Coinvariant stresses, Lefschetz properties and random complexes

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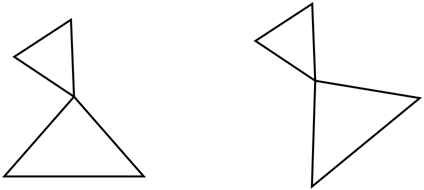
Rigidity theory of bar and joint structures

A **framework** is a pair (G, p) where G is a graph and $p : V(G) \to \mathbb{R}^n$ is an embedding of G in $\mathbb{R}^n$. A framework (G, p) is **flexible** if there exists a nontrivial continuous motion of the vertices that preserves the edge lengths of (G, p), and **rigid** otherwise.



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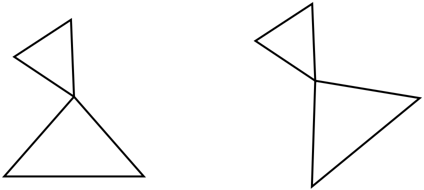


It turns out that to study rigidity of frameworks with generic embeddings, one only has to study the rank of a specific matrix M. Elements in the kernel of M are called **stresses**.

The setup

A **simplicial complex** Δ is a collection of subsets of $[n]$ such that

$$\tau \subset \sigma \in \Delta \implies \tau \in \Delta$$

Definition

Given a simplicial complex Δ on $[n]$ vertices, its **Stanley-Reisner** ideal is the ideal

$$I_\Delta = (x_{i_1} \dots x_{i_s} : \{i_1, \dots, i_s\} \notin \Delta)$$

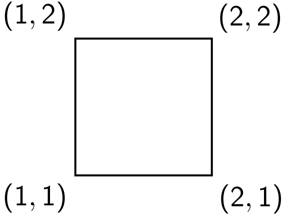
Definition

Given a homogeneous ideal $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ such that $\dim \frac{R}{I} = d$, a **linear system of parameters (lsop)** is a sequence of linear forms $\theta_1, \dots, \theta_d$ such that

$$\dim \frac{R}{I + (\theta_1, \dots, \theta_d)} < \infty$$

Lee's amazing idea (an example)

Given a simplicial complex Δ of dimension d and an embedding p of Δ in $\mathbb{R}^{d+1}$, we may view p as $d+1$ linear forms. In our case, these will be a lsop of I_Δ.



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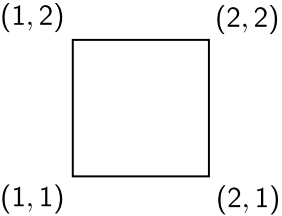

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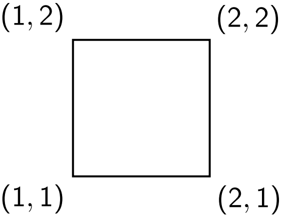

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Computing stresses = computing coefficients of f $(\deg f = 2)$

The summary

Let $A(\Delta) = \frac{R}{I_\Delta + (\theta_1, \ldots, \theta_{d+1})}$

	Algebra	Combinatorics
Data from geometric complex	Linear system of parameters	Vertex coordinates
Dimension of space of stresses	Hilbert series of $A(\Delta)$ (h-vector of Δ)	Dimension of solution space for the system of PDEs
Stresses	Elements of $A(\Delta)$	Solutions to system of differential equations

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A key problem when trying to work on the topics above is that computations can be sensitive to $\theta_1, \ldots, \theta_{d+1}$

The nonlinear case: symmetric polynomials

On the algebra side, most of the theory does not rely on $\theta_1, \dots, \theta_{d+1}$ being linear.

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \dots x_{i_k} \quad \textbf{\textit{k}-th elementary symmetric polynomial}$$

Proposition (DEP, GS, S, HM, AR)

If Δ is a simplicial complex of dimension d, the set of polynomials $e_1, \dots, e_{d+1}$ is a system of parameters of I_Δ.

From now on, let $\mathbb{K}^{co}(\Delta)$ denote the following finite dimensional vector space

$$\mathbb{K}^{co}(\Delta) = \frac{\mathbb{K}[x_1, \dots, x_n]}{I_\Delta + (e_1, \dots, e_{d+1})} \qquad \mathrm{HS}(\mathbb{K}^{co}(\Delta), q) = h_\Delta(q)[q]_d!$$

A starting point: a very familiar example

Let $\mathcal{S}^{\mathrm{co}}(\Delta)$ be the space of solutions to the system

$$\begin{cases} \sum_{1 \le i_1 < \cdots < i_k \le n} f_{x_{i_1} \dots x_{i_k}} = 0 \quad \forall k \\ f_{x_{i_1} \dots x_{i_s}} = 0 \quad \forall \{i_1, \dots, i_s\} \notin \Delta \end{cases}$$

we call it the space of **coinvariant stresses of Δ**

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Example

If Δ is the boundary of a simplex, coinvariant stresses correspond to solutions of

$$\sum_{1\leq i_1<\cdots<i_k\leq n} f_{x_{i_1}\dots x_{i_k}} = 0 \quad \forall k$$

It is known that there is a unique polynomial of degree $\binom{n}{2}$ satisfying the condition above:

$$\prod_{1\leq i<j\leq n} (x_i - x_j) \quad \text{The Vandermonde determinant}$$

From the boundary of a simplex to arbitrary complexes

Theorem (Top coinvariant stresses and top homology, (-, 2025))

Let Δ *be a* d*-dimensional simplicial complex and* $c_1 F_1 + \cdots + c_s F_s (\neq 0) \in \tilde{H}_d(\Delta; \mathbb{K})$*. Then*

$$c_1 x_{F_1} V(F_1) + \cdots + c_s x_{F_s} V(F_s) \in \mathcal{S}^{co}(\Delta),$$

where $x_{F_j} = \prod_{i \in F_j} x_i$ *and*

$$V(F_j) = \prod_{\substack{i<j \\ \{i,j\} \in \Delta}} (x_i - x_j)$$

Corollary (-, 2025)

If Δ *is a* d*-dimensional* $\mathbb{K}$*-homology sphere, then the unique polynomial of degree* $\binom{d+2}{2}$ *in* $\mathcal{S}^{co}(\Delta)$ *is the one above.*

Some (unexpected?) consequences of coinvariant stresses

Let Δ be a d-dimensional simplicial complex and

$$A_\Delta = \frac{\mathbb{K}[x_1, \ldots, x_n]}{I_\Delta + (x_1^{d+2}, \ldots, x_n^{d+2})}$$

Theorem (WLP and coinvariant stresses (-, 2025))

If $\tilde{H}_d(\Delta; \mathbb{K}) \neq 0$ and $f_{d-1} \geq f_d$, then A_Δ fails the weak Lefschetz property (WLP)

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Theorem (Failure should be expected (-, 2025))

Given a generalized Erdős–Rényi model for complexes of dimension $d > 0$, there exists an open interval $(c_d, d+1) \neq \emptyset$ such that

$$\lim_{n \to \infty} \mathbb{P}(A_\Delta \text{ fails the WLP}) = 1$$

when the probability parameter p is in $(c_d, d+1)$

Some questions

When Δ is the boundary of a simplex the ring $\mathbb{K}^{co}(\Delta)$ has several nice properties from combinatorial, algebraic and geometric perspectives.

Question (Coinvariant algebraic g-theorem)

Let Δ be a $\mathbb{Q}$-homology sphere. Does the ring $\mathbb{K}^{co}(\Delta)$ satisfy the strong Lefschetz property?