Refined h-Polynomial of Associahedral Complexes

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k SETS OF n VARIABLES

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \end{pmatrix}$$

G GROUP OF $n \times n$ INVERTIBLE MATRICES

Polynomial $f(X) \in \mathbb{C}[X]$

$\mathbb{G}$-Action

$f(Xg)$

$g \in \mathbb{G}$

GL_k-Action

$f(MX)$

$M \in GL_k$

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$$\mathcal{S}_G^{(k)} := \mathbb{C}[X] \big/ I_G$$

I_G ideal generated by constant term free G-invariant polynomials

$$f(Xg) = f(X)$$

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$$\mathcal{E}_n^{(k)} := \mathcal{E}_{\mathbb{S}_n}^{(k)}$$

$$A_n^{(k)}$$

ALTERNATING COMPONENT OF $\mathcal{E}_n^{(k)}$

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Dimension as a function of k

$$\dim(\mathcal{E}_1^{(k)}) = 1$$

$$\dim(\mathcal{E}_2^{(k)}) = 1 + k$$

$$\dim(\mathcal{E}_3^{(k)}) = 1 + 2k + k^2 + \binom{k+1}{2} + \binom{k+2}{3}$$

$$\vdots$$

Dimension as a function of n

$$\dim(\mathcal{E}_n^{(1)}) = n!$$

$$\dim(\mathcal{E}_n^{(2)}) = (n+1)^{n-1}$$

(Haiman 2000's)

$$\dim(\mathcal{E}_n^{(3)}) \overset{?}{=} 2^n (n+1)^{n-1}$$

Dimension as a function of n

$$\dim(A_n^{(1)}) = 1$$

$$\dim(A_n^{(2)}) = \frac{1}{n+1}\binom{2n}{n}$$

$$\dim(A_n^{(3)}) \overset{?}{=} \frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

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TRIANGULAR PARTITIONS

(UNDER ANY LINE)

IN COLLABORATION WITH
MIKHAIL MAZIN
ARXIV: 2203.15942

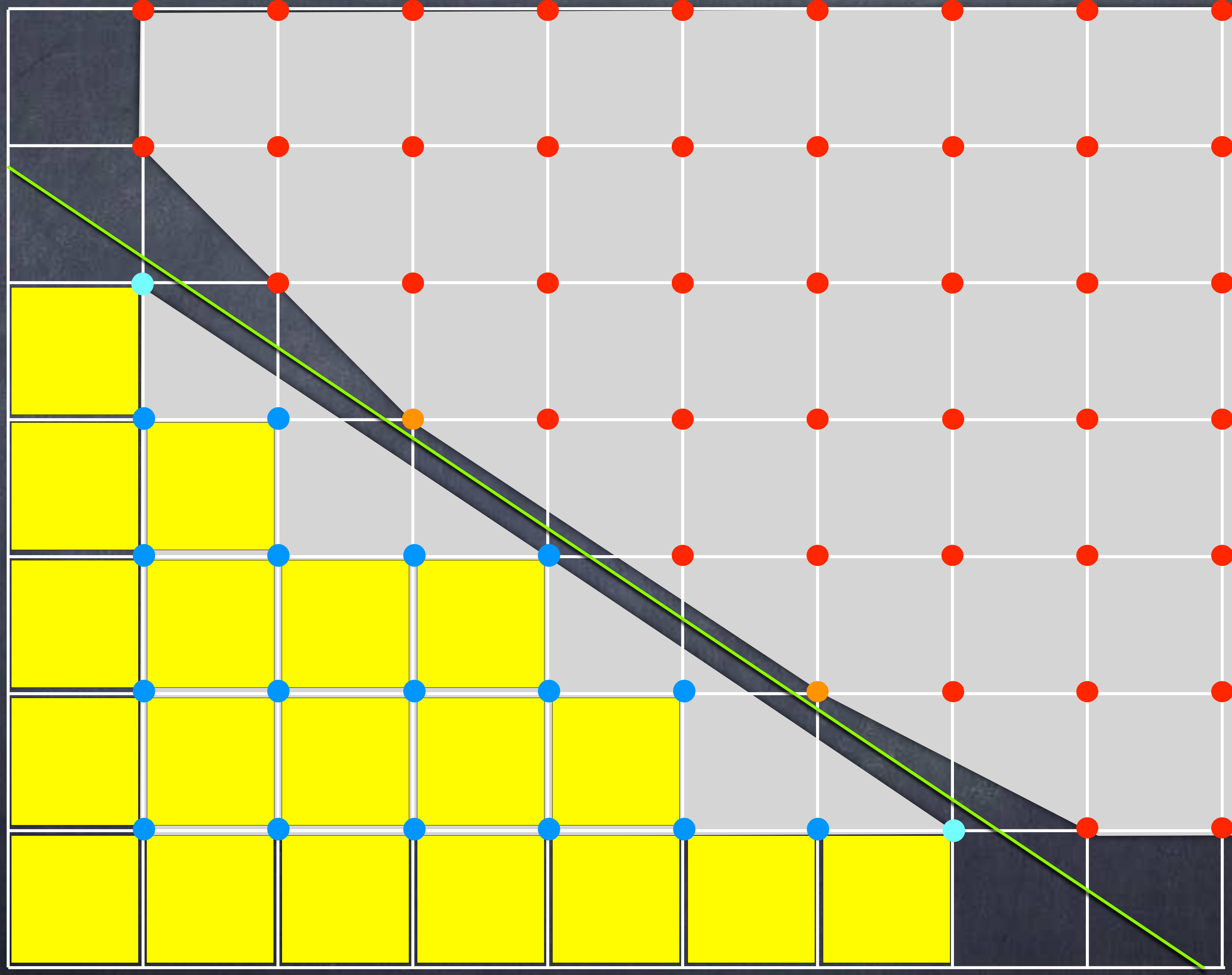
FRANÇOIS BERGERON

Blasiak, Haiman, Morse
Pun, Seelinger

A series of papers

ArXiv: 2112.07070
2112.07063
2102.08815
2102.07931

- Cohomology Ring of the Flag Variety
- Macdonald Polynomials and Operators
- Diagonal Harmonics
- Diagonal Coinvariant Space
- Hilbert Schemes of Points in the Plane
- Cherednik Hecke Algebras.
- Refined Knot invariants
- Elliptic Hall Algera.
- Rectangular Catalan Combinatorics
- ....



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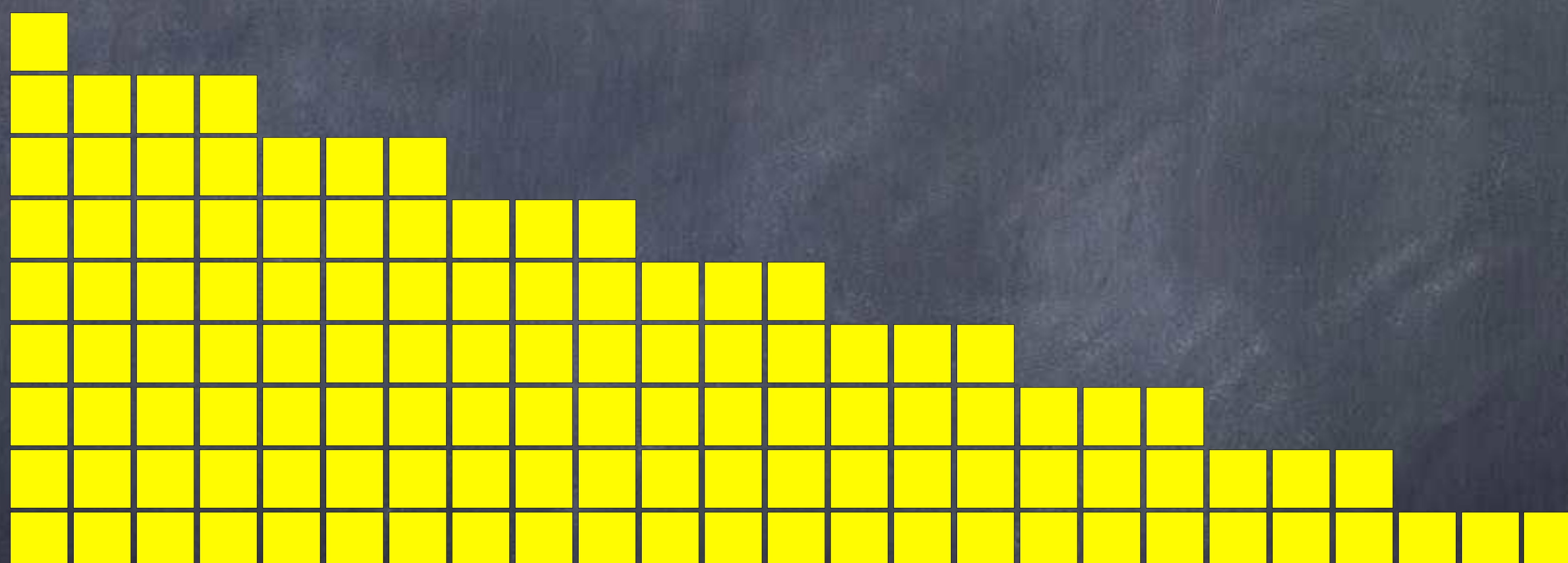
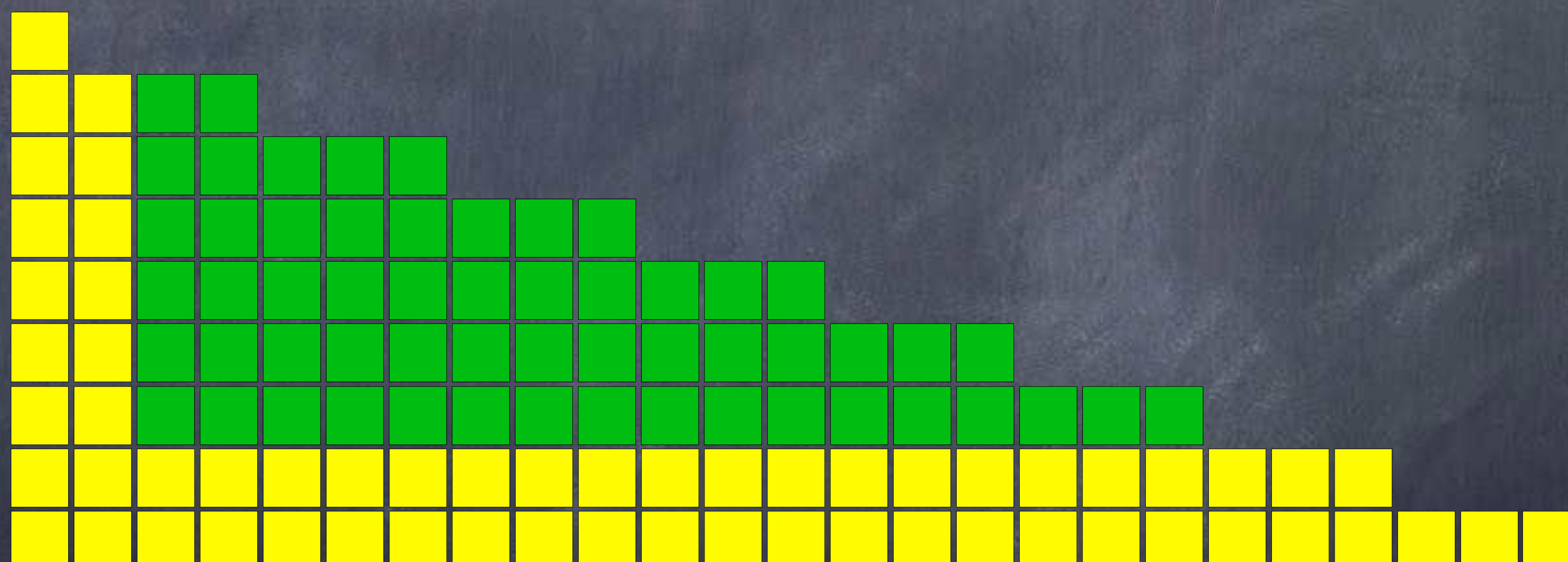
Being Triangular is Hereditary

τ



Being Triangular is Hereditary

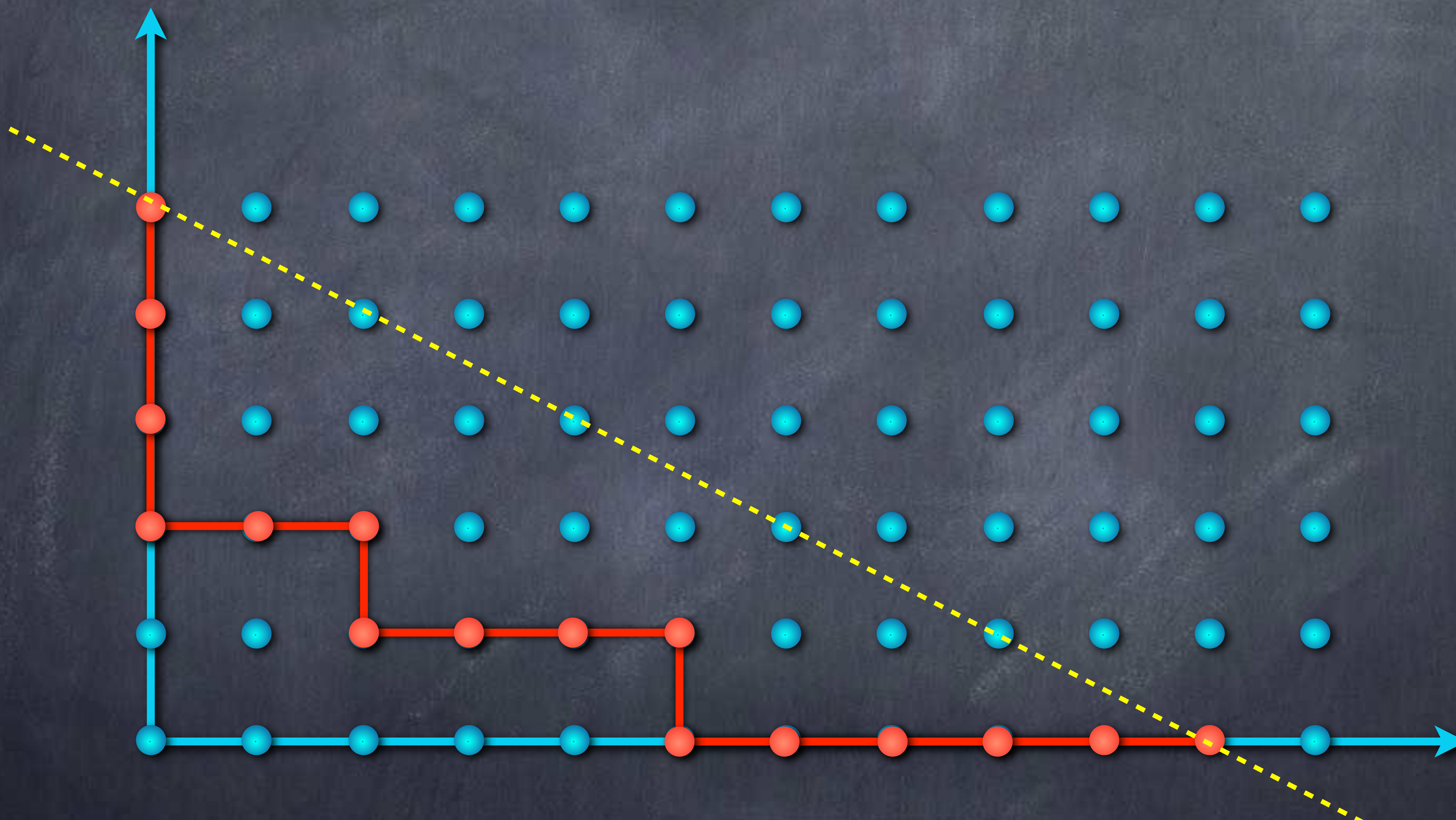
τ



DYCK PATHS

(SUB-PARTITIONS)

Dyck Paths



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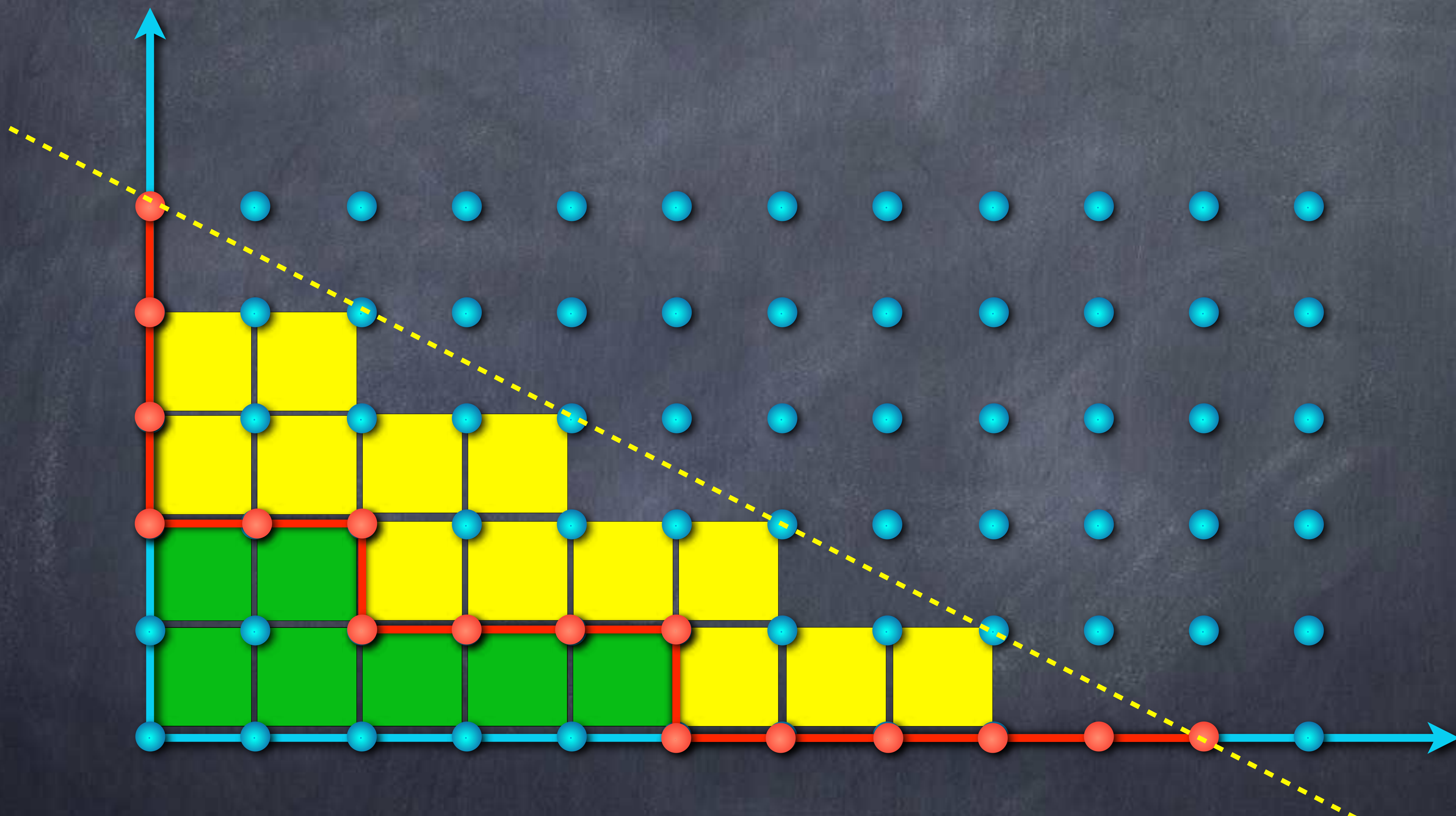

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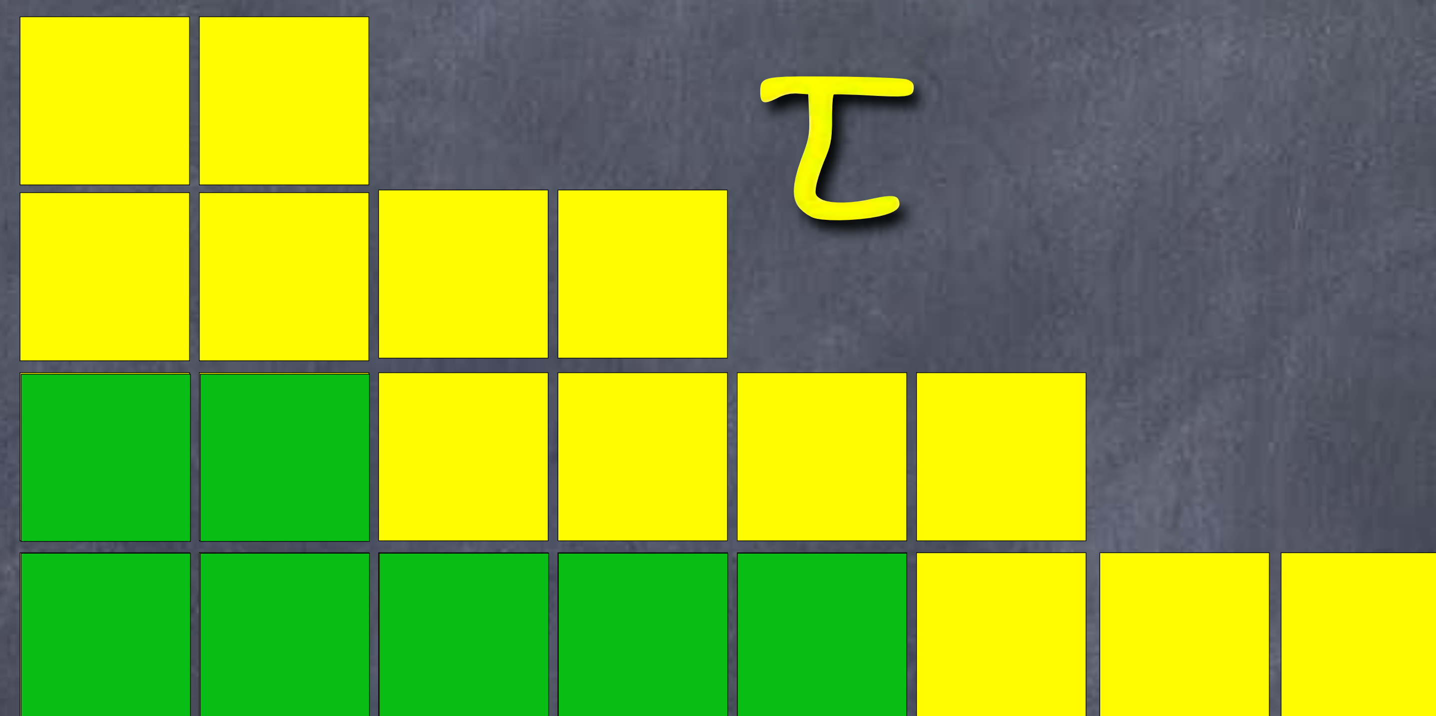
τ-Dyck Paths

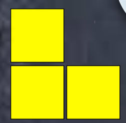


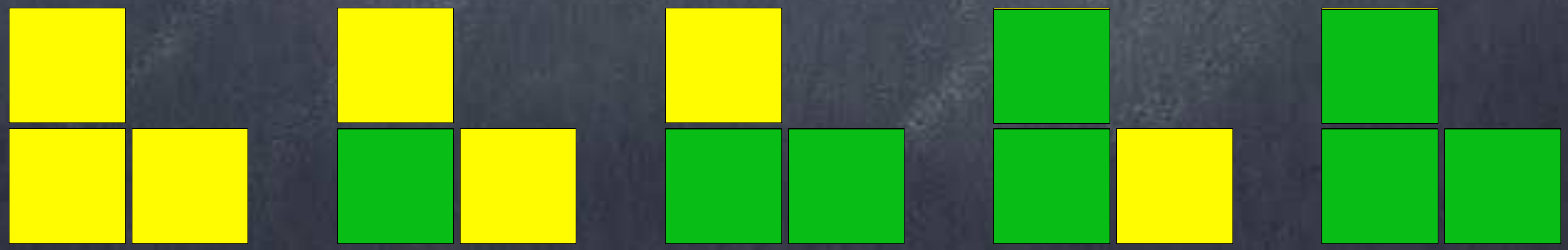
$$\alpha \subseteq \tau$$

q-ENUMERATION

$$A_\tau(q) := \sum_{\alpha \subseteq \tau} q^{|\tau| - |\alpha|}$$

$$A_{\square}(q) := q^3 + q^2 + q + q + 1$$





I-TAMARI ORDER

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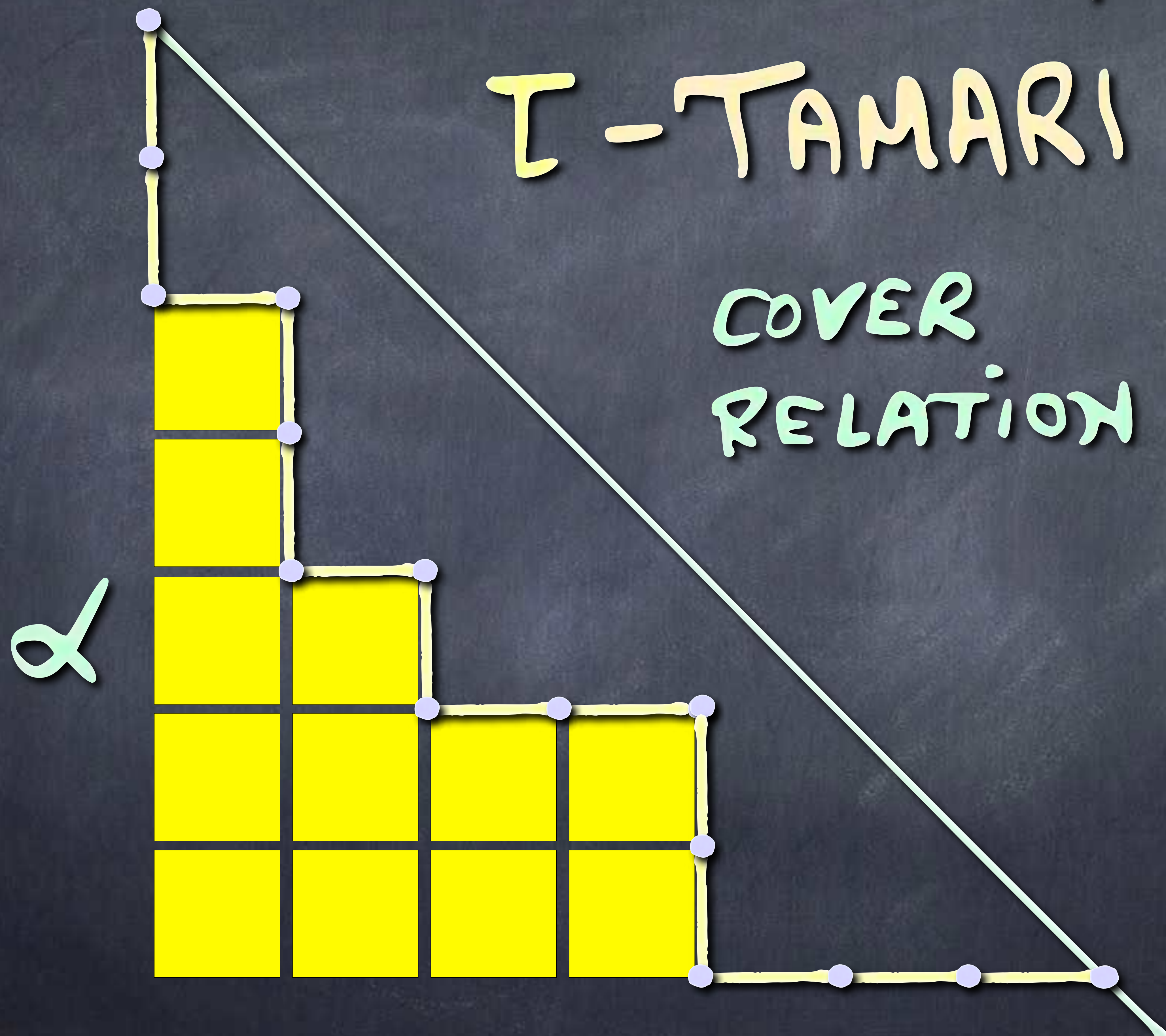
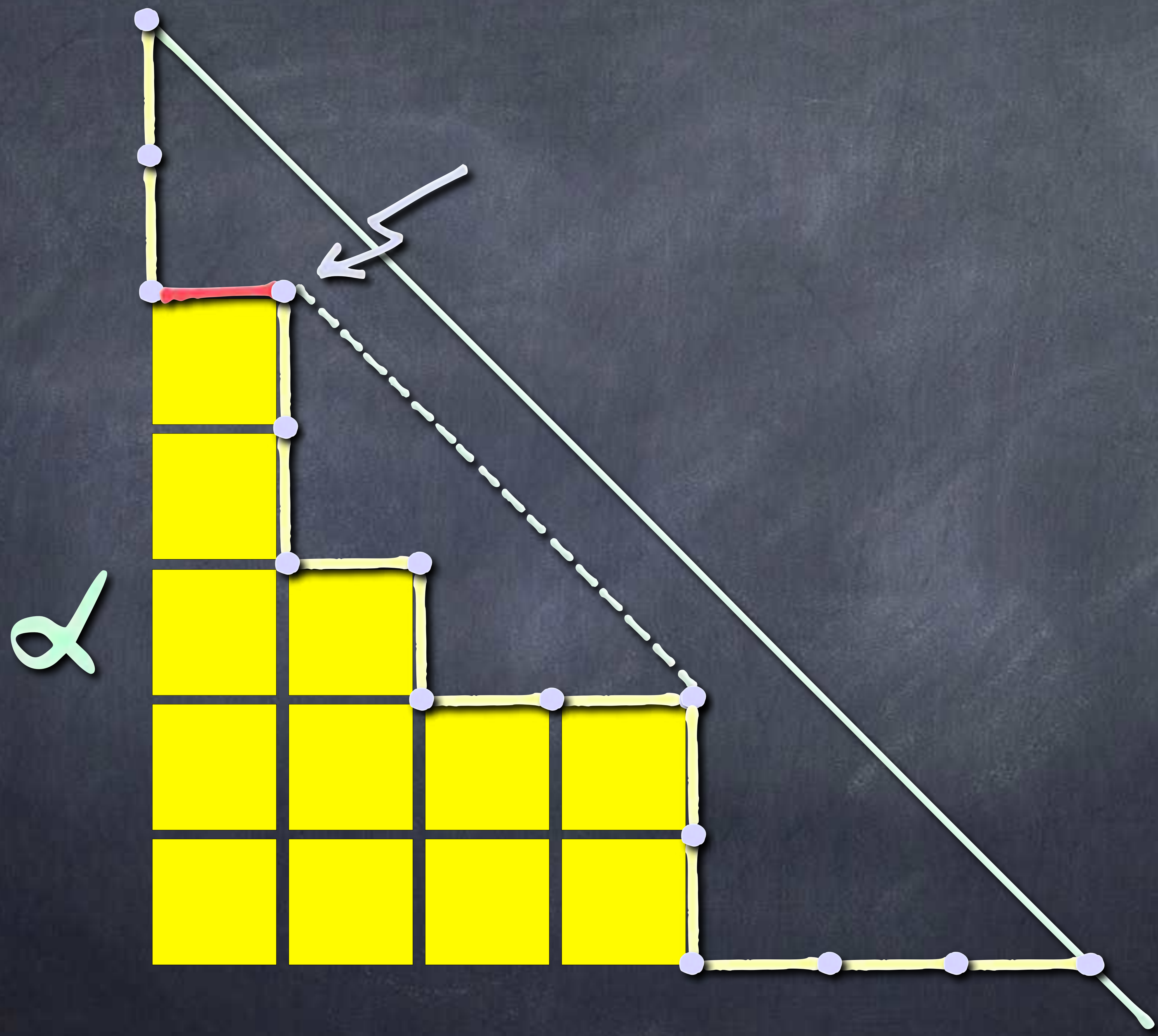
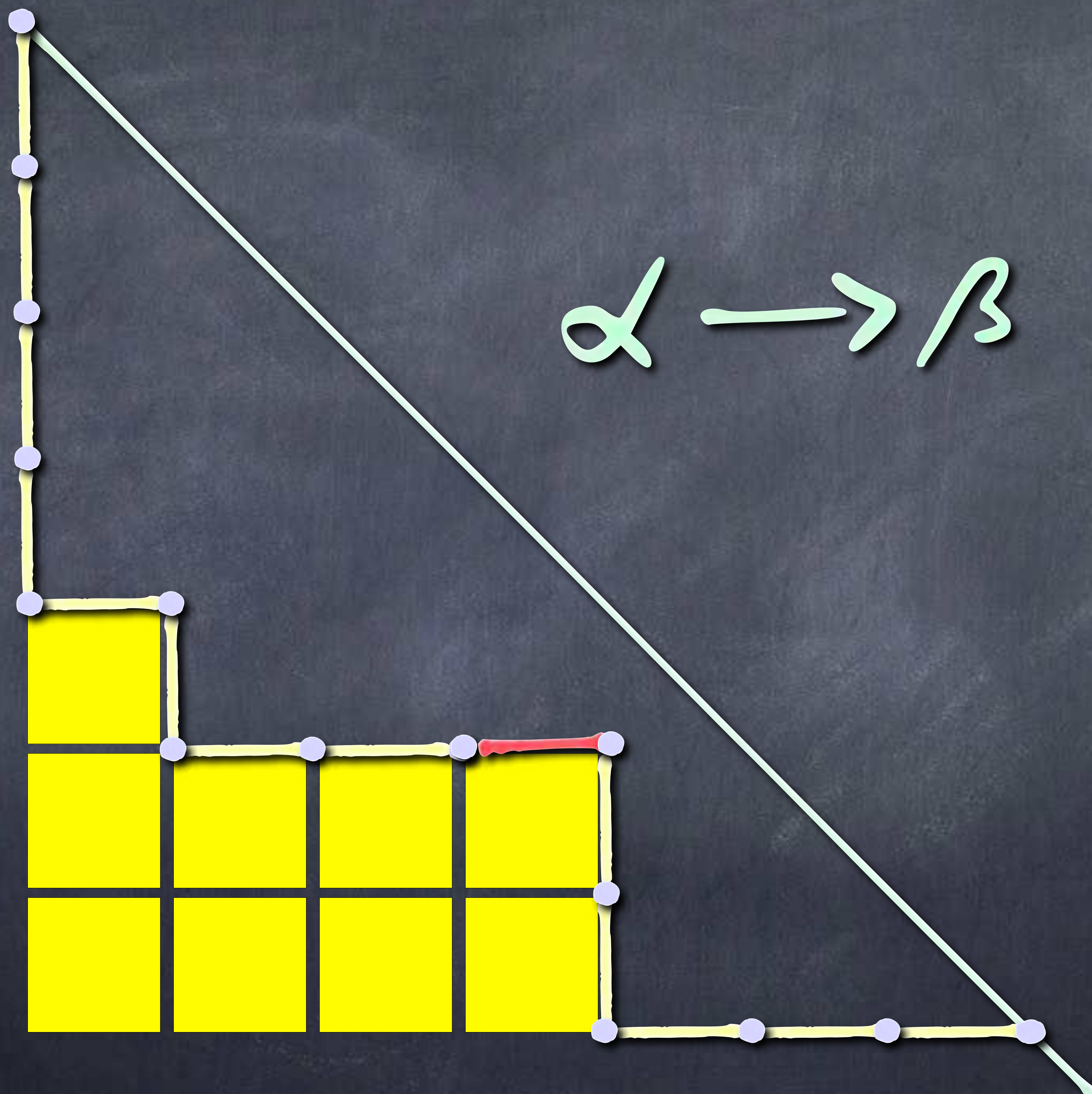
I-TAMARI

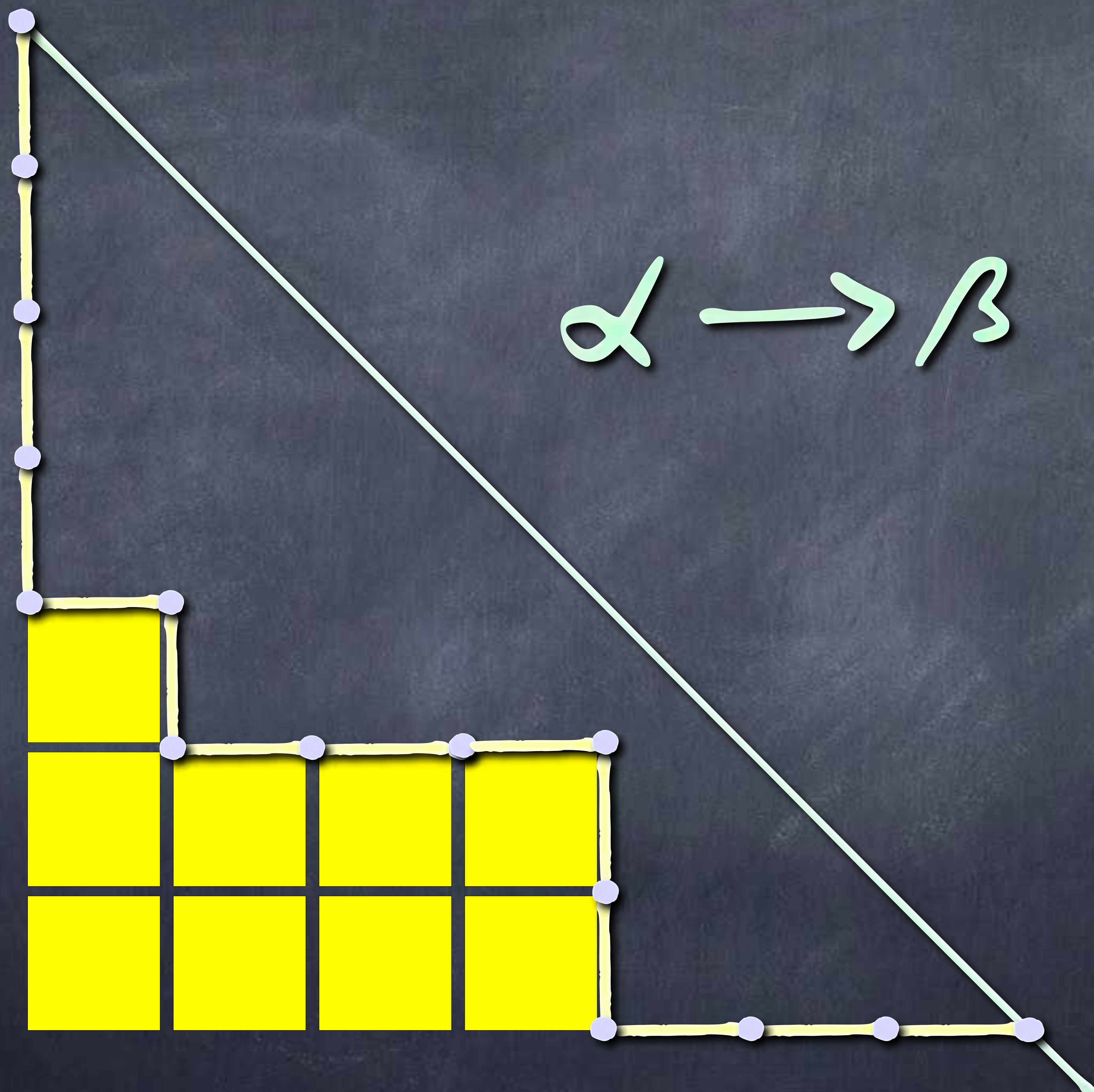
COVER RELATION

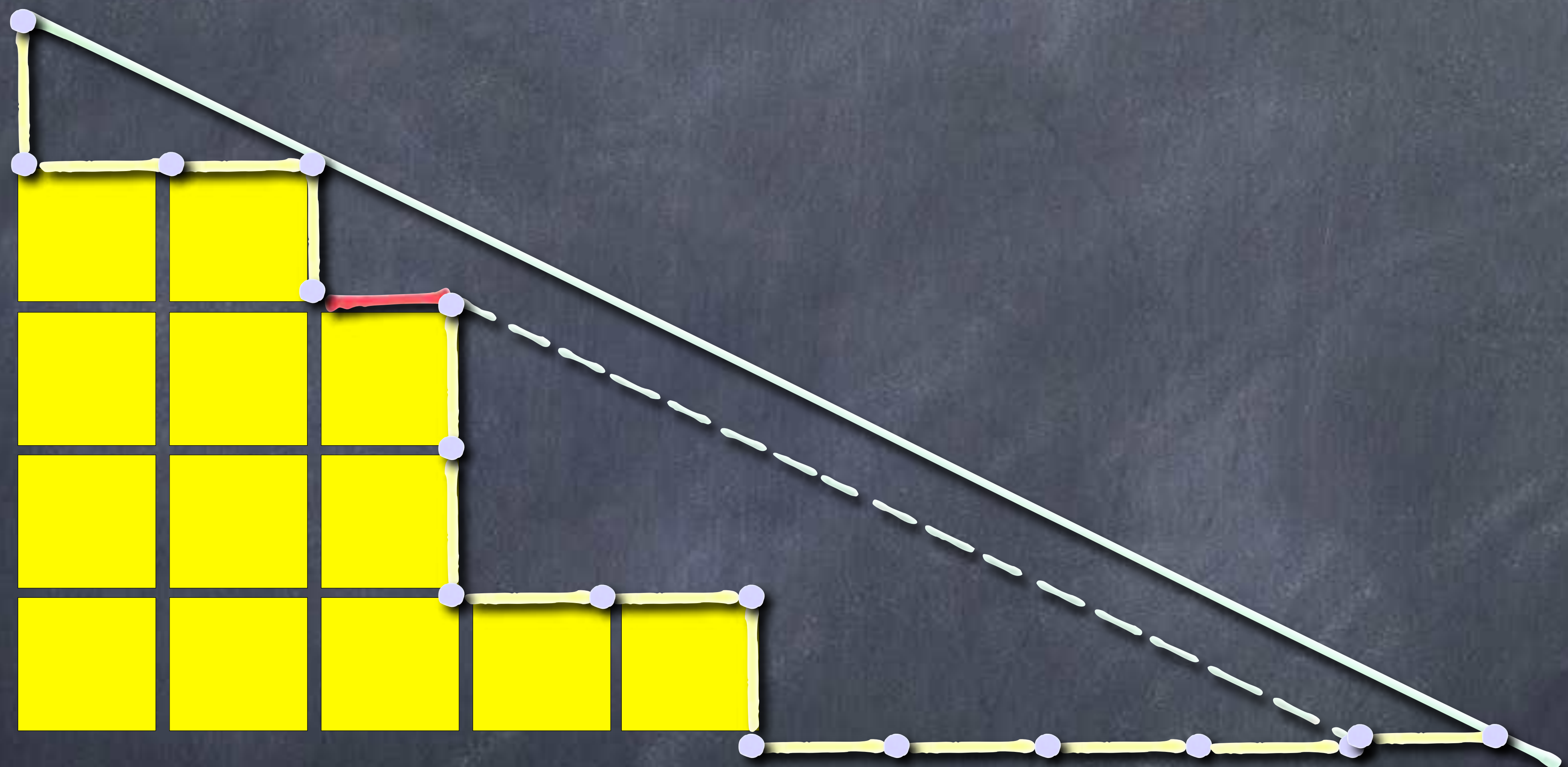



α



$\alpha \longrightarrow \beta$


α ⟶ β



FACE INTERVAL

$$[\alpha, \bigvee_{i=1}^{d} \beta_i]$$

d-DIMENSIONAL

PRODUCT OF ASSOCIAHEDRONS

FACE INTERVAL

$$[\alpha, \bigvee_{i=1}^{d} \beta_i]$$

THM (CEBALLOS - PADROL - SARMIENTO)
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ASSOCIAHEDRAL COMPLEXES

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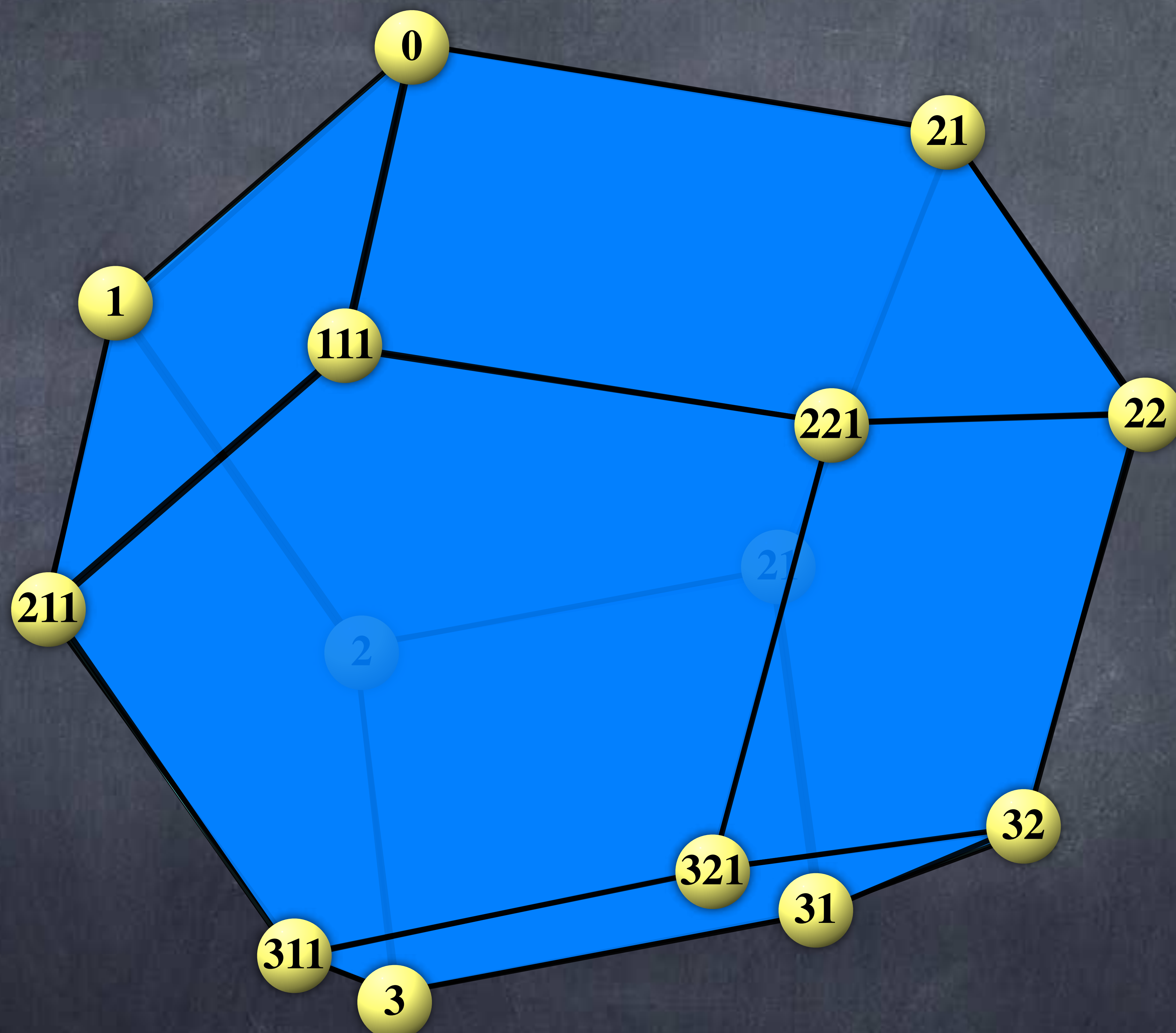
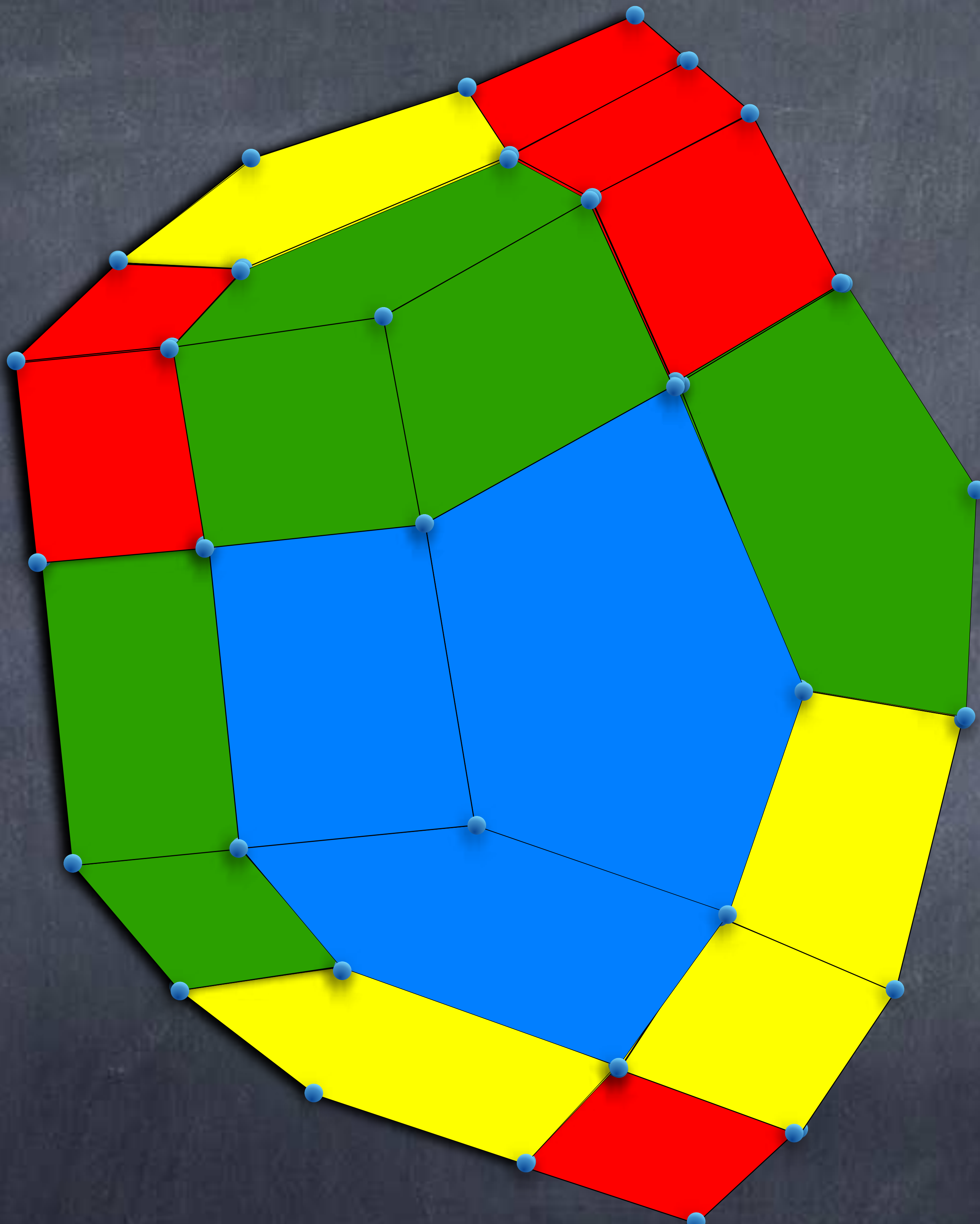

0
21
1
111
221
22
211
2
21
32
321
31
311
3



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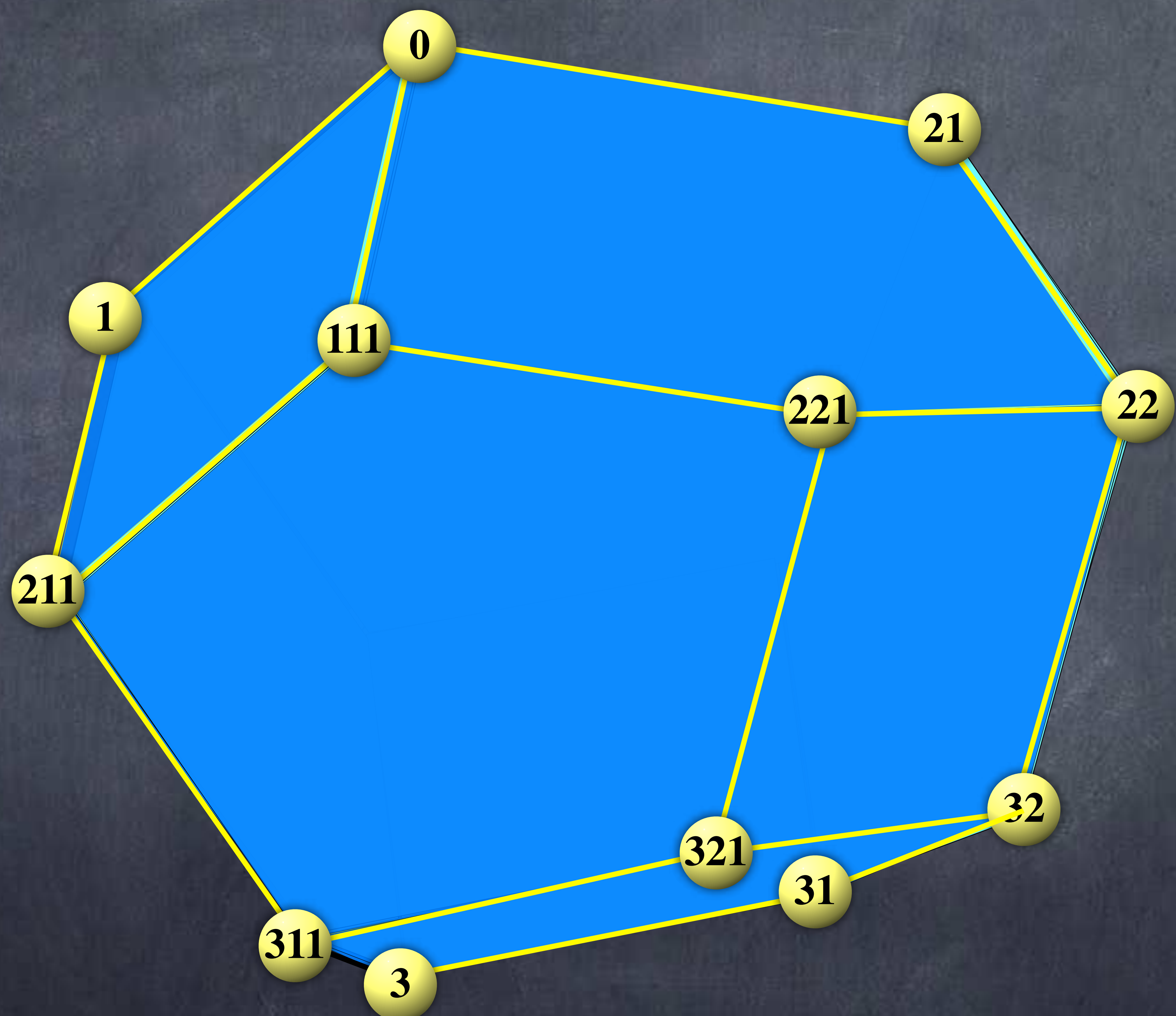
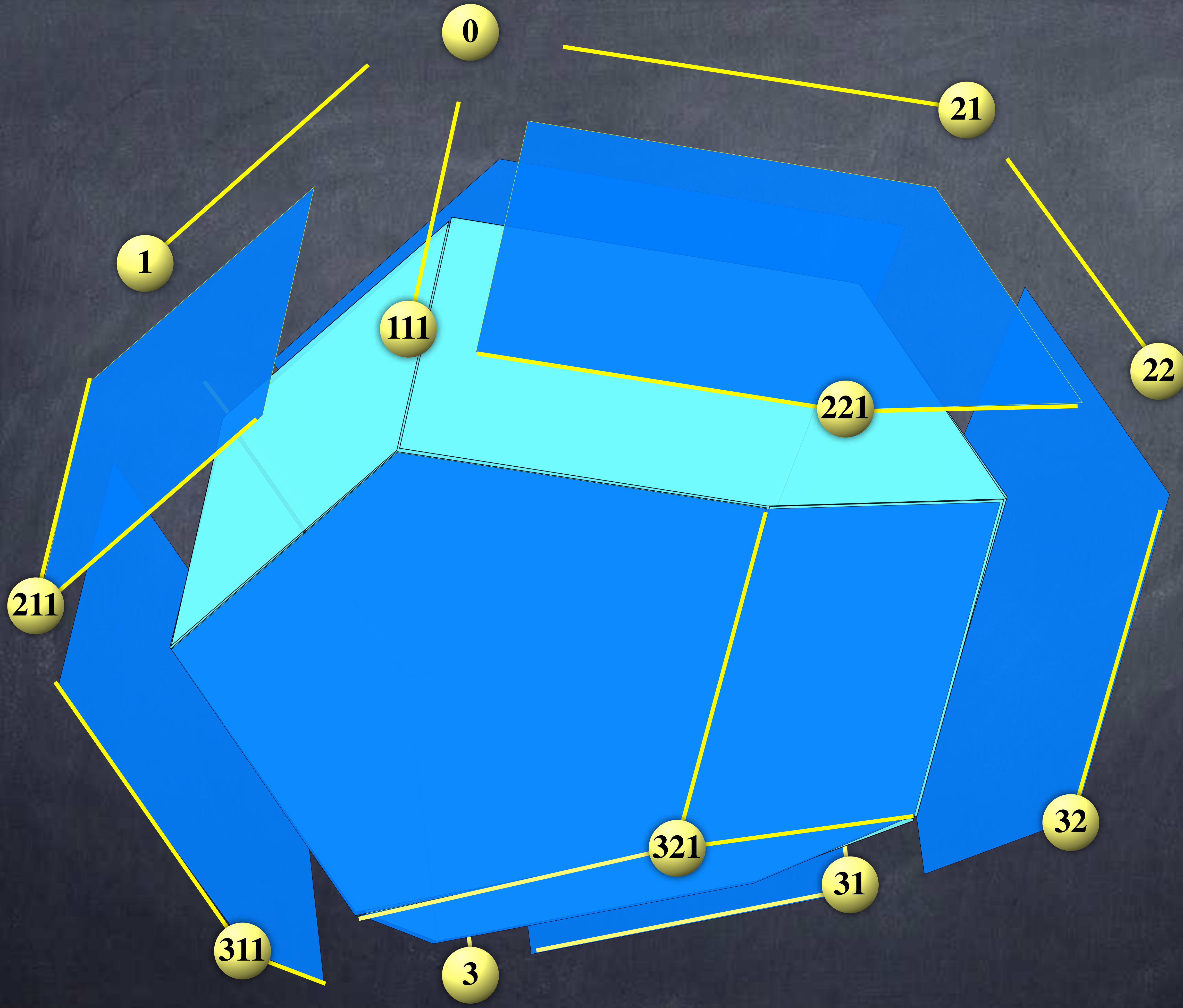
REFINED h-POLYNOMIAL



0
21
1
111
221
22
211
32
321
31
311
3



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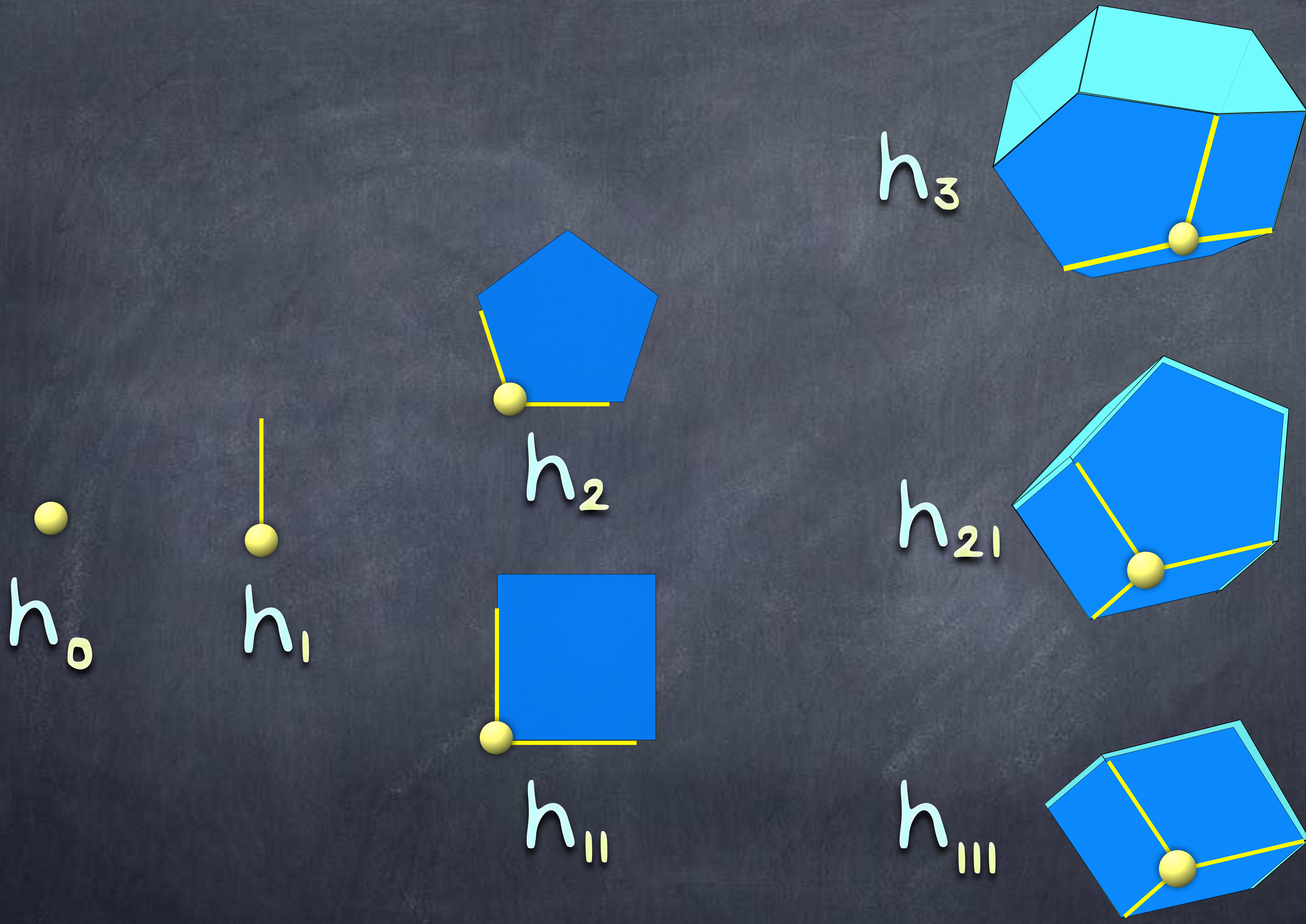
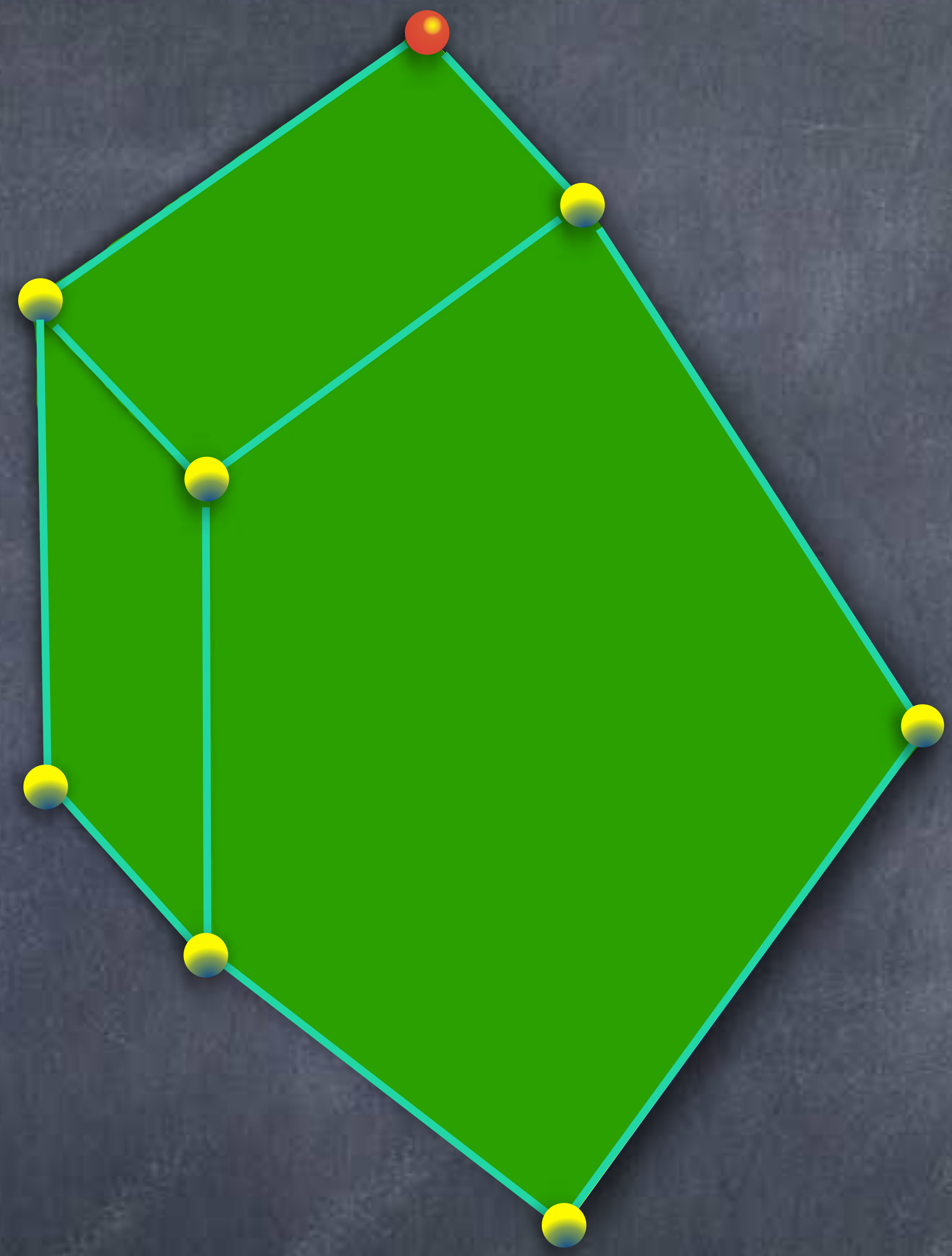
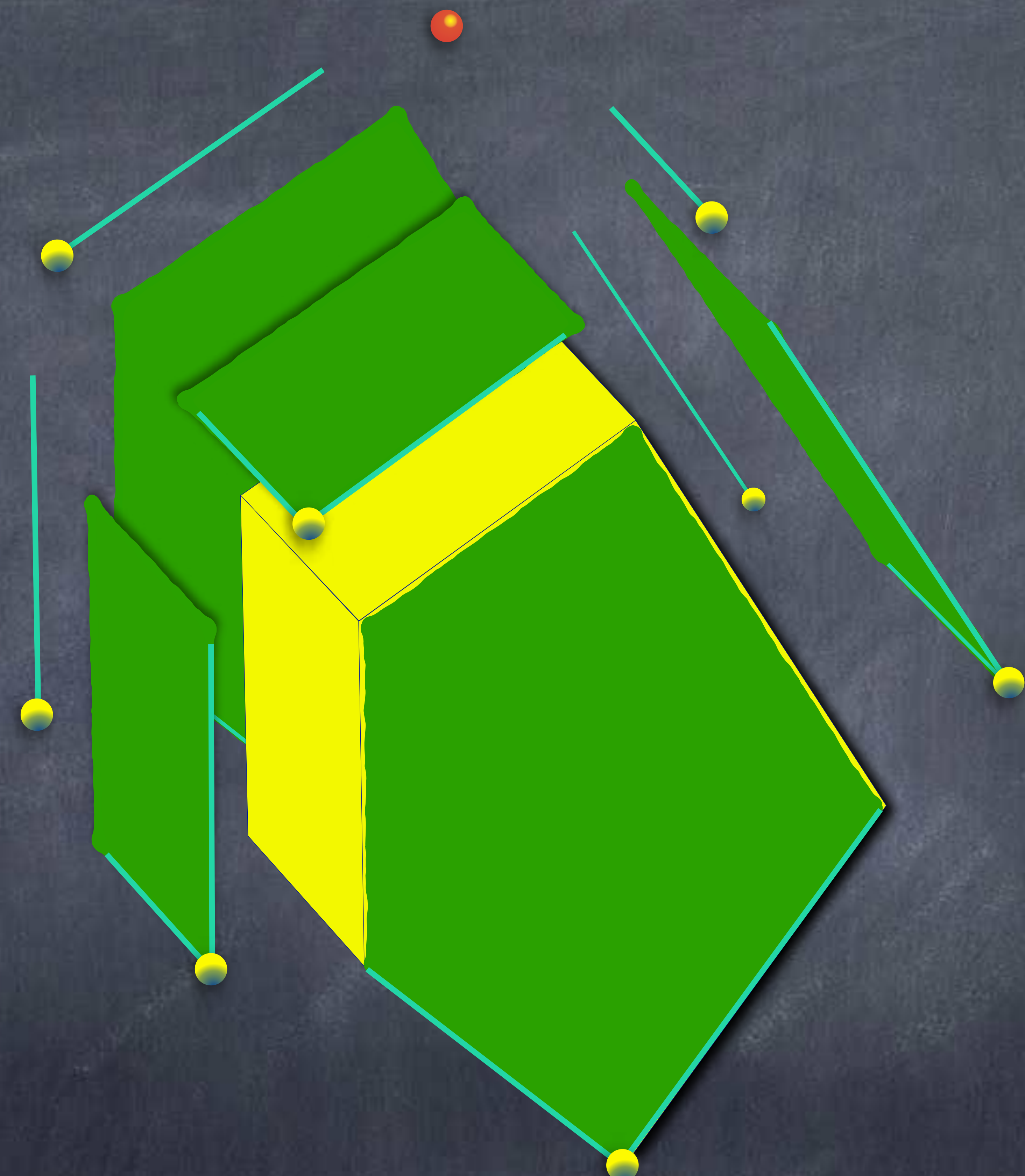

h_3
h_2
h_{21}
h_0
h_1
h_{11}
h_{111}





$$1 + 4h_1 + 3h_{11} + h_2 + h_{21}$$

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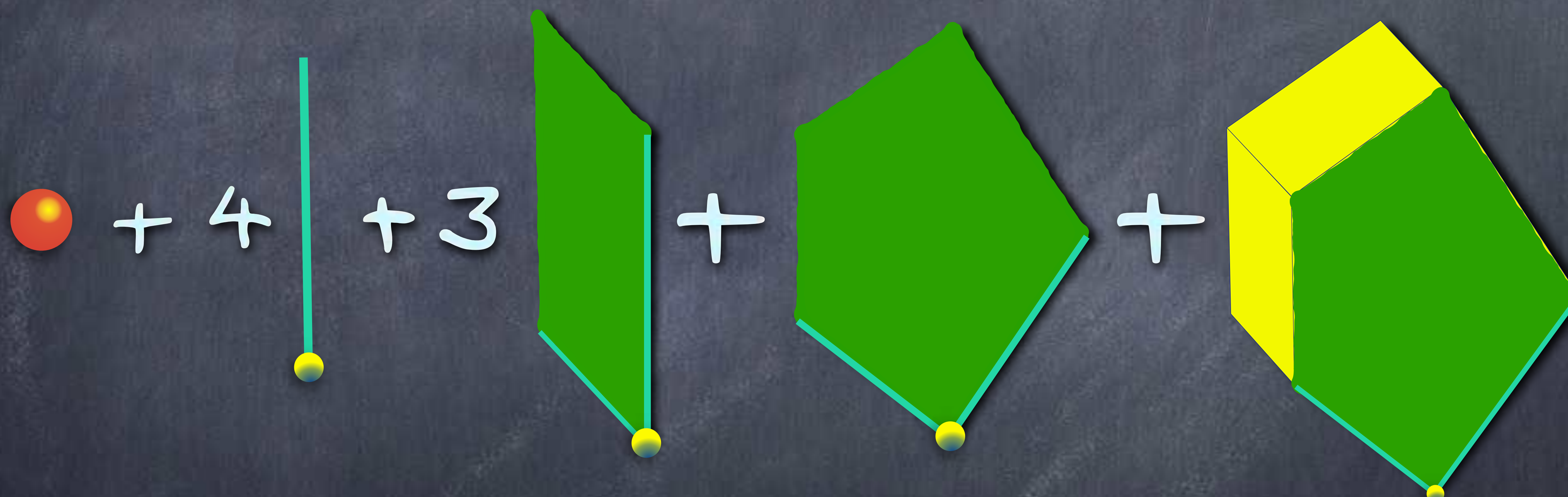
$$1 + 4h_1 + 3h_{11} + h_2 + h_{21}$$



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$$K_\tau = \sum_{\alpha \leq \tau} h_{\mu(\alpha)}$$

$$\mu(\alpha) = \mu_1 \mu_2 \cdots \mu_\ell$$

PRODUCT OF ASSOCIAHEDRONS

$$h_\mu \longmapsto x^{|\mu|}$$

THE USUAL h-POLYNOMIAL

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$$\kappa_0 = 1,$$
$$\kappa_1 = 1 + h_1,$$
$$\kappa_2 = 1 + 2h_1,$$
$$\kappa_{21} = 1 + 3h_1 + h_2,$$
$$\kappa_3 = 1 + 3h_1,$$
$$\kappa_{31} = 1 + 4h_1 + h_{11} + h_2,$$
$$\kappa_4 = 1 + 4h_1,$$
$$\kappa_{32} = 1 + 5h_1 + h_{11} + 2h_2,$$
$$\kappa_{41} = 1 + 5h_1 + 2h_{11} + h_2,$$

$$\kappa_5 = 1 + 5h_1,$$
$$\kappa_{321} = 1 + 6h_1 + 2h_{11} + 4h_2 + h_3,$$
$$\kappa_{42} = 1 + 6h_1 + 3h_{11} + 2h_2,$$
$$\kappa_{51} = 1 + 6h_1 + 3h_{11} + h_2,$$
$$\kappa_6 = 1 + 6h_1,$$
$$\kappa_{421} = 1 + 7h_1 + 5h_{11} + 4h_2 + h_{21} + h_3,$$
$$\kappa_{52} = 1 + 7h_1 + 5h_{11} + 2h_2,$$
$$\kappa_{61} = 1 + 7h_1 + 4h_{11} + h_2,$$
$$\kappa_7 = 1 + 7h_1$$

PROPOSITION

IF

$$\tau = \big((n-1)r, (n-2)r, \ldots, 2r, r\big)$$

THEN

$$\mathcal{K}_\tau = \sum_j \sum_{\mu \vdash j} r \binom{r(n+1)-j-1}{\ell(\mu)-1} \binom{n+1}{j+1} \frac{(\ell(\mu)-1)!}{c_1!\, c_2! \cdots c_k!} h_\mu$$

c_i : # OF PARTS OF SIZE i IN μ

(COEFFICIENTS ARE POLYNOMIAL IN n AND r)

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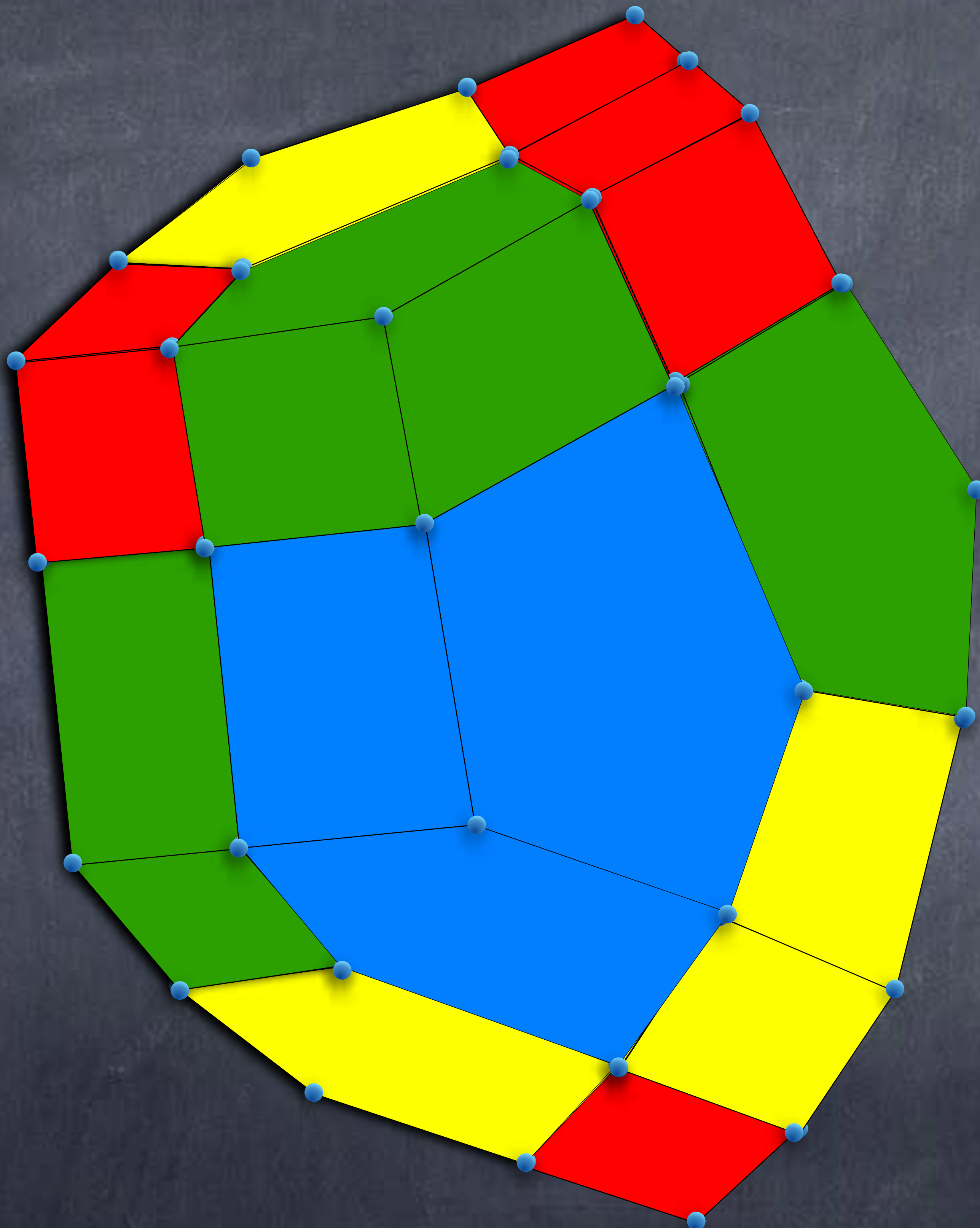
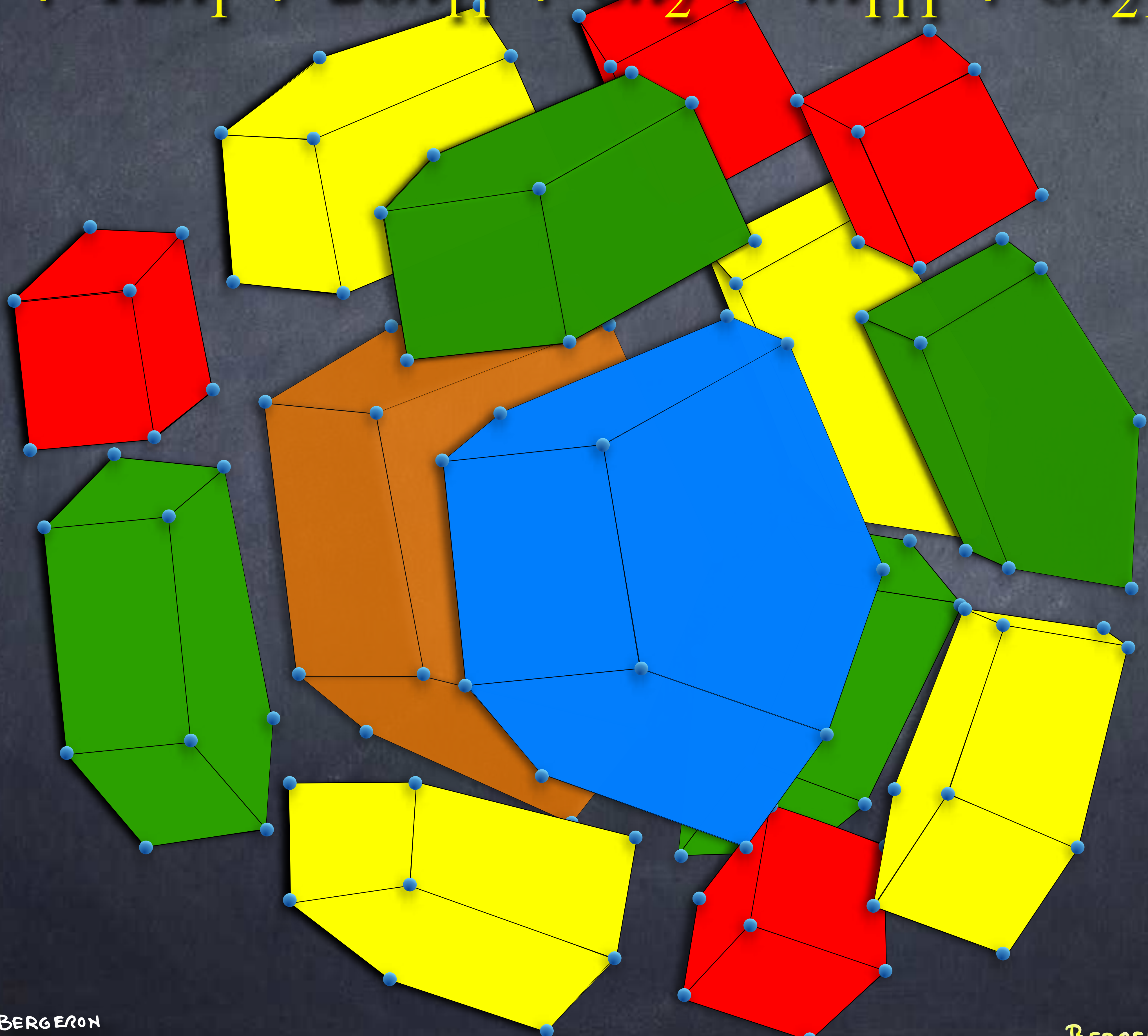


$$1 + 12h_1 + 20h_{11} + 8h_2 + 4h_{111} + 8h_{21} + 2h_3$$



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$$1 + 12h_1 + 20h_{11} + 8h_2 + 4h_{111} + 8h_{21} + 2h_3$$

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