The Iterated Local Model for Social Networks

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Atlantic Graph Theory Seminar
Last Talk of 2020!
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Outline

Complex Networks

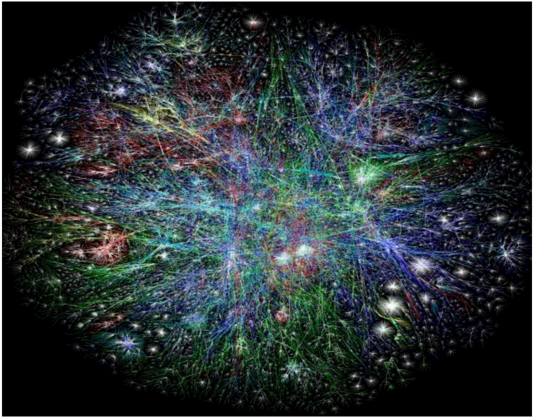


Figure: The Web Graph

Complex Networks

Five main properties:

1. Large-scale
2. Evolving over time
3. Power law degree distribution
4. Small world property
5. Densification

Power Law Degree Distribution

Degree Distribution: $\{N_{k,G} : 0 \leq k \leq n\}$

$$N_{k,G} = |\{x \in V(G) : \deg_G(x) = k|$$

Power Law: for $1 < \beta \in \mathbb{R}$, and interval of $k \in \mathbb{N}$

$$\frac{N_{k,G}}{n} \approx k^{-\beta}.$$

Small World Property

The average distance is

$$L(G) = \frac{\sum_{u,v \in V(G)} d(u,v)}{\binom{|V(G)|}{2}}$$

The clustering coefficient of G is defined as follows:

$$C(G) = \frac{1}{|V(G)|} \sum_{x \in V(G)} C_x(G), \quad \text{where} \quad C_x(G) = \frac{\left|E\left(G[N_G(x)]\right)\right|}{\binom{\deg(x)}{2}}.$$

Densification

A sequence of graphs $\{G_t : t \in \mathbb{N}\}$ densifies over time if

$$\lim_{t\to\infty} \frac{|E(G_t)|}{|V(G_t)|} \to \infty$$

Preferential Attachement

Preferential Attachment Model: Barabasi and Albert 1999

- Fix $m \in \mathbb{N}$
- Begin with K_2
- Add a new vertex with m edges, neighbors chosen by:

$$\frac{deg_{G_t} v_s}{2(mt+1)}$$

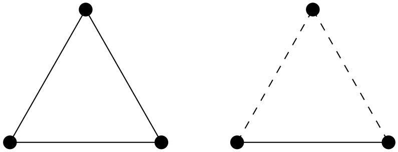
ACL - Preferential Attachement

ACL PA Model: Aiello, Chung, Lu 2001

- Fix $p \in (0, 1)$
- Begin with G_0 single vertex and loop
- Take a vertex step with probability p and an edge step with probability $1 - p$
  - Vertex Step: Add a new vertex with edge uv with u chosen by:
  $$\frac{deg_{G_t} u}{2(mt + 1)}$$
  - Edge Step: Add edge $u_1 u_2$ with both u_i's chosen independently by:
  $$\frac{deg_{G_t} u_i}{\sum_{v \in G_t} \deg(v)}$$

Structural Balance Theory

Representing adversarial relationships with $(-)$ and friendly relationships with $(+)$, Structural Balance Theory says triads seek a positive product of edge signs, called closure.



ILT

Iterated Local Transitivity Model (ILT) (2009, Bonato, Hadi, Horn, Prałat, Wang)

Input: G_0

To form G_t at time t clone each $x \in V(G_{t-1})$ by adding a new node x' such that

$$N_{G_t}(x') = N_{G_{t-1}}[x]$$

Example of ILT

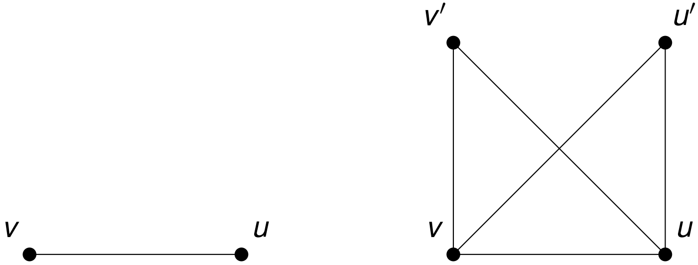


Figure: Example of one time step of ILT with $G_0 = K_2$.

Iterated Local Anti-Transitivity

Iterated Local Anti-Transitivity Model (ILAT) (2017, Bonato, Infeld, Pokhrel, Prałat)

Input: G_0

To form G_t at time t anti-clone each $x \in V(G_{t-1})$ by adding new node x^* such that

$$N_{G_t}(x^*) = V(G_{t-1}) \backslash N_{G_{t-1}}[x]$$

Example of ILAT

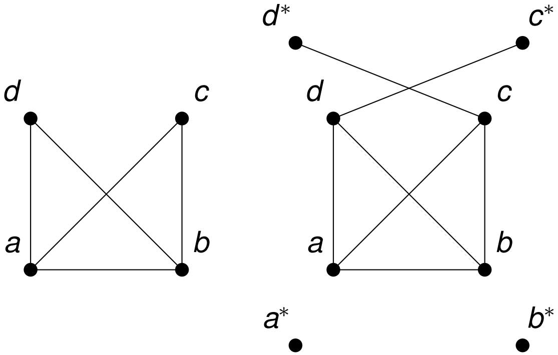


Figure: Example of one time step ILAT.

ILM

Iterated Local Model (ILM) (2019+, Bonato, Chuangpishit, English, Kay, M.)

Input: G_0 and $S = \{b_i\}_{i \in \mathbb{N}}$, where $b_i \in \{0, 1\}$

To form $\mathrm{ILM}_{t,S}(G_0)$ at time t:

- if $b_t = 1$ add a clone x' for each $x \in V(G_{t-1})$ with

$$N_{G_t}(x') = N_{G_{t-1}}[x]$$

- if $b_t = 0$ add an anti-clone x^* for each $x \in V(G_{t-1})$ with

$$N_{G_t}(x^*) = V(G_{t-1}) \setminus N_{G_{t-1}}[x]$$

Example of ILM

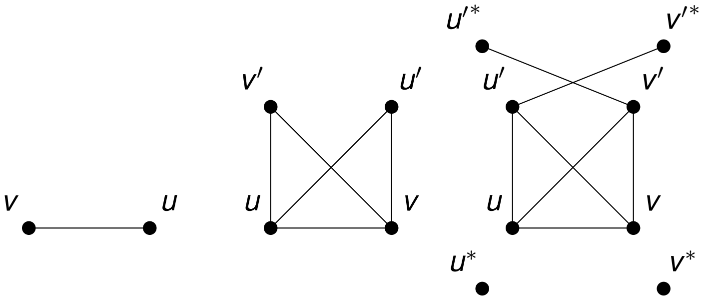


Figure: Example of ILM using $G_0 = K_2$ and $S = \{1, 0, \ldots\}$

Size, Evolution, and Densification

Theorem (2019+, BCEKM) Given any graph, G_0, and any binary sequence, S, with at least one zero, then at time step t

$$|E(\mathrm{ILM}_{t,S}(G))| = \Theta\left(2^{t+\beta}\left(\frac{3}{2}\right)^{t-\beta}\right) = \Theta\left(2^{\beta}\left(\frac{3}{2}\right)^{t-\beta} n_t\right)$$

Where τ is the first index such that $s_\tau = 0$, and β is the largest index such that $s_\beta = 0$.

Size, Evolution and Densification

So far, ILM exhibits 3 of the 4+1 complex network properties

1. Large Scale
2. Evolving over time
5. Densification

Low Diameter

Theorem (2019+, BCEKM) Given $G \neq K_1$ be a graph that is not the disjoint union of two cliques, and a sequence with at least two zeroes, then

$$\mathrm{diam}(\mathrm{ILM}_{t,S}(G)) = 3$$

Low Diameter

Lemma $2 \leq \text{diam}(G) = \text{diam}(\text{LT}(G))$ and $2 \leq \text{radius}(G) = \text{radius}(\text{LT}(G))$.

Proof For any $u, v \in V(G)$ with $uv \notin E(G)$

$\text{dist}_G(u, v) = \text{dist}_{LT(G)}(u, v)$ and $\text{dist}_G(u, v) = \text{dist}_{LT(G)}(u', v')$

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When $uv \in E(G)$, $\text{dist}_{LT(G)}(u', v') = 2$

Low Diameter

Proof Sketch

- LAT(G) has radius at least 3 since $\text{dist}(x, x^*) \geq 3$
- Lemma: If $\gamma(G) \geq 3$ then $\text{diam}(\text{LAT}(G)) \leq 3$

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- Find x, y whose closed neighborhoods partition the vertex set

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- Find x, y whose closed neighborhoods partition the vertex set
- Pairs of vertices in $N[x]$ or in $N[y]$ are distance 2.
- If $u \in N[x]$ has a neighbour in $N[y]$
- Otherwise, we get this picture using a counting argument and case analysis

Clustering

Theorem (2019+, BCEKM) Given a sequence with bounded gaps between zeroes, and k a constant such that there are no gaps of length k,

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The clustering coefficient is bounded away from zero.

Small World

With non-zero clustering and low diameter, ILM exhibits its 4^{th} and final property

4. Small World Property

Induced Subgraphs

Theorem (2019+, BCEKM) If F is a graph, then there exists some constant $t_0 = t_0(F)$ such that for all $t \geq t_0$, all graphs G, and all binary sequences S, F is an induced subgraph of $\mathrm{ILM}_{t,S}(G)$.

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Proof (sketch):
Show $ILT_k(K_1)$ contains induced copy of F

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- One transitive step will increase the r for the induced copy of ILT
- Two anti-transitive steps will similarly increase the induced copy of ILT
- In any binary sequence of length $2r$ there exist r 0s or r 1s.

Hamiltonicity

Theorem For $G \neq K_1$ and S a binary sequence with at least two non-consecutive zeros, then $\text{ILM}_{t,S}(G) = G_t$ is Hamiltonian.

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Definition Let $\zeta(G)$ be the first value such that $ILT_{\zeta(G)}$ is Hamiltonian, and let ζ_n be the maximum over all graphs of order n.

Theorem For all $n \geq 3$

$$\log_2(n-1) \leq \zeta_n \leq \lceil \log_2(n-1) \rceil + 1$$

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- Using $\Delta(G_t) = \frac{n_t}{2} - 1$, the compliment of G_t is Hamiltonian by Dirac's theorem
- Four clone vertices form a clique in the HC of $\overline{G_t}$
- Find two cycles that partition the vertex set and perform an edge switch

Other Structural Properties

For the model with certain restrictions on the input sequence and graph:

- $\chi(G) + t - 1 \leq \chi\left(\text{ILM}_{t,S}(G)\right) \leq \chi(G) + t$
- $\gamma(\text{ILM}_{t,S}(G)) \leq 3$

Future Directions

- Graph Limits
- Domination number in remaining cases
- Randomization of the model
- Improve Clustering, still unknown for ILAT

Thank You