Chapter 2: Probability

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Outline

- Basics of Probability
- Computing Probabilities of Events
- Conditional Probability and Bayes' Rule

Basics of Probability

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Terminologies

- Experiment: a process that produces a set of possible outcomes.
- Sample space: the collection of all possible outcomes.
- Events: a subset of sample space.
- Probability: the chance that an uncertain event will occur (always between 0 and 1)

Experiment

An Experiment is any action or process whose outcome subject to uncertainty.

- Examples:
  - Rolling a dice
  - Tossing a coin
  - Forecasting the weather of tomorrow

Sample Space

The sample space of an experiment, usually denoted as S, is the set of all possible outcomes of that experiment.

- Examples:
  - S = {1, 2, 3, 4, 5, 6}
  - S = {H, T}
  - S = {Sunny, Rain, Cloudy, Snow, . . . }

Each outcome in a sample space is called an element or a member of the sample space, or simply a sample point.

Question: What’s the sample space of flipping two coins?

Event

An event is a set of outcomes of an experiment (a subset of the sample space, $A \subset S$).

- We say that an event A occurs if the outcome (the result) of the experiment is an element of A.
- Simple event: contains one outcome. (e.g. A = {Sunny})
- Compound event: contains at least two outcomes.

  (e.g. B = {Sunny, Cloudy})
- Impossible event or Null event ($\varnothing$): contains no outcome. (e.g. get a 7 when rolling a six-sided dice)
- Sure event: contains the whole sample space. (e.g. get a number less than 10 when rolling a dice)

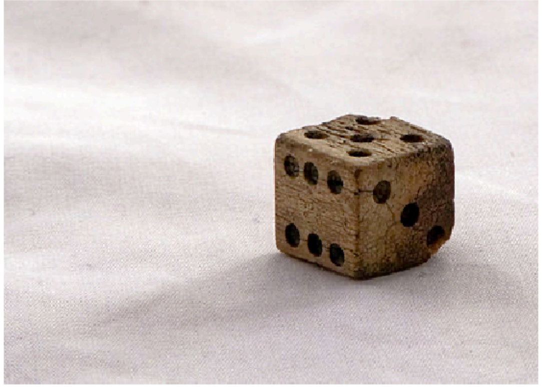
Operations with events

An event is just a set, so relationships and results from elementary set theory can be used to study events.

- Complement of an event A (denoted by A′ or A^c) is the set of all outcomes in S but not in A.
- Union of two events A and B (A ∪ B) is the set of outcomes in either A or B or both.
- Intersection of two events A and B (A ∩ B) is the set of outcomes in both A and B.
- A and B are mutually exclusive or disjoint events when $A \cap B = \varnothing$.

Exercise

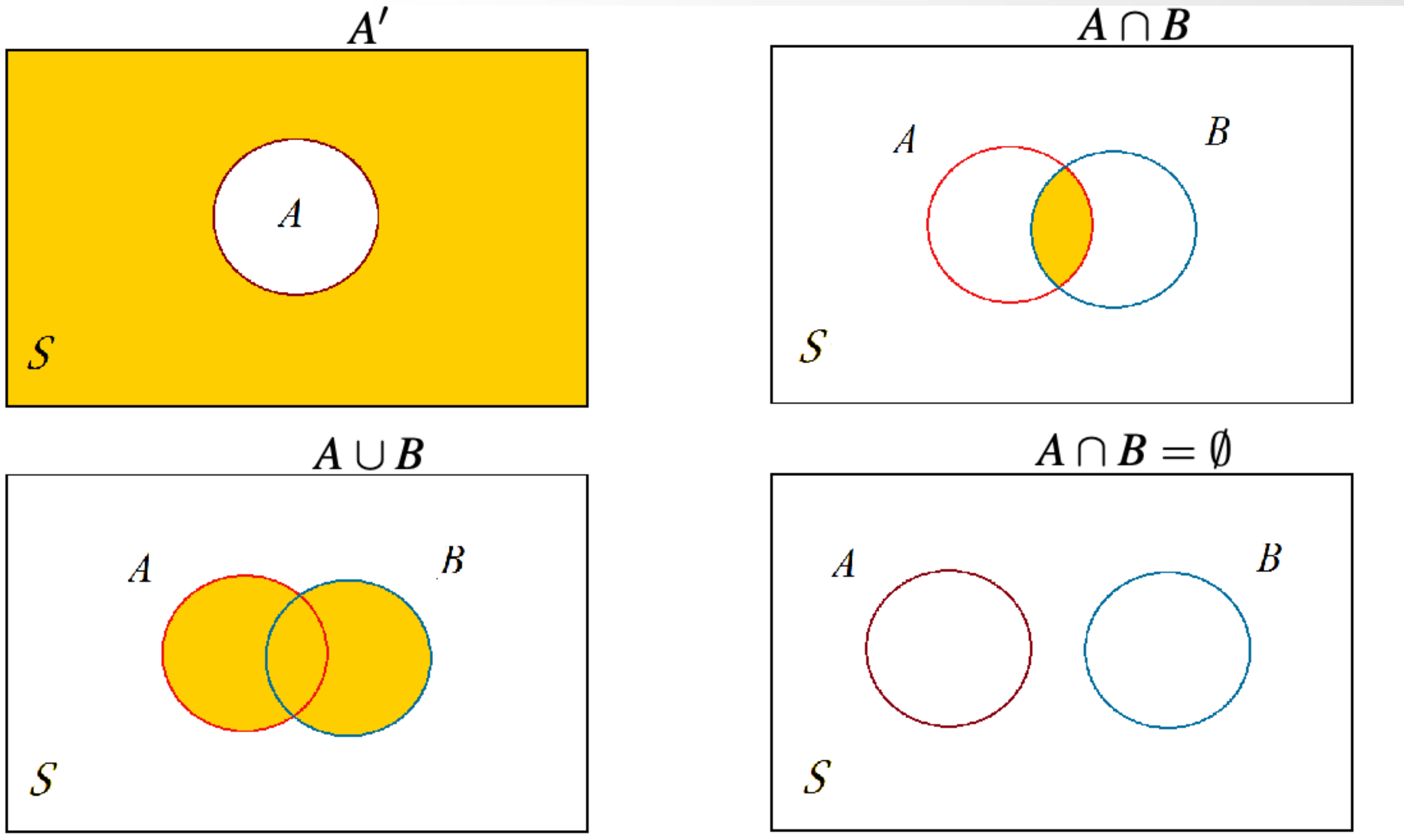
Rolling a dice



- $S = ?$
- A: event of getting an odd number
- B: event of getting a number less than 4
- $A \cap B = ?$
- $A \cup B = ?$
- Are A and B disjoint events?

Venn Diagrams

A graphical way of representing the relationships between events.



Probability

Given an experiment and a sample space S, the objective of probability is to assign each event A a number P(A), called the probability of the event A, which will give a precise measure of the chance that A will occur.

Axioms of Probability:

- Axiom 1: $P(A) \geq 0$ for any event A
- Axiom 2: $P(S) = 1$
- Axiom 3: If $A_1, A_2, \cdots, A_k$ are pairwise disjoint events, then

$$P\left(A_1 \cup A_2 \cup \cdots \cup A_k\right) = \sum_{i=1}^{k} P\left(A_i\right), \quad \text{finite set}$$

$$P\left(A_1 \cup A_2 \cup \cdots\right) = \sum_{i=1}^{\infty} P\left(A_i\right), \quad \text{infinite set}$$

Properties of Probability

- $P(\emptyset) = 0$
- For 2 disjoint events: $P(A \cup B) = P(A) + P(B)$
- $P(A) = 1 - P(A')$
- $P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- For any three events A, B, and C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Exercise

In an insurance agency

- 60% of the customers have car insurance
- 40% of the customers have home insurance
- 20% of the customers have both insurance

Question: What is the probability that a customer from this agency has

a) at least one insurance?

b) the car insurance but not the home insurance? Neither insurance?

c) exactly one insurance?

Computing Probabilities of Events

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Outline

- **Probability of an event**

- **Equally Likely Outcomes and Counting Techniques**
  - Product Rule
  - Permutation
  - Combination

Probability of an event

- Generally, to find the probability of an event A, we sum all the probabilities assigned to the sample points in A. This sum is called the probability of A and is denoted by P(A).

Given a sample space of size n,

$$S = \{s_1, s_2, \ldots, s_n\}$$

the probability assigned to each outcome in S as

$$P(s_i) = p_i,\ i = 1, 2, \ldots,\ n$$

Then the probability of event A is given by

$$P(A) = \sum_{s_i \in A} p_i .$$

Equally Likely Outcomes

- For an experiment has n equally likely outcomes (e.g. rolling a fair dice …),

$$S = \{s_1, s_2, \ldots, s_n\}$$

$$P(\{s_1\}) = P(\{s_2\}) = \cdots = P(\{s_n\}) = \frac{1}{n}$$

Let A be an event containing n_A outcomes, then $P(A) = \frac{n_A}{n}$

Exercise: A coin is tossed twice. What is the probability that at least 1 head occurs?

Counting techniques

counting techniques can be used to count the number of outcomes in the sample space (or in some events) without listing each element.

Product Rule

- In a sequence of k experiments in which the first one has n_1 outcomes, the second event has n_2 , the third has n_3 , and so forth, the total number of outcomes for the k experiments will be

$$n_1 \times n_2 \times \cdots \times n_k$$

Exercise

1. A person has 4 pairs of pants, 3 shirts and 2 hats, how many different ways to dress?

2. A player is rolling three fair identical dice separately. Let A be the event that all three dice gave the same number. What is the probability of event A?

Permutation

- Any ordered sequence of k objects taken from a set of n distinct objects is called **a permutation of size k from n objects.**

- Notation: $P_{k,n}$

- Formula: $P_{k,n} = n(n-1)(n-2)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$,

  where $n!$ is n factorial, $n! = n(n-1)(n-2)\cdots \times 3 \times 2 \times 1$

  Example: How many ways that 6 teachers can be assigned to 4 different courses, if no teacher is assigned to more than one course.

Combination

- Given a set of n distinct objects, any unordered subset of k objects is called a combination.

Notation: $\binom{a}{b}$ or $C_{k,n}$

Formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P_{k,n}}{k!}$. Read as "n choose k"

Note: $\binom{n}{n} = 1, \binom{n}{0} = 1, \binom{n}{1} = n, \binom{n}{k} = \binom{n}{n-k}$

Example: How many ways to choose 3 students from a class of 10.

The major difference between Combination and Permutation is the ordering. In permutation, the order matters; while in combination, the order doesn't matter.

- Example:

1. How many ways to choose 3 different numbers from 1 to 9?
   - Order doesn't matter here, because choosing {1,2,3} is same as choosing {3,2,1}.

2. How many 3-digit numbers you can make with 3 different numbers from 1 to 9?
   - Order matters, because $123 \neq 321$.

Exercise

- How many 9-digit numbers that contain exactly four number 4, three number 3, and two number 2? (Ex: 444433322, 432432434, . . .)

Exercise

(a) In how many ways can 6 people be lined up to get on a bus?

(b) If 3 specific persons, among 6, insist on following each other, how many ways are possible?

(c) If 2 specific persons, among 6, refuse to follow each other, how many ways are possible?

Conditional Probability and Bayes' Rule

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Outline

- Conditional Probability
- Bayes' Theorem
- Independent Events

Conditional Probability

Let A and B be two events. Then, the conditional probability of A given that B has occurred, P(A | B), is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

The additional information we are given that event B has occurred, allows us to reduce the sample space to just outcomes in the event B.

e.g. a player randomly drawing a card from deck of 52,

1) what's the probability that the player drew the ace of hearts?
2) If we knew the player drew an ace, what is the probability that it is the ace of hearts?

Exercise

Two machines produce the same type of products. Machine A produces 8, of which 2 are identified as defective. Machine B produces 10, of which 1 is defective. The sales manager randomly selected 1 out of these 18 for a demonstration.

- (1) What's the probability that he selected product from machine A?
- (2) What's the probability that the selected product is defective?
- (3) What's the probability that the selected product is defective and from A?
- (4) If the selected product turned to be defective, what's the probability that this product is from machine A?

Multiplication Rule

The multiplication rule is used to calculate the joint probability of two events. It is simply a rearrangement of the conditional probability formula:

$$P(A \cap B) = P(A \mid B)P(B)$$

or $$P(A \cap B) = P(B \mid A)P(A)$$

Law of total probability

If $A_1, A_2, \ldots, A_k$ are disjoint and exhaustive events, then

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

for any event B.

Bayes' Theorem

- General form

If $A_1, A_2, \ldots, A_k$ are disjoint and exhaustive events, then

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i)P(A_i)}$$

for $j = 1, 2, \ldots, k$ and any event B with positive probability.

- It connects two conditional probabilities $P(A_j | B)$ and $P(B | A_j)$
- It can be used to find the "causes" of the outcome. Given event B occurred, what is the probability that the cause of B is A_j

Exercise

At a certain gas station, 40% of the customers use regular gas (A_1), 35% use plus gas (A_2), and 25% use premium (A_3). Of those customers using regular gas, only 30% fill their tanks. Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their tanks. Let B be the customers fill their tanks.

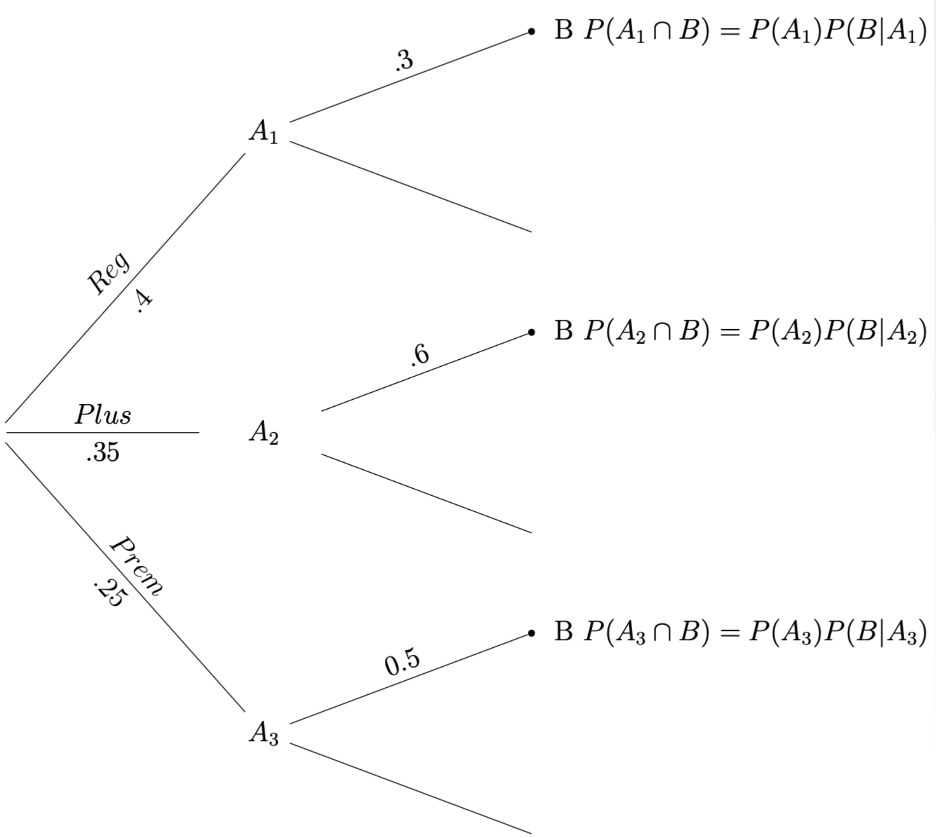
(a) What is the probability that the next customer will request plus gas and fill the tank ($A_2 \cap B$)?

(b) What is the probability that the next customer fills the tank?

(c) If the next customer fills the tank, what is the probability that regular gas is requested? Plus? Premium?

The tree diagram is in next page

- The calculation can be visualized as multiplying probabilities along the branches of a probability tree.



Independence

The literal meaning of Independent Events is that one event occurs does not affect the probability of other events occurring.

- Two events A and B are independent if $P(A \mid B) = P(A)$. Otherwise A and B are dependent.

Properties

- Events A and B are independent $\Leftrightarrow P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$
- Events A and B are independent $\Leftrightarrow P(A \cap B) = P(A)P(B)$

Question: If A and B are disjoint, are they independent?

Independence of more than two events

- Events $A_1, A_2, \ldots, A_n$ are mutually independent if for any subset of n events, $A_{i1}, A_{i2}, \ldots, A_{ik}$, for $k \leq n$, we have

$$P(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}) = P(A_{i1})P(A_{i12}) \cdots P(A_{ik})$$

Example:

Three events A_1, A_2 and A_3 are mutually independent $\leftrightarrow$

$P\left(A_1 \cap A_2\right) = P\left(A_1\right) P\left(A_2\right), P\left(A_1 \cap A_3\right) = P\left(A_1\right) P\left(A_3\right),$

$P\left(A_2 \cap A_3\right) = P\left(A_2\right) P\left(A_3\right)$ and $P\left(A_1 \cap A_2 \cap A_3\right) = P\left(A_1\right) P\left(A_2\right) P\left(A_3\right)$

Exercise

The frequency table below is based on a survey of 200 people about their gender and dominant hand:

	Right-handed	Left-handed	Total
Female	60	20	80
Male	90	30	120
Total	150	50	200

If we randomly select a person who participated in this survey,

a) what is the probability that this person is a female?
b) what is the probability that this person is Left-handed?
c) Are gender and dominant-hand independent?
d) Given this person is Right-handed, what is the probability he is a male?