Monarchs for excluded minor classes

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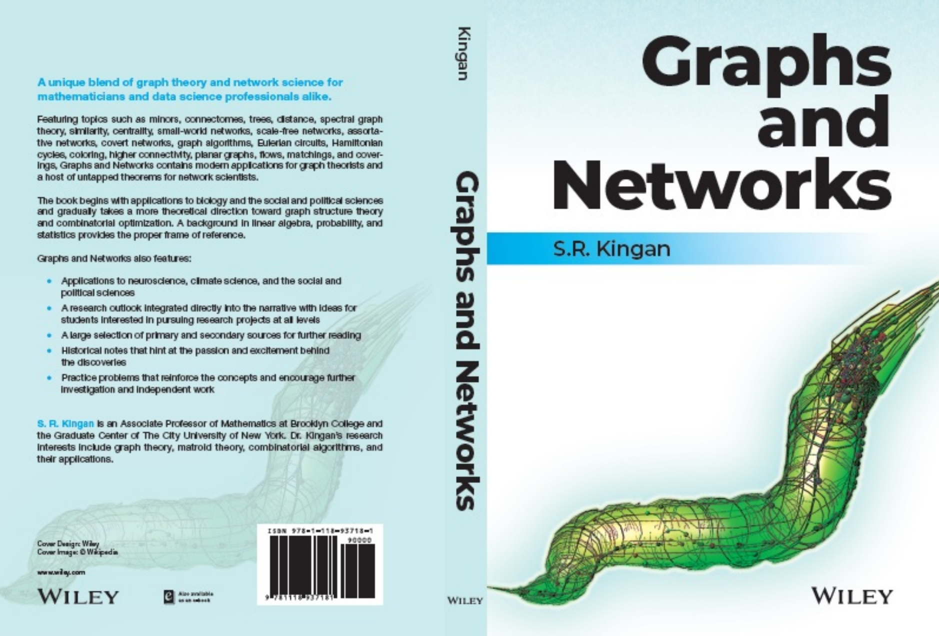

A unique blend of graph theory and network science for mathematicians and data science professionals alike.
Featuring topics such as minors, connectomes, trees, distance, spectral graph theory, similarity, centrality, small-world networks, scale-free networks, assortative networks, covert networks, graph algorithms, Eulerian circuits, Hamiltonian cycles, coloring, higher connectivity, planar graphs, flows, matchings, and coverings, Graphs and Networks contains modern applications for graph theorists and a host of untapped theorems for network scientists.
The book begins with applications to biology and the social and political sciences and gradually takes a more theoretical direction toward graph structure theory and combinatorial optimization. A background in linear algebra, probability, and statistics provides the proper frame of reference.
Graphs and Networks also features:
• Applications to neuroscience, climate science, and the social and political sciences
• A research outlook integrated directly into the narrative with ideas for students interested in pursuing research projects at all levels
• A large selection of primary and secondary sources for further reading
• Historical notes that hint at the passion and excitement behind the discoveries
• Practice problems that reinforce the concepts and encourage further investigation and independent work
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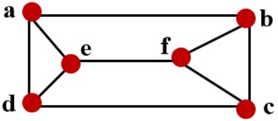
1. What is a matroid?

A graph consists of a set of points called vertices and a set of unordered pairs of points called edges.



$$V(G) = \{a, b, c, d, e, f\}$$
$$E(G) = \{ab, bc, cd, da, ae, ef, de, fb, fc\}$$

The degree of a vertex is the number of edges incident to it.

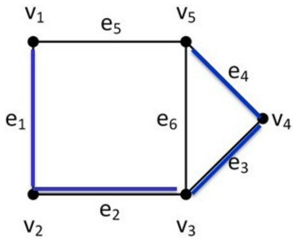
A cycle is a closed path.
Example: $\{a, e, d, a\}$ or $\{a, b, c, d, a\}$

A graph is connected if there is a path from every vertex to every other vertex. Otherwise it is disconnected.

A connected acyclic graph is called a tree.
In a tree with n vertices and m edges $m = n - 1$.

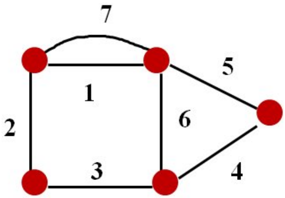
Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A spanning tree T of G is a subgraph that is a tree and $V(T) = V(G)$



Example: Spanning tree $= \{e_1, e_2, e_3, e_4\}$
The rank of a graph on n vertices: $r(G) = n - 1$.

Consider the following graph (with multiple edges):



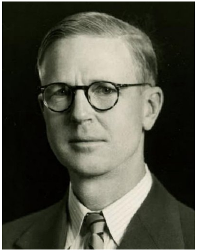
Let E be the set of edges of the graph

$$E = \{1, 2, 3, 4, 5, 6, 7\}$$

Let $\mathcal{C}$ be the set of cycles of the graph

$$\mathcal{C} = \{\{1, 7\}, \{4, 5, 6\}, \{1, 2, 3, 6\}, \{2, 3, 6, 7\}, \{1, 2, 3, 4, 5\}, \{2, 3, 4, 5, 7\}\}$$

H. Whitney (1935). On the abstract properties of linear dependence, *Amer. J. Math.* **54**, 150 – 168.

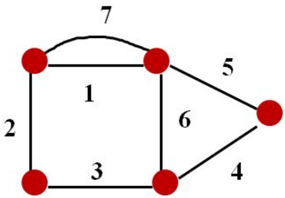


Hassler Whitney (1907-1989)

Whitney observed that cycles of a graph satisfy:

1. If C_1 and C_2 are cycles such that $C_1 \subseteq C_2$, then $C_1 = C_2$; and
2. If $e \in C_1 \cap C_2$, then there is a cycle C_3 such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Example 1:



$E = \{1, 2, 3, 4, 5, 6, 7\}$

$\mathcal{C} = \{\{1, 7\}, \{4, 5, 6\}, \{1, 2, 3, 6\}, \{2, 3, 6, 7\}, \{1, 2, 3, 4, 5\}, \{2, 3, 4, 5, 7\}\}$

The two properties again:

1. If C_1 and C_2 are cycles such that $C_1 \subseteq C_2$, then $C_1 = C_2$; and
2. If $e \in C_1 \cap C_2$, then there is a cycle C_3 such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Example 2: Let A be a matrix over a field.

$$\mathbf{A} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}_{\mathbf{F}_2} \end{array}$$

$$E = \{1, 2, 3, 4, 5, 6, 7\}$$

Independent sets of columns of A

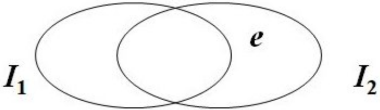
ϕ	{1, 2}	{3, 4}	{1, 2, 3}	{2, 4, 5}	{1, 2, 3, 4}	{2, 4, 6, 7}
{1}	{1, 3}	{3, 5}	{1, 2, 4}	{2, 4, 6}	{1, 2, 3, 5}	{2, 5, 6, 7}
{2}	{1, 4}	{3, 6}	{1, 2, 5}	{2, 4, 7}	{1, 2, 4, 5}	{3, 4, 5, 7}
{3}	{1, 5}	{3, 7}	{1, 2, 6}	{2, 5, 6}	{1, 2, 4, 6}	{3, 4, 6, 7}
{4}	{1, 6}	{4, 5}	{1, 3, 4}	{2, 5, 7}	{1, 2, 5, 6}	{3, 5, 6, 7}
{5}	{2, 3}	{4, 6}	{1, 3, 5}	{2, 6, 7}	{1, 3, 4, 5}	
{6}	{2, 4}	{4, 7}	{1, 3, 6}	{3, 4, 5}	{1, 3, 4, 6}	
{7}	{2, 5}	{5, 6}	{1, 4, 5}	{3, 4, 6}	{1, 3, 5, 6}	
	{2, 6}	{5, 7}	{1, 4, 6}	{3, 4, 7}	{2, 3, 4, 5}	
	{2, 7}	{6, 7}	{1, 5, 6}	{3, 5, 6}	{2, 3, 4, 6}	
			{2, 3, 4}	{3, 5, 7}	{2, 3, 4, 7}	
			{2, 3, 5}	{3, 6, 7}	{2, 3, 5, 6}	
			{2, 3, 6}	{4, 5, 7}	{2, 3, 5, 7}	
			{2, 3, 7}	{4, 6, 7}	{2, 4, 5, 7}	

A maximal independent set is called a **basis**.

A minimal dependent set is called a **circuit**.

Whitney observed that (linearly) independent sets of columns of a matrix satisfy the following properties:

1. If I_2 is an independent set and $I_1 \subseteq I_2$, then I_1 is independent; and
2. If $|I_1| < |I_2|$, then there exists $e \in I_2 - I_1$ such that $I_1 \cup e$ is independent.



Reason: Suppose $I_1 \cup e$ is dependent for all $e \in I_2 - I_1$.
Let S be the span of $I_1 \cup I_2$.

$$dim(S) \geq |I_2|$$

By assumption S is contained in the span of I_1. So $dim(S) \leq |I_1|$.

$$|I_2| \leq dim(S) \leq |I_1| < |I_2|.$$

This is a contradiction

Whitney proved that these two axioms systems are equivalent.

Let E be a set and $\mathcal{C}$ be a set of subsets of E such that

1. If $C_1 \subseteq C_2$, then $C_1 = C_2$; and
2. If $e \in C_1 \cap C_2$, then there is a cycle C_3 such that $C_3 \subseteq (C_1 \cup C_2) - e$; and
3. $\phi \notin \mathcal{C}$

Then $(E, \mathcal{C})$ is a matroid. Members of $\mathcal{C}$ are called **circuits (cycles)**.

Let E be a set and $\mathcal{I}$ be a set of subsets of E such that

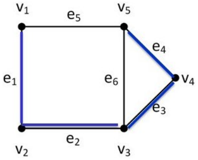
1. If I_2 is an independent set and $I_1 \subseteq I_2$, then I_1 is independentt;
2. If $|I_1| < |I_2|$, then there exists $e \in I_2 - I_1$ such that $I_1 \cup e$ is independent; and
3. $\phi \in \mathcal{I}$

Then $(E, \mathcal{I})$ is a matroid. Members of $\mathcal{I}$ are called **independent sets.**

In a graph, the independents sets are acylic subgraphs.

2. From Graphs to Binary Matroids

Consider the graph



It has incidence matrix

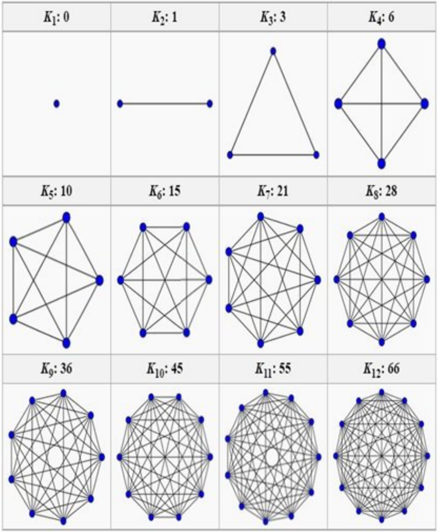
	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	0	0	0	1	0
v_2	1	1	0	0	0	0
v_3	0	1	1	0	0	1
v_4	0	0	1	1	0	0
v_5	0	0	0	1	1	1

Doing elementary row operations we get

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	0	0	0	1	0
v_2	0	1	0	0	1	0
v_3	0	0	1	0	1	1
v_4	0	0	0	1	1	1
v_5	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~

A graph is a special kind of $\{0,1\}$-matrix. A binary matroid is <u>any</u> $\{0,1\}$ matrix.

An n-vertex graph (with rank $r = n - 1$) is a substructure of K_n



Wikipedia

The size of K_n is $\binom{n}{2}$.

A (simple) binary matroid is a substructure of the rank r binary projective geometry PG(r-1, 2).

PG(2,2)

1	2	3	4	5	6	7
1	0	0	0	1	1	1
0	1	0	1	0	1	1
0	0	1	1	1	0	1

PG(3,2)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	1	0	0	0	1	1	1	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1	0	1	1	0	0	1	1
0	0	0	1	1	1	0	1	1	0	1	0	1	0	1

PG(4,2)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	1	0	0	0	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	1	0	1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	0	0	0	1	1	1	0	1	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

PG(5,2)

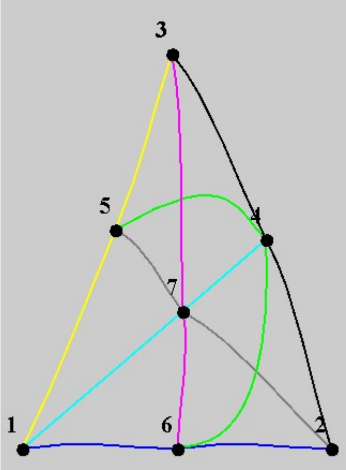
7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	1
0	1	1	1	0	0	0	1	1	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1
1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	1	1	0	0	1	1	0	0
1	1	0	1	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1

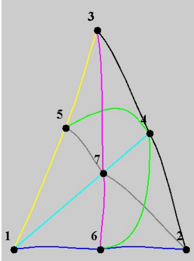
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
1	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
1	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

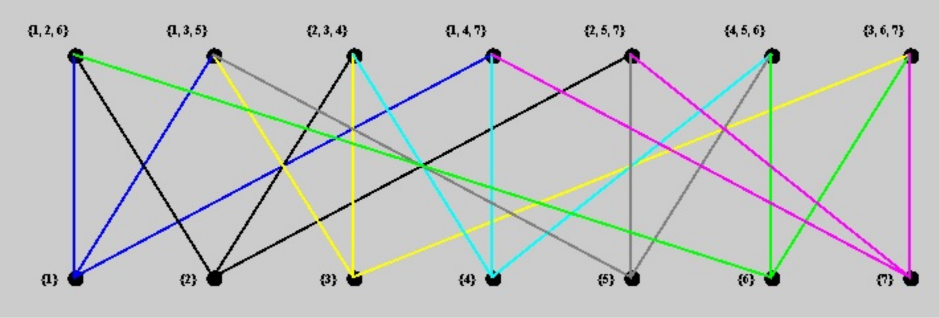
The size of $PG(r-1,2)$ is $2^r - 1$.

Example 4: Fano Matroid:

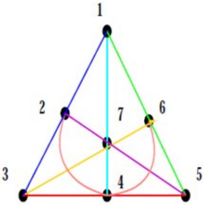
$$PG(2,2) = F_7 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

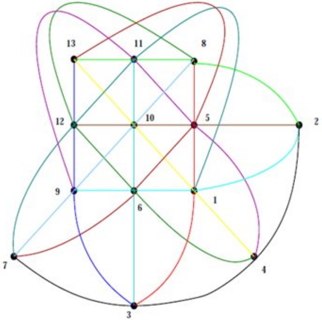



3
5
4
7
1
6
2


{1, 2, 6}
{1, 3, 5}
{2, 3, 4}
{1, 4, 7}
{2, 5, 7}
{4, 5, 6}
{3, 6, 7}
{1}
{2}
{3}
{4}
{5}
{6}
{7}

Example 5: $PG(2,2)$ and $PG(2,3)$

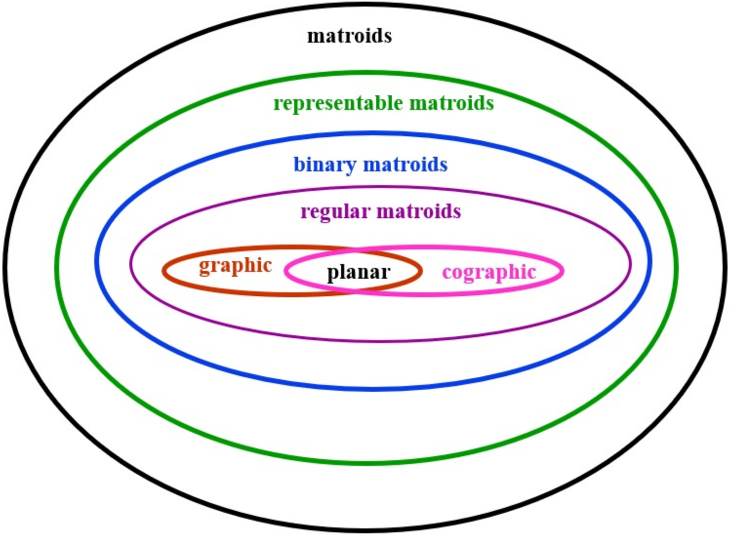




Hassler Whitney (1935) "On the Abstract properties of linear dependence,"

Garrett Birkhoff (1935) "Abstract linear dependence in lattices."

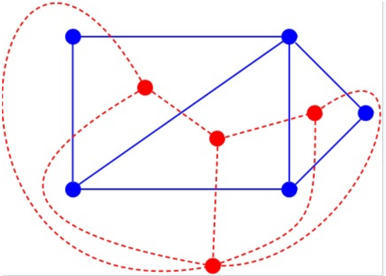
Saunders MacLane (1936) "Some interpretations of abstract linear dependence in terms of projective geometry."


matroids
representable matroids
binary matroids
regular matroids
graphic
planar
cographic

3. Duality

A graph is **planar** if it can be drawn on the plane with no crossing edges.

A planar graph has a **geometric dual**.



The red graph is the dual graph of the blue graph, and *vice versa*.

Wikipedia

Non-planar graphs do not have geometric duals.

Every matroid has a companion matroid called its dual.

If M is a matroid on E and $\mathcal{B}$ is the set of bases, then

$$\mathcal{B}^* = \{E - B \ : \ B \in \mathcal{B}\}$$

also satisfies the Basis Axioms and therefore gives rise to a matroid M^* called the dual matroid.

The dual of an n-element rank r binary matroid

$$A = [I_r|D]$$

over a prime field F_p is the rank $n - r$ binary matroid

$$A^* = [-D^T|I_{n-r}].$$

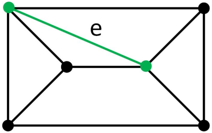
Example: Fano Matroid:

$$F_7 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_3 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \end{array}\right]$$

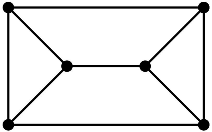
$$F_7^* = \left[\begin{array}{c|ccc} & 0 & 1 & 1 \\ I_4 & 1 & 0 & 1 \\ & 1 & 1 & 0 \\ & 1 & 1 & 1 \end{array}\right]$$

4. Minors

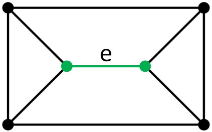
A minor of a graph is obtained by deleting edges and contracting edges.



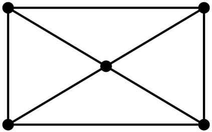
G



G\e

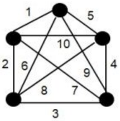


G



G/e

A minor of a graph is obtained by deleting edges and contracting edges.



1	2	3	4	5	6	7	8	9	10
1	0	0	0	1	1	0	0	1	0
0	1	0	0	1	1	1	0	1	1
0	0	1	0	1	0	1	1	1	1
0	0	0	1	1	0	0	1	0	1

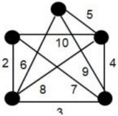
K_5

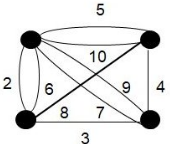
1	2	3	4	5	6	7	8	9	10
1	0	0	0	1	1	0	0	1	0
0	1	0	0	1	1	1	0	1	1
0	0	1	0	1	0	1	1	1	1
0	0	0	1	1	0	0	1	0	1

$K_5 \backslash 1$

1	2	3	4	5	6	7	8	9	10
1	0	0	0	1	1	0	0	1	0
0	1	0	0	1	1	1	0	1	1
0	0	1	0	1	0	1	1	1	1
0	0	0	1	1	0	0	1	0	1

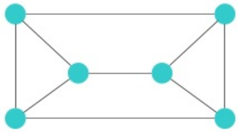
$K_5/1$

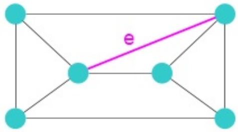




The operations that reverse deletions and contractions are edge additions and vertex splits.

A graph G with an edge e added between non-adjacent vertices is denoted by $G + e$ and called a (simple) edge addition of G.

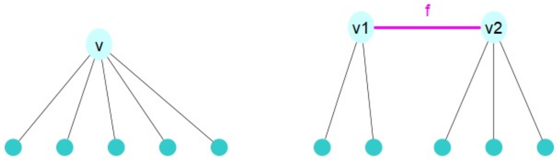




Suppose G is a 3-connected graph with a vertex v such that $deg(v) \geq 4$. To split vertex v:

- Divide $N(v)$ into two disjoint sets S and T, both of size at least 2.
- Replace v with two distinct vertices v_1 and v_2, join them by a new edge $f = v_1v_2$; and
- Join each neighbor of v in S to v_1 and each neighbor in T to v_2.

The resulting graph is called a vertex split of G and is denoted by $G \circ_{S,T} f$.



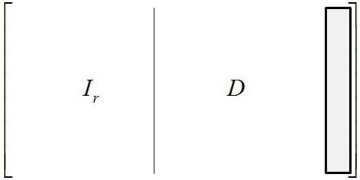
We can get a different graph depending on the assignment of neighbors of v to v_1 and v_2. By a slight misuse of notation, we can say $G \circ f$.

A matroid M with an element e added so that e is not a loop or parallel element is denoted by $M + e$ and is called a simple single-element extension of M.

$$A = [I_r|D].$$

Columns of A may be viewed as a subset of the columns of $PG(r-1, 2)$.

$M + e$ is represented by $[I_r|D']$, where D' is the same as D, but with one additional column corresponding to the new element e. The new column is distinct from the existing columns and has at least two non-zero elements.



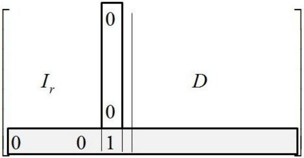
Simple extension

A matroid M with an element f added so that f is not a coloop nor a series element is denoted by $M \circ f$ and called a cosimple single-element coextension of M.

$$A = [I_r | D].$$

The rows of A may be viewed as a subset of the rows of $PG(n-r+1, 2)$.

$M \circ f$ is represented by $[I_{r+1} | D'']$, where D'' is the same as D, but with one additional row. The new row is distinct from the existing rows and has at least two non-zero elements.



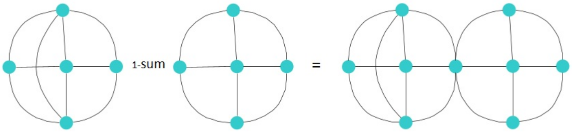
Cosimple coextension

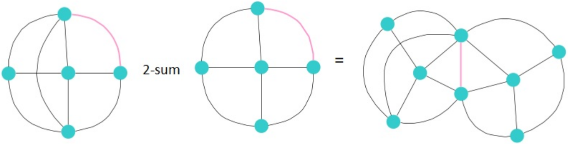
5. Higher Connectivity

A graph is 3-connected if at least 3 vertices must be removed to disconnect the graph.

Theorem

A graph that is not 3-connected can be constructed from its 3-connected proper minors using 1-sums and 2-sums.





Let M be a matroid with ground set E. For every $A \subseteq E$, the connectivity function λ is defined as

$$\lambda(A) = r(A) + r(E - A) - r(M)$$

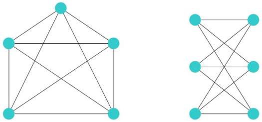
M is k-connected if $\lambda(A) \geq k - 1$ for $|A|, |E - A| \geq k$

M is 3-connected if $\lambda(A) \geq 2$ for $|A|, |E - A| \geq 3$

Theorem (Bixby 1972)

A matroid that is not 3-connected can be constructed from its 3-connected proper minors using 1-sums and 2-sums.

Let $\mathcal{G}$ be a class of graphs closed under minors. We say H is a minimal excluded minor for $\mathcal{G}$ if $H \notin \mathcal{G}$, but every proper minor of H is in $\mathcal{G}$

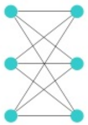


Theorem 1. (Kuratowski-Wagner 1937)

A graph is planar if and only if it has no K_5 or $K_{3,3}$ minor.

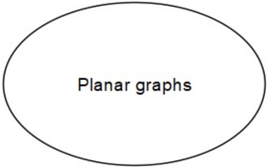
Kuratowski, K. (1930). Sur le probleme des courbes gauches en topologie. *Fund. Math.* **15**, 271–283.

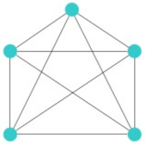
Structure result

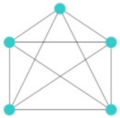


Corollary 2.

Let G be a simple 3-connected graph. Then G has no $K_{3,3}$ minor if and only if G is planar or $G \cong K_5$.

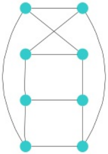


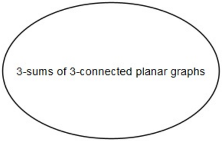




Theorem 3. (Wagner 1937)

Let G be a 3-connected graph. Then G has no K_5-minor if and only if $G \cong V_8$ or G is the <u>3-sum of 3-connected planar graphs</u>.





Decomposition result

$$U_{2,4} = \left[\begin{array}{c|cc} I_2 & 1 & 1 \\ & 1 & 2 \end{array}\right]$$

This matrix has entries from F_3.

Theorem 4. (Tutte, 1958)

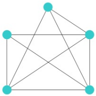
A matroid is binary if and only if it has no $U_{2,4}$ minor.

$$F_7 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_3 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \end{array}\right]$$

Theorem 5. (Tutte, 1959)

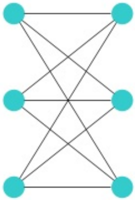
A matroid is regular if and only if it has no F_7 or F_7^* minors .

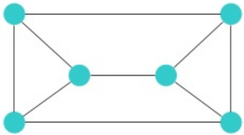
Structure results

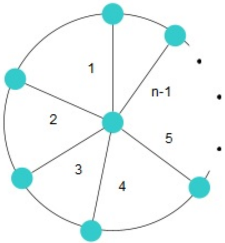


Theorem 6. (Wagner 1960)

Let G be a simple 3-connected graph. Then G has no $K_5\backslash e$ minor if and only if $G \cong K_{3,3}$, $(K_5\backslash e)^*$ or W_{n-1} for $n \geq 4$.

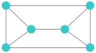






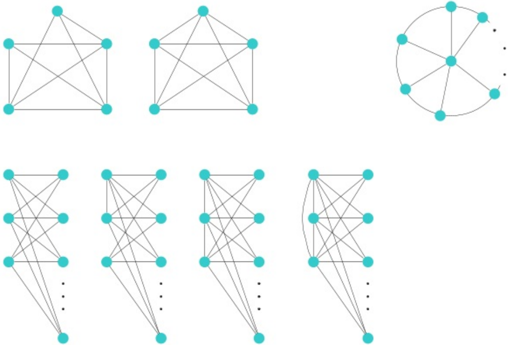
Monarch result

G. A. Dirac (1963). Some results concerning the structure of graphs, *Canad. Math. Bull.* **6**, 183–210.

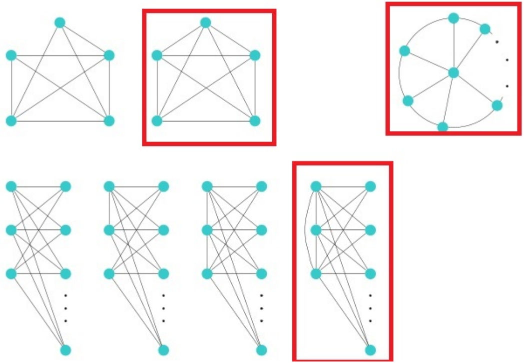


Theorem 7. (Dirac 1963)

Let G be a simple 3-connected graph. Then G has no prism minor if and only if G is isomorphic to $K_5 \backslash e$, K_5, W_{n-1} for $n \geq 4$, $K_{3,n-3}$, $K'_{3,n-3}$, $K''_{3,n-3}$, or $K'''_{3,n-3}$ for $n \geq 6$.



A rank r 3-connected graph in a minor closed class $\mathcal{G}$ is called a rank r monarch for $\mathcal{G}$ if it has no further edge additions in $\mathcal{G}$.



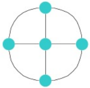
K_5, W_{n-1}, and $K'''_{3,n-3}$ are the monarchs

What other decomposition and monarch results have been classified?

H	$\|E(H)\|$	3-connected H-free graphs
K_4	6	$\emptyset$
W_4	8	$\{K_4\}$
$K_5 \setminus e$	9	$\{K_{3,3}, Prism\} \cup \{W_n\}$
$Prism$	9	$\{K_5\} \cup \{W_n\} \cup \{K_{3,n}\}^{\downarrow}$
$K_{3,3}$	9	$\{K_5\} \cup$ {3-connected planar graphs}
$Prism + e$	10	$\{K_5, Prism\} \cup \{W_n\} \cup \{K_{3,n}\}^{\downarrow}$
$K_{3,3} + e$	10	$\{K_{3,3}, K_5\} \cup$ {3-connected planar graphs}
W_5	10	$\{K_5^{\perp}, Cube, Oct, Pyramid\}^{\downarrow} \cup \{K_{3,n}\}^{\downarrow}$
K_5	10	$\{V_8\} \cup$ {3-sums of 3-connected planar graphs}
$Cube/e$	11	Augmentations of graphs in $\mathcal{G}_{3.3}$
$K_5^{\perp}$	11	$\{K_5, V_8\} \cup$ {3-sums of 3-connected planar graphs}
$K_{3,3}^{\nabla}$	11	$\{K_6\}^{\downarrow} \cup \{K_{3,n}\}^{\downarrow} \cup$ {3-connected planar graphs}
$K_{3,3}^{\ddagger}$	11	$\mathcal{G}_{4.3} \cup$ {3-connected planar graphs}
$W_5 + e$	11	$\mathcal{G}_{4.6} \cup \{W_n\} \cup \{K_{3,n}\}^{\downarrow}$
$Oct \setminus e$	11	$\{V_8, K_5^{\Delta}, Cube\}^{\downarrow} \cup \mathcal{S}$
$(W_5+e)^*$	11	$\{K_6, K_{4,4}, Petersen\}^{\downarrow} \cup \mathcal{G}_{4.12} \cup \{W_n\}$

G. Ding and C. Liu (2013) Excluding a small minor, *Discrete Applied Mathematics* 161 355-368.

Oxley, J. G. (1987). The binary matroids with no 4-wheel minor, *Trans. Amer. Math. Soc.* **154**, 63-75.



Theorem 9 (Oxley, 1987).

Let M be a 3-connected binary matroid. Then M has no minor isomorphic to W_4 if and only if M is isomorphic to $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,3}$, W_3, F_7, F_7^*, or Z_r, Z_r^*, $Z_r \backslash a_r$ or $Z_r \backslash c_r$, for $r \geq 4$.

Monarch:

$$
\begin{array}{ccc}
 & \begin{array}{ccc} b_1 & \cdots & b_r \end{array} & \begin{array}{cccccc} a_1 & a_2 & \cdots & a_{r-1} & a_r & c_r \end{array} \\
Z_r = & \left[\begin{array}{c} \\ \\ I_r \\ \\ \\ \end{array} \right. & \left| \begin{array}{cccccc} 0 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 1 \end{array} \right]
\end{array}
$$

The proof of Oxley's 1987 result classifying the matroids with no W_4-minor comes down to the following result:

$$P_9 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad P_9^* = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 \end{array}\right]$$

Theorem 9' (Oxley, 1987).

Let M be a binary non-regular 3-connected matroid. Then M has no P_9 or P_9^* if and only if M is isomorphic to F_7, F_7^*, or Z_r, Z_r^*, $Z_r \backslash a_r$ or $Z_r \backslash c_r$, for $r \geq 4$.

Monarch:

$$Z_r = \begin{array}{c} \begin{array}{ccccccccc} b_1 & \cdots & b_r & a_1 & a_2 & \cdots & a_{r-1} & a_r & c_r \end{array} \\ \left[\begin{array}{c|cccccc} & 0 & 1 & \cdots & 1 & 1 & 1 \\ & 1 & 0 & \cdots & 1 & 1 & 1 \\ I_r & 1 & 1 & \cdots & 1 & 1 & 1 \\ & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & 1 & 1 & \cdots & 0 & 1 & 1 \\ & 1 & 1 & \cdots & 1 & 0 & 1 \end{array}\right] \end{array}$$

S. R. Kingan (2018) A short proof of binary matroids with no 4-wheel minor, *Australasian Journal of Combinatorics*, 72(2), 201–205.

S. R. Kingan (2021). Finding monarchs in excluded minor classes of matroids, *Australasian Journal of Combinatorics, 79(3), 302–326.*

Theorem 10(SRK, 2021).

A binary 3-connected non-regular matroid M has no P_9^*-minor if and only if M is isomorphic to F_7, $PG(3,2)$, R_{16}, Z_r for $r \geq 4$, Ω_r for $r \geq 5$, or one of their 3-connected deletion-minors.

Monarchs

$$F_7 = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_3 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \end{array}\right]$$

$$Z_r = \begin{array}{c} \begin{array}{ccccccccc} b_1 & \cdots & b_r & a_1 & a_2 & \cdots & a_{r-1} & a_r & c_r \end{array} \\ \left[\begin{array}{c|cccccc} & 0 & 1 & \cdots & 1 & 1 & 1 \\ & 1 & 0 & \cdots & 1 & 1 & 1 \\ I_r & 1 & 1 & \cdots & 1 & 1 & 1 \\ & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & 1 & 1 & \cdots & 0 & 1 & 1 \\ & 1 & 1 & \cdots & 1 & 0 & 1 \end{array}\right] \end{array}$$

$$PG(3,2) = \left[\begin{array}{c|ccccccccccc} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

New Monarchs:

$$R_{16} = \left[\begin{array}{c|ccccccccccc} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c|ccccc|cc|cc|c|cccc|ccccccc|c|cc} b_1 \; \ldots \; b_r & a_1 & a_2 & a_3 & a_4 & a_5 & c_5 & d_5 & c_6 & d_6 & \ldots & c_{r-2} & d_{r-2} & c_{r-1} & d_{r-1} & c_r & d_r & e_r & f_r & g_{r,1} & g_{r,2} & g_{r,3} & \ldots & g_{r,r-5} & g_{r,r-4} \\ \hline & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & 0 & 0 \\ I_r & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 \\ & \vdots & & & & & & & & \vdots & \ddots & & & \vdots & \vdots & & & & & & & & \ddots & & \\ & & & & & & & & & & & 1 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 1 \end{array}$$

Ω_r with $4r - 5$ elements, where $r \geq 5$

$EX[M_1, \ldots, M_t]$: Class with no minors isomorphic to $M_1, \ldots, M_t$

Class	Rank r Monarch	Size
Graphs	K_{r+1}	$\binom{r+1}{2}$
Binary matroids	$PG(r-1, 2)$	$2^r - 1$
Planar graphs	? ? ?	$3r - 3$
Prism-free graphs (Dirac 1963)	K_5, W_r, $K'''_{3,p}$	$3r - 3$
Binary non-regular $EX[P_9, P_9^*]$ (Oxley 1989)	Z_r	$\binom{r+1}{2}$
Binary non-regular $EX[P_9^*]$ (SRK 2021)	Z_r and $\mathbf{\Omega_r}$	$\binom{r+1}{2}$

Table: Monarchs for excluded minor classes

7. Technique: Strong Splitter Theorem

The monarch Ω_r again:

$b_1 \ldots b_r$	a_1	a_2	a_3	a_4	a_5	c_5	d_5	c_6	d_6	…	c_{r-2}	d_{r-2}	c_{r-1}	d_{r-1}	c_r	d_r	e_r	f_r	$g_{r,1}$	$g_{r,2}$	$g_{r,3}$	…	$g_{r,r-5}$	$g_{r,r-4}$
	0	1	1	1	1	1	0	1	0		1	0	1	0	1	0	1	0	1	1	1		1	1
	1	0	1	1	1	1	0	1	0		1	0	1	0	1	0	1	0	1	1	1		1	1
	1	1	0	1	0	0	1	0	1		0	1	0	1	0	1	1	1	1	1	1		1	1
	1	1	1	1	0	0	1	0	1	…	0	1	0	1	0	1	0	1	1	1	1	…	1	1
	0	0	0	1	1	0	0	0	0		0	0	0	0	0	0	0	1	0	0	0		0	0
I_r	0	0	0	0	0	1	1	0	0		0	0	0	0	0	0	0	0	0	1	0		0	0
	0	0	0	0	0	0	0	1	1		0	0	0	0	0	0	0	0	0	0	1		0	0
	⋮								⋮	⋱			⋮	⋮								⋱		
											1		0	0	0	0	0	0	0	0	0		1	0
	0	0	0	0	0	0	0	0	0		0	0	1	1	0	0	0	0	0	0	0		0	1

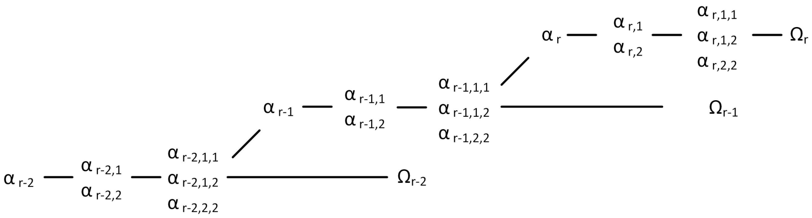
Monarch Ω_r with $4r - 5$ elements, where $r \geq 5$

Let $\mathcal{M}$ be a class of matroids closed under minors. A rank r 3-connected matroid in $\mathcal{M}$ that has no further 3-connected extensions in $\mathcal{M}$ is called a rank r monarch for $\mathcal{M}$.

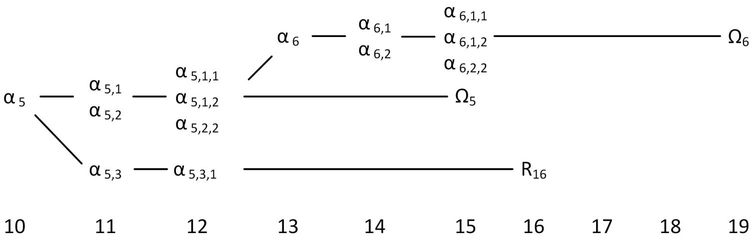
b_1	…	b_r	a_1	a_2	a_3	a_4	a_5	c_5	d_5	c_6	d_6	c_7	d_7	…	c_{r-1}	d_{r-1}
			0	1	1	1	1	1	0	1	0	1	0		1	0
			1	0	1	1	1	1	0	1	0	1	0		1	0
			1	1	0	1	0	0	1	0	1	0	1		0	1
			1	1	1	1	0	0	1	0	1	0	1	…	0	1
			0	0	0	1	1	0	0	0	0	0	0		0	0
	I_r		0	0	0	0	0	1	1	0	0	0	0		0	0
			0	0	0	0	0	0	0	1	1	0	0		0	0
			0	0	0	0	0	0	0	0	0	1	1		0	0
								⋮						⋱	⋮	⋮
															0	0
			0	0	0	0	0	0	0	0	0	0	0	…	1	1

Seed α_r with $3r-5$ elements, where $r \geq 5$

A rank r 3-connected matroid in $\mathcal{M}$ is called a rank r seed for $\mathcal{M}$ if removal of any element results in a matroid that is not 3-connected or not in $\mathcal{M}$.



Growth pattern of the seed and monarch



Base case of the induction argument

S. R. Kingan and M. Lemos (2014). Strong Splitter Theorem, Annals of Combinatorics, 18 – 1, 111 – 116.

Let $\mathcal{M}$ be a class of matroids closed under minors.

The Strong Splitter Theorem implies that every 3-connected rank r monarch in $\mathcal{M}$ is a simple extension of a 3-connected rank r matroid M_r, where M_r is obtained from a 3-connected rank $r-1$ matroid M_{r-1} in the following ways:

(1) $M_r = M_{r-1} \circ f$;

(2) $M_r = M_{r-1} + e \circ f$, where f is added in series to an element in M_{r-1}; or

(3) $M_r = M_{r-1} + \{e_1, e_2\} \circ f$, where $\{e_1, e_2, f\}$ is a triad.

There is no reason to asssume *a priori* that M_r is unique for a specific excluded minor class. However, if M_r happens to be unique, we get a recursive way of defining it, and consequently a recursive way of defining the corresponding rank r monarch.

Thank you for your attention!