Theorem

For $t \geq 2p+1$, $C_3^t C_5^p$ is
graceful when $(t,p) \equiv (0,0), (0,3), (1,0), (3,3) \pmod 4$ and
near graceful when $(t,p) \equiv (0,1), (0,2), (1,2), (3,1) \pmod 4$.

$C_3^t C_5^p$

- Example: Suppose we want to label $C_3^9 C_5^4$.
- Recall that we already labelled a C_5^4 with $(0, 19, 1, 2, 10)$, $(0, 20, 5, 7, 16)$, $(0, 17, 3, 6, 12)$, and $(0, 11, 4, 8, 13)$.
- Each of these is of the form $(0, b_{a_i} + 4, a_i, b_i, b_{b_i} + 4)$. We replace these by quintuples of the form $(0, b_{a_i} + 4 + 27, a_i, b_i, b_{b_i} + 4 + 27)$. We get $(0, 46, 1, 2, 37)$, $(0, 47, 5, 7, 43)$, $(0, 44, 3, 6, 39)$, and $(0, 38, 4, 8, 40)$.
- Now consider the Langford sequence of order 9 with defect 5, $(13, 11, 9, 7, 5, 12, 10, 8, 6, 5, 7, 9, 11, 13, 6, 8, 10, 12)$.
- From this get the triples of the form $(0, a_i + 13, b_i + 13)$. We get $(0, 18, 23)$, $(0, 22, 28)$, $(0, 17, 24)$, $(0, 21, 29)$, $(0, 16, 25)$, $(0, 20, 30)$, $(0, 15, 26)$, $(0, 19, 31)$, $(0, 14, 27)$.
- These are the vertex labels we want.

$C_3^t C_5^p$

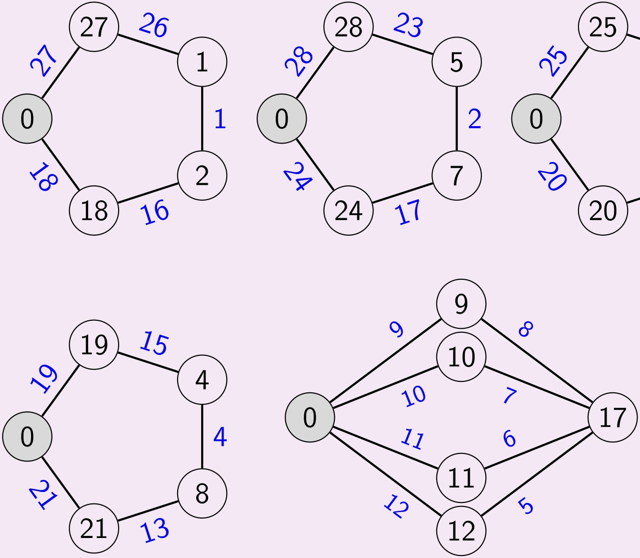

0 46 1 2 37
46 45 1 35 37
0 47 5 7 43
47 42 2 36 43
0 44 3 6 39
44 41 3 33 39
0 38 4 8 40
38 34 4 32 40
0 18 23
18 5 23
0 22 28
22 6 28
0 17 24
17 7 24
0 21 29
21 8 29
0 16 25
16 9 25
0 20 30
20 10 30
0 15 26
15 11 26
0 19 31
19 12 31
0 14 27
14 13 27

GC_5^p

We can use this mechanism (of shifting the labels on the C_5's to get a gap that we can then fill) to label other windmills.

- Example: Suppose we want to label $K_{2,4}C_5^4$.
- We already labelled C_5^4 with $(0, 19, 1, 2, 10)$, $(0, 20, 5, 7, 16)$, $(0, 17, 3, 6, 12)$, and $(0, 11, 4, 8, 13)$.
- This time shift the quintuples by 8, $(0, b_{a_i} + 4 + 8, a_i, b_i, b_{b_i} + 4 + 8)$. We get $(0, 27, 1, 2, 18)$, $(0, 28, 5, 7, 24)$, $(0, 25, 3, 6, 20)$, and $(0, 19, 4, 8, 21)$.
- We can label $K_{2,4}$ avoiding the already used vertex and edge labels using vertex labels $\{0, 17\}$ for one partite set and $\{9, 10, 11, 12\}$ for the other.

GC_5^p



0 27 1 2 18
27 26 1 16 18
0 28 5 7 24
28 23 2 17 24
0 25 3 6 20
25 22 3 14 20
0 19 4 8 21
19 15 4 13 21
0 9 10 11 12 17
9 10 11 12 8 7 6 5

$C_3^t C_6^h$

We also have some partial results for $C_3^t C_6^h$. These all come from the same constructive technique.

Theorem

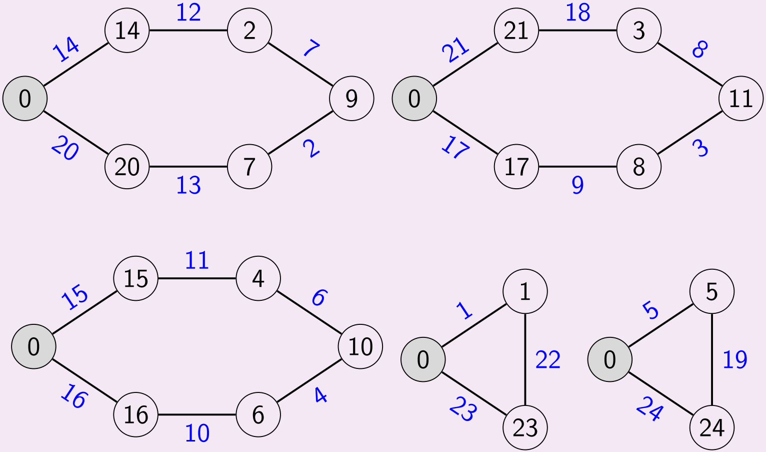
Let $h \le 2t+1$ and $t+2h = 5q+r$, where $0 \le r \le 4$. Also let $q \equiv k \pmod 4$, where $0 \le k \le 3$. $C_3^t C_6^h$ is graceful when $(r,k) =$ (0,0), (0,1), (1,0), (1,3), (2,2), (2,3), (3,1), (3,2), (4,0), (4,1) and near graceful otherwise.

$n = t + 2h$	Graceful	Near graceful
$5k$	$k \equiv 0, 1 \pmod 4$	$k \equiv 2, 3 \pmod 4$
$5k+1$	$k \equiv 0, 3 \pmod 4$	$k \equiv 1, 2 \pmod 4$
$5k+2$	$k \equiv 2, 3 \pmod 4$	$k \equiv 0, 1 \pmod 4$
$5k+3$	$k \equiv 1, 2 \pmod 4$	$k \equiv 0, 3 \pmod 4$
$5k+4$	$k \equiv 0, 1 \pmod 4$	$k \equiv 2, 3 \pmod 4$

$C_3^t C_6^h$

- Example: Suppose we want to label $C_3^2 C_6^3$.
- We can label C_3^8 using the triples of the form $(0, i, b_i + 8)$ from the Skolem sequence $(8, 6, 4, 2, 7, 2, 4, 6, 8, 3, 5, 7, 3, 1, 1, 5)$.
- We get $(0, 1, 23)$, $(0, 2, 14)$, $(0, 3, 21)$, $(0, 4, 15)$, $(0, 5, 24)$, $(0, 6, 16)$, $(0, 7, 20)$, $(0, 8, 17)$.
- We pair up six of these triples and convert each pair $(0, i, b_i + 8)$ and $(0, j, b_j + 8)$ to the 6-tuple $(0, b_i + 8, i, i + j, j, b_j + 8)$.
- We convert $(0, 2, 14)$ and $(0, 7, 20)$ to $(0, 14, 2, 9, 7, 20)$; $(0, 3, 21)$ and $(0, 8, 17)$ to $(0, 21, 3, 11, 8, 17)$; $(0, 4, 15)$ and $(0, 6, 16)$ to $(0, 15, 4, 10, 6, 16)$.
- These 6-tuples and the remaining triples are the vertex labels we want.

$C_3^t C_6^h$


0
14
2
9
7
20
14
12
7
2
13
20
0
21
3
11
8
17
21
18
8
3
9
17
0
15
4
10
6
16
15
11
6
4
10
16
0
1
23
1
22
23
0
5
24
5
19
24

Thank you!