DG-Sensitivity: Matching and Pruning

Henry Potts-Rubin

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(joint with Hugh Geller & Des Martin)

Resolutions

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A (Q-free) **resolution** $(\mathbb{F}, \partial^{\mathbb{F}})$ of Q/I is an exact sequence

$$\cdots \to \mathbb{F}_i \xrightarrow{\partial_i^{\mathbb{F}}} \mathbb{F}_{i-1} \to \cdots \to \mathbb{F}_1 \xrightarrow{\partial_1^{\mathbb{F}}} Q \to Q/I \to 0$$

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Since each $\partial_i^{\mathbb{F}}$ is a map between free modules, it may be represented as a matrix. If the entries of each $\partial_i^{\mathbb{F}}$ are in $(x_1, \ldots, x_n)$, call $\mathbb{F}$ **minimal**.

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Not minimal. *The* minimal Q-free resolution of Q/I is

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The Taylor Resolution of a Monomial Ideal

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which has basis $\{e_F\}$, where F is an i-element subset of $\{f_1, \ldots, f_t\}$.

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$$\partial(e_F) = \sum_{f_j \in F} (-1)^{\sigma(j,F)} \frac{m_F}{m_{F_j}} e_{F_j},$$

where $F_j := F \smallsetminus \{f_j\}$, $m_F = \operatorname{lcm}\{f_k : f_k \in F\}$, and $\sigma(j, F) = |\{f_k \in F \mid k < j\}|$.

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Big Question

When does a resolution of a module admit the structure of a dg algebra?

DG Algebra Structure on the Taylor Resolution

Example (Gemeda, 1976)

$$e_V \cdot e_W = \begin{cases} (-1)^{\sigma(V,W)} \dfrac{m_V m_W}{m_{V \cup W}} e_{V \cup W}, & V \cap W = \emptyset, \\ 0, & V \cap W \neq \emptyset, \end{cases}$$

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Notice: $V, W \subseteq V \cup W$

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Example of a complex that *does not* admit the structure of a dg algebra:

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Example of a complex that *does not* admit the structure of a dg algebra:

- The minimal $\mathbb{k}[x, y, z, w]$-free resolution of $\dfrac{\mathbb{k}[x, y, z, w]}{(x^2, xy, yz, zw, w^2)}$ (Avramov, 1981)

Questions

Refined Question

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Related Question

Given a dg algebra resolution $\mathbb{F}$ of Q/I, what processes can we perform on $\mathbb{F}$ to *construct a new resolution* $\mathbb{F}'$ (possibly of a *different module*, possibly over a *different ring*) such that $\mathbb{F}'$ admits the structure of a dg algebra?

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Related Question

Given a dg algebra resolution $\mathbb{F}$ of Q/I, what processes can we perform on $\mathbb{F}$ to *construct a new resolution* $\mathbb{F}'$ (possibly of a *different module*, possibly over a *different ring*) such that $\mathbb{F}'$ admits the structure of a dg algebra? And when does such a process preserve/yield minimality?

Process: Matching

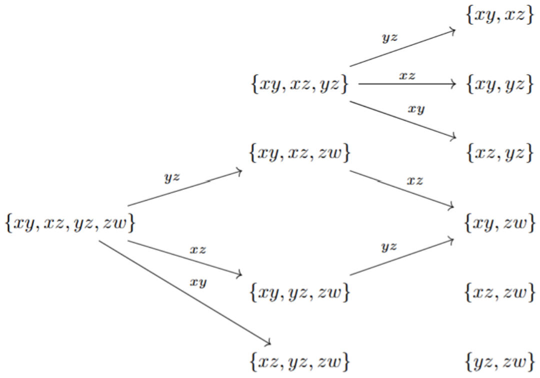
Example

$I = (xy, xz, yz, zw)$

Process: Matching

Example

$I = (xy, xz, yz, zw)$ (arrows identify non-minimality)



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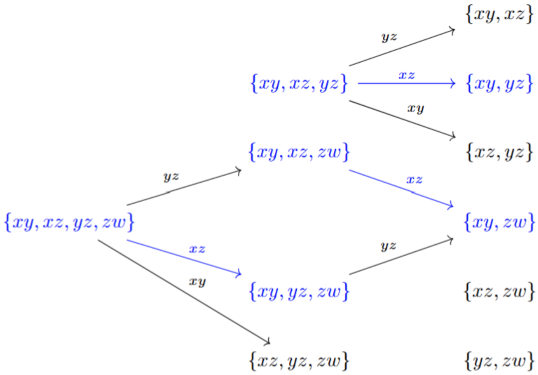
Example

$I = (xy, xz, yz, zw)$, Morse matching



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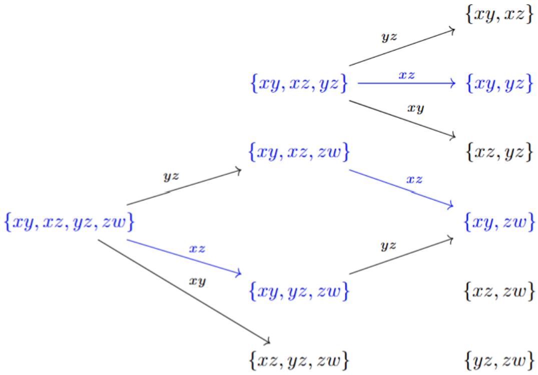

$$\mathbb{J} := \bigoplus_{V = \text{ blue source}} 0 \to Qe_V \to Q\partial(e_V) \to 0 \subseteq \mathbb{T}$$

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If the collection of blue sources is closed under taking supersets, then $\mathbb{T}/\mathbb{J}$ admits the structure of a dg algebra.

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Theorem (Geller-Martin-P, 2025+)

Let G be a graph such that Q/I_G is minimally resolved by a Lyubeznik resolution, where I_G is the edge ideal of G. Then, the minimal free resolution of Q/I_G admits the structure of a differential graded algebra.

Process: Pruning

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Pruning Monomial Ideals

Theorem (Boocher, 2012)

For a monomial ideal I of a polynomial ring Q, (iteratively) pruning the minimal Q-free resolution of Q/I obtains the minimal R-free resolution of R/I'.

Main Result

Theorem (Geller-Martin-P, 2025+)

For *squarefree* monomial ideals, (iteratively) pruning the minimal free resolution of Q/I is DG-sensitive.

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If Δ' is a facet-induced subcomplex of Δ and R is the ambient polynomial ring of Δ', then the minimal R-free resolution of $R/I(\Delta')$ admits the structure of a differential graded algebra.

Classifications

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Let Γ be a tree of diameter d. The minimal Q-free resolution of Q/I_Γ admits the structure of a differential graded algebra if and only if $d \leq 4$.

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Let C_n be the cycle on n vertices. The minimal Q-free resolution of Q/I_{C_n} admits the structure of a differential graded algebra if and only if $n \leq 5$.

Thank You!

Thanks for listening!

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