



The HOMFLY Polynomial of a Forest Quiver

Amanda Schwartz

University of Michigan, Ann Arbor



Introduction

Postnikov introduced plabic graphs in order to study a stratification of the totally nonnegative Grassmannian into positroid cells [3]. These graphs have proved to be useful intermediate objects in establishing connections between cluster algebras and knot theory. For example, in [1] Galashin and Lam studied connections between invariants of links and quivers which can be associated to these plabic graphs. We establish further connections by defining the HOMFLY polynomial of a forest quiver and relating it to the HOMFLY polynomial of certain plabic links [4].

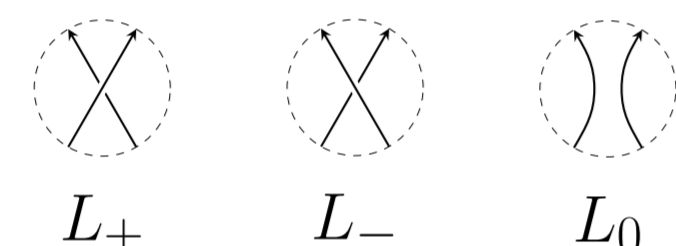
Plabic links generalize the class of positroid links, which are links that can be associated to positroids or various objects in bijection with them. The invariants of these links in certain cases can provide information about associated objects such as positroid varieties.

The HOMFLY Polynomial of a Link

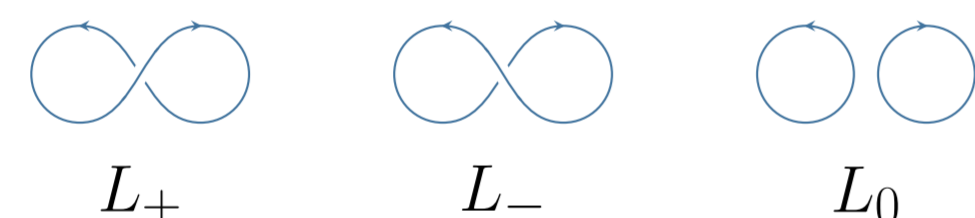
The *HOMFLY polynomial* is a link invariant which is a Laurent polynomial in the variables a and z . It is defined by the skein relation

$$aP(L_+) - a^{-1}P(L_-) = zP(L_0) \quad (1)$$

and setting $P(\text{Unknot}) = 1$ where L_+ , L_- , and L_0 are links whose diagrams are the same except locally at one location where they are related as follows:



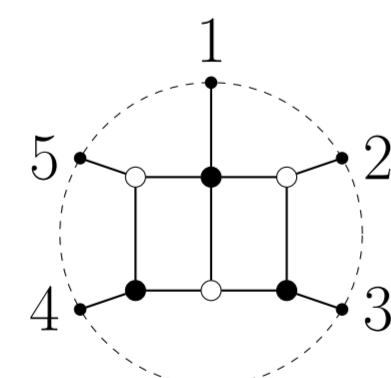
Example.



$$P(\text{2-component unlink}) = \frac{aP(\text{Unknot}) - a^{-1}P(\text{Unknot})}{z} = \frac{a - a^{-1}}{z}$$

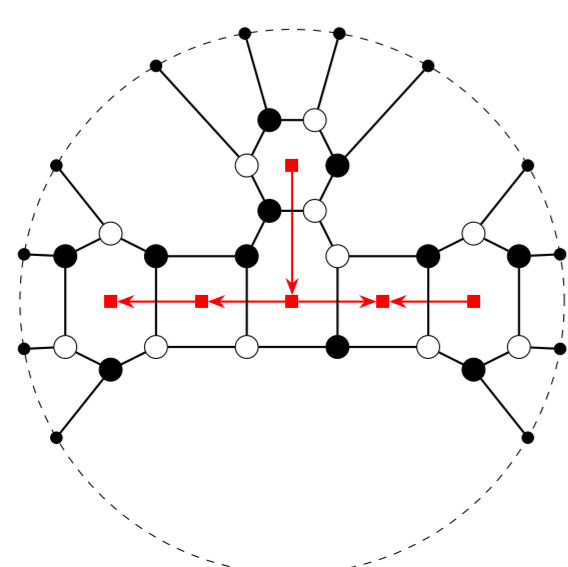
Plabic Graphs and Their Links and Quivers

A *plabic graph* G is a planar, bicolored graph which is embedded in the disk and whose vertices are colored black and white.



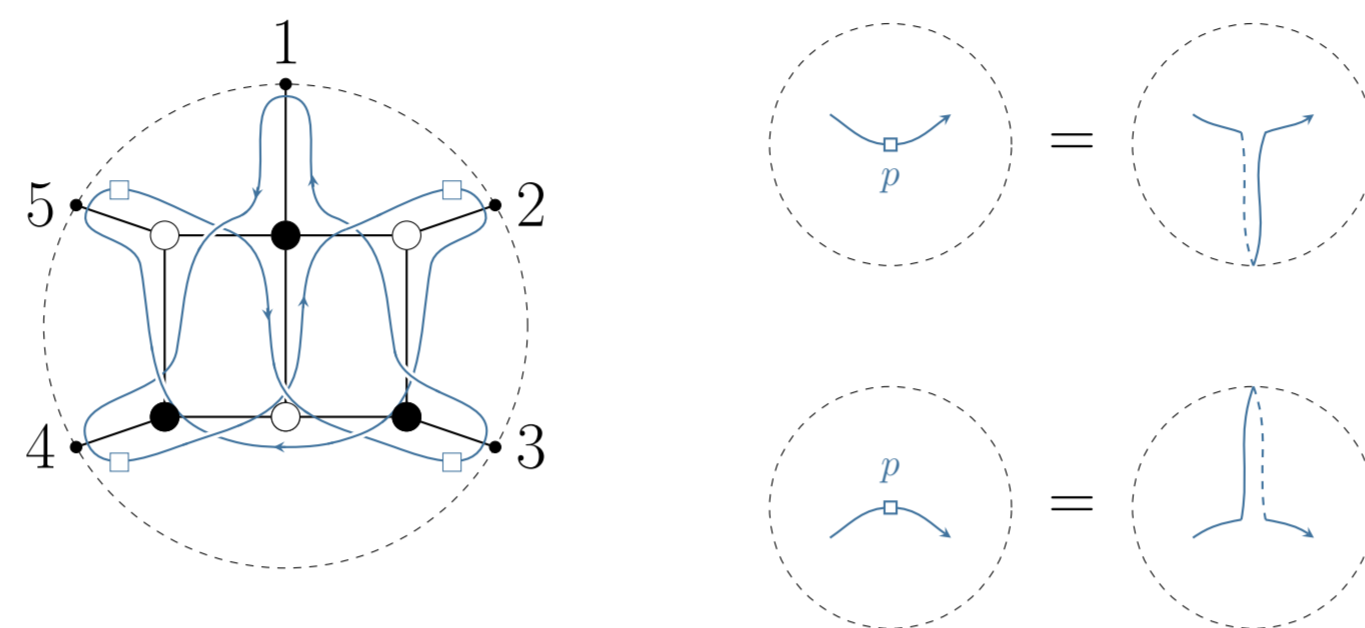
One can associate a directed, planar graph Q_G to each plabic graph G as follows:

- Place one vertex v_F inside each interior face F of G .
- Place an edge between vertices v_F and $v_{F'}$ for each edge e in G with opposite colored endpoints such that F and F' are adjacent to e .
- Orient the edge in Q_G so that as one travels along it, the white vertex in e is to the left.



If Q_G has no directed 1- or 2-cycles, it is a *quiver*.

One can draw a link diagram for a plabic link L_G^{plab} of G by traveling along the edges in G , turning maximally right at each black vertex and maximally left at each white vertex. The crossings are determined according to the arguments of the tangent vectors in the complex plane.



Motivations to Study Polynomial Invariants of Plabic Links

Theorem [2] For an acyclic, mutable quiver Q with n vertices the point count over \mathbb{F}_q of the associated cluster variety \mathcal{A} is given by

$$\#\mathcal{A}(\mathbb{F}_q) = R(Q; q) = \sum_{k \geq 0} a_k (q-1)^{n-2k} q^k \quad (2)$$

where a_k is the number of independent sets of size k in the quiver.

Theorem [1] For leaf recurrent plabic graphs G , the function $R(Q_G; q)$ can be obtained from a substitution into the HOMFLY polynomial of L_G^{plab} .

Guiding Question: *Can we go directly from the combinatorial objects, e.g. a plabic graph or its quiver, to the link invariants?*

The HOMFLY Polynomial of a Forest Quiver

Definition (S.) Let Q be a forest quiver. The *HOMFLY polynomial* of Q , denoted $f(Q)$, is defined recursively by setting

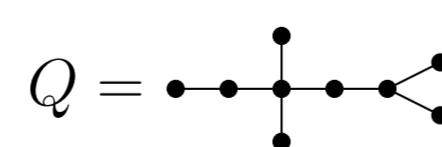
- $f(Q) = 1$ if Q is empty,
- $f(Q) = \frac{z+z^{-1}}{a} - \frac{z^{-1}}{a^3}$ if Q is a single vertex,
- $f(Q) = \frac{z}{a}f(Q - \{v\}) + \frac{1}{a^2}f(Q - \{v, \tilde{v}\})$ if v is a leaf in Q which is adjacent to \tilde{v} , and
- $f(Q) = f(Q_1) \cdot f(Q_2)$ if $Q = Q_1 \sqcup Q_2$.

The Alexander Polynomial $\Delta(Q)$ is obtained from $f(Q)$ by setting $a = 1$ and $z = t^{1/2} - t^{-1/2}$.

Connections Between the HOMFLY Polynomials

Theorem (S.) Let G be a connected plabic graph whose quiver Q_G is a forest quiver. Then $P(L_G^{\text{plab}}) = f(Q_G)$.

Example.



$$\Delta(Q) = t^{-9/2}(t^9 - t^8 - 3t^7 + 7t^6 - 8t^5 + 8t^4 - 7t^3 + 3t^2 + t - 1)$$

$$f(Q) = \frac{z^9 + 9z^7 + 28z^5 + 39z^3 + 28z + 11z^{-1} + 2z^{-3}}{a^9} - \frac{z^7 + 11z^5 + 36z^3 + 47z + 28z^{-1} + 7z^{-3}}{a^{11}} + \frac{5z^3 + 20z + 23z^{-1} + 9z^{-3}}{a^{13}} - \frac{z + 6z^{-1} + 5z^{-3}}{a^{15}} + \frac{z^{-3}}{a^{17}}$$

The Alexander Polynomial of a Forest Quiver

For the Alexander polynomial of a forest quiver, there is a closed-form expression.

Theorem (S.) Let Q be a forest quiver with n vertices. Then the Alexander polynomial of Q is given by

$$\Delta(Q) = \frac{(-1)^n}{t^{n/2}} \sum_{k=0}^{\lfloor n/2 \rfloor} c_k(Q) t^k (1-t)^{n-2k} \quad (3)$$

where $c_k(Q)$ is the number of independent sets of size k in the line graph of Q , i.e. the number of subsets of k distinct edges in Q which do not share any endpoints.

Example. Consider the Dynkin diagram E_6 .

$$c_0(E_6) = 1 \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

$$c_1(E_6) = 5 \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

$$c_2(E_6) = 5 \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

$$c_3(E_6) = 1 \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

Then the Alexander polynomial $\Delta(E_6)$ is

$$\Delta(E_6) = t^{-3} \left(t^0(1-t)^6 + 5t^1(1-t)^4 + 5t^2(1-t)^2 + t^3(1-t)^0 \right) = t^{-3} \left(t^6 - t^5 + t^3 - t + 1 \right)$$

Future Work

- Can one find a closed-form expression for the HOMFLY polynomial of a forest quiver, similar to the closed-form expression for the Alexander polynomial?
- Can the definition of the HOMFLY polynomial f be extended to a larger class of quivers?

References

- [1] Pavel Galashin and Thomas Lam. *Plabic links, quivers, and skein relations*. 2022. arXiv: 2208.01175 [math.CO]. URL: <https://arxiv.org/abs/2208.01175>.
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- [4] Amanda Schwartz. *The HOMFLY Polynomial of a Forest Quiver*. 2024. arXiv: 2410.00399 [math.CO]. URL: <https://arxiv.org/abs/2410.00399>.

Acknowledgments

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