A new definition for m-Cambrian lattices

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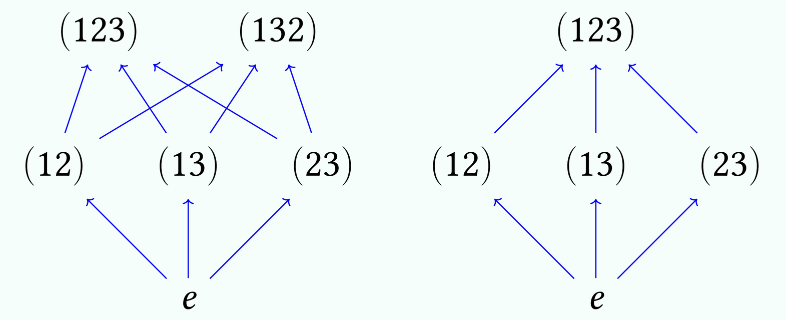
The Cambrian lattices, introduced by Reading, generalize the Tamari lattice to any choice of Coxeter element in any finite Coxeter group. They are further generalized to the m-Cambrian lattices. [Reading, 2006, Stump et al., 2020, Tamari, 1962] We propose a new definition for m-Cambrian lattices, where the objects are m-multichains in the noncrossing partition lattice, with an explicit comparison criterion.

Noncrossing partitions and Cambrian lattice

Definition

A **(standard) Coxeter element** in a Coxeter group W is a product of all simple reflections in some order. Coxeter elements are always maximal in the absolute order $\mathrm{Abs}(W)$.
The **noncrossing partition lattice** $\mathrm{NCL}(W, c)$ is the interval $[e, c]$ in the absolute order $\mathrm{Abs}(W)$.



Example

The absolute order on $\mathfrak{S}_3$ (left) and the noncrossing partition lattice $\mathrm{NCL}(\mathfrak{S}_3, (123))$ (right).

Proposition [C. Athanasiadis, T. Brady, and C. Watt, '07]

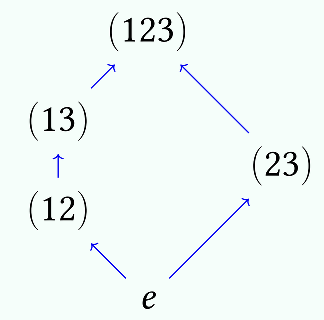
The choice of a Coxeter element c gives a total order $\mathrm{R}(c)$ on all reflections. Each c-noncrossing partition possesses a unique $\mathcal{R}$-word whose letters appear in increasing order. They are thus in bijection with subwords of the word $R(c)^2$ which are reduced $\mathcal{R}$-words for c, which we call **1-factorizations of c**.

	s (12)	u (13)	t (23)	s (12)	u (13)	t (23)
e	x		x			
(23)	x					x
(123)				x		x
(12)		x		x		
(13)			x		x	

$\underbrace{s\ u\ t\ \underbrace{s\ u\ t}_{w}}_{c}$

Definition

A **rotation** of a 1-factorization of c consists in moving a cross in the first copy of $\mathrm{R}(c)$ to the second copy, conjugating every cross in between.
The **Cambrian lattice** $\mathrm{Camb}(W, c)$ is the poset on 1-factorizations obtained as the transitive closure of **rotations**.



Example

The Cambrian lattice $\mathrm{Camb}(\mathfrak{S}_3, (123))$.

The m-Cambrian lattices

Definition

An **m-factorization** of a Coxeter element c is a subword of $\mathrm{R}(c)^{m+1}$ which is a reduced $\mathcal{R}$-word for c.

Proposition [D. Armstrong, '09]

Multichains with m elements in $\mathrm{NCL}(W, c)$ are in bijection with m-factorizations of c.

12 13 23	12 13 23	12 13 23	12 13 23	12 13 23	
x (13)			x (12)		$e \leqslant_{\mathcal{R}} (12) \leqslant_{\mathcal{R}} (12) \leqslant_{\mathcal{R}} (12)$
		x (23)		x (13)	$(13) \leqslant_{\mathcal{R}} (13) \leqslant_{\mathcal{R}} (123) \leqslant_{\mathcal{R}} (123)$

$\underbrace{\ \cdots\ \underbrace{\ \cdots\ \underbrace{\ \cdots\ }_{w_1}}_{w_2}}_{w_m}$

Example

Two 4-noncrossing partitions in $\mathrm{NCL}(\mathfrak{S}_3, (123))$.

Definition

For two m-noncrossing partitions $w_{(m)}$ and $w'_{(m)}$, we set $w_{(m)} \leqslant_{(m)} w'_{(m)}$ if

1. **Vertical condition**: For all $1 \leqslant i \leqslant m$, $w_i \leqslant w'_i$ in $\mathrm{Camb}(W, c)$;
2. **Diagonal condition**: For all $1 \leqslant i < m$, $w_i \leqslant_{\mathcal{R}} w'_{i+1}$ in $\mathrm{NCL}(W, c)$.

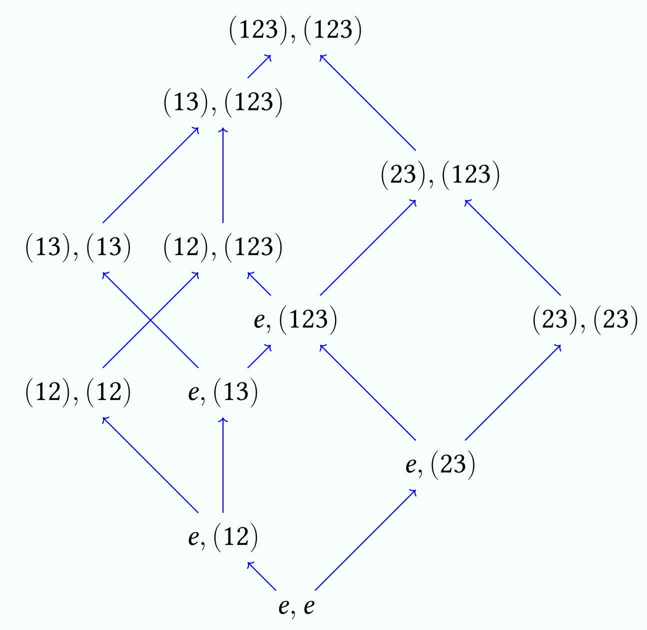
$$\begin{tikzcd}
w'_1 \arrow[r, "\leqslant_{\mathcal{R}}"] & w'_2 \arrow[r, "\leqslant_{\mathcal{R}}"] & w'_3 \arrow[r, "\leqslant_{\mathcal{R}}"] & \cdots \arrow[r, "\leqslant_{\mathcal{R}}"] & w'_{m-1} \arrow[r, "\leqslant_{\mathcal{R}}"] & w'_m \\
w_1 \arrow[u, "\leqslant"] \arrow[r, "\leqslant_{\mathcal{R}}"'] \arrow[ur, "\leqslant_{\mathcal{R}}"] & w_2 \arrow[u, "\leqslant"] \arrow[r, "\leqslant_{\mathcal{R}}"'] \arrow[ur, "\leqslant_{\mathcal{R}}"] & w_3 \arrow[u, "\leqslant"] \arrow[r, "\leqslant_{\mathcal{R}}"'] \arrow[ur, "\leqslant_{\mathcal{R}}"] & \cdots \arrow[r, "\leqslant_{\mathcal{R}}"'] \arrow[ur, "\leqslant_{\mathcal{R}}"] & w_{m-1} \arrow[u, "\leqslant"] \arrow[r, "\leqslant_{\mathcal{R}}"'] \arrow[ur, "\leqslant_{\mathcal{R}}"] & w_m \arrow[u, "\leqslant"]
\end{tikzcd}$$

Example

Since $e \leqslant (13)$, $(12) \leqslant (13)$, $(12) \leqslant (123)$, and $(12) \leqslant (123)$ in $\mathrm{Camb}(\mathfrak{S}_3, (123))$, the vertical condition is satisfied.
Since $e \leqslant_{\mathcal{R}} (12)$, $(13) \leqslant_{\mathcal{R}} (13)$, and $(13) \leqslant_{\mathcal{R}} (123)$ in $\mathrm{NCL}(\mathfrak{S}_3, (123))$, the diagonal condition is also satisfied.

Theorem [C., Fang, Henriet, '24+]

The binary relation $\leqslant_{(m)}$ is a partial order on m-noncrossing partitions, which is isomorphic to the m-Cambrian lattice $\mathrm{Camb}^{(m)}(W, c)$.



Example

The 2-Cambrian lattice $\mathrm{Camb}^{(2)}(\mathfrak{S}_3, (123))$.

c-increasing chains and greedy algorithm

Definition

If $w \lessdot w'$ is a covering relation, define the **flip reflection** $r(w, w')$ as the selected letter of w that is sent to the next copy.
A saturated chain $w_0 \lessdot w_1 \lessdot \ldots \lessdot w_m$ is an **c-increasing chain** if $r(w_i, w_{i+1})$ is smaller than $r(w_{i+1}, w_{i+2})$ in $\mathrm{R}(c)$ for all i.

Proposition: Unicity

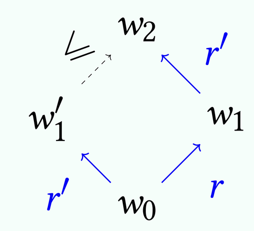
If $w \leqslant w'$ in the m-Cambrian lattice, if there exists a c-increasing chain from w to w', then the smallest letter of w that is not a letter of w' is the smallest flip reflection in any saturated chain from w to w'. Thus, a c-increasing chain from w to w' is **unique**.

Proposition: Existence

If $w \leqslant w'$ in the m-Cambrian lattice, there **exists** a c-increasing chain from w to w'.

Local reordering lemma

Let $w_0 \lessdot w_1 \lessdot w_2$ in $\mathrm{Camb}^{(m)}(W, c)$ with $r(w_0, w_1) > r(w_1, w_2) = r'$. Then $r' \in w_0$ and setting w'_1 the upper cover of w_0 such that $r(w_0, w'_1) = r'$, we have $w'_1 \leqslant w_2$.



Proposition

There is a greedy algorithm to decide comparability in the m-Cambrian lattice. It consists of reading the letters of $\mathrm{R}(c)^{m+1}$ from left to right, and try to flip each letter in turn.

Proposition

If the flip root of a covering relation appears in the i-th copy of $\mathrm{R}(c)$, then only the entry w_{m-i} of the m-noncrossing partition is modified.

Corollary

The existence of such an increasing c-chain is equivalent to the comparison scheme of $\leqslant_{(m)}$.

Further direction and open questions

Corollary

Since the unique c-increasing chain of each interval is lexicography smaller than all other chains, the m-Cambrian lattices are EL-shellable.

Corollary

We can easily generate the Cambrian lattices thanks to the greedy algorithm, and all m-Cambrian lattices thanks to the new comparison criterion.

Corollary

We can define a binary relation on Cambrian intervals, such that $[w_1, w'_1] \leqslant [w_2, w'_2]$ if $w_1 \leqslant_{\mathcal{R}} w_2$, $w_1 \leqslant_{\mathcal{R}} w'_2$, and $w_2 \leqslant_{\mathcal{R}} w'_2$ in the noncrossing partition lattice $NCL W, c$.
It is transitive and antisymmetric, and its m-multichains are in bijection with intervals in the m-Cambrian lattice.

$$\begin{tikzcd}
w'_1 \arrow[r, "\leqslant_{\mathcal{R}}"] & w'_2 \\
w_1 \arrow[u, "\leqslant"] \arrow[r, "\leqslant_{\mathcal{R}}"'] \arrow[ur, "\leqslant_{\mathcal{R}}"] & w_2 \arrow[u, "\leqslant"]
\end{tikzcd}$$

Question

Can we use this 'almost' poset on Cambrian chains to understand the conjecture stating that theere are as many intervals in the linear type A m-Cambrian lattice as in the m-Tamari lattice? [Bousquet-Mélou et al., 2012, Stump et al., 2020]

Question

The noncrossing partition lattice corresponds to the shard order (or core label order) of the Cambrian lattice. Can we mimic this m-construction by replacing the Cambrian lattice by some other lattices, e.g. semidistributive (and trim?) lattices?

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