Toric Geometry and Analytic Combinatorics

Toric Compactifications for Critical Points At Infinity

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What ACSV cares about

Given a rational multivariate function

$$F(\mathbf{z}) = Q(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}, \quad \mathbf{z}^{\mathbf{r}} = z_1^{r_1}\cdots z_d^{r_d}$$

its coefficients $a_{\mathbf{r}} \in \mathbb{C}$ are determined by a multivariate Cauchy integral

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d}\int_{T\subset\mathcal{M}:\mathbb{C}_*^d\setminus\mathcal{V}} \mathbf{z}^{-\mathbf{r}}F(\mathbf{z})\frac{d\mathbf{z}}{\mathbf{z}}$$

where the direction $\mathbf{r} = (r_1, \cdots, r_d) \in \mathbb{R}^d$.

Analytic combinatorics in several variables (ACSV) concerns the asymptotic behaviour of

$a_{\mathbf{r}}$ as $|\mathbf{r}| \to \infty$ with $\mathbf{r}/|\mathbf{r}| \to \hat{\mathbf{r}} \in \mathbb{R}^d_{\geq 0}$.

Binomial coefficients example

Consider $F(x, y) = \dfrac{1}{1 - x - y}$ and $\mathbf{r} = (r, s)$,

$$F(x, y) = \sum_{r,s \geq 0} \binom{r+s}{r} x^r y^s$$

$$a_{\mathbf{r}} = \binom{r+s}{r} \sim \frac{(r+s)^{r+s}}{r^r s^s} \sqrt{\frac{r+s}{2\pi rs}}$$

	r=1	r=2	r=3	r=4	r=5
s=1	2	3	4	5	6
s=2	3	6	10	15	21
s=3	4	10	20	35	56
s=4	5	15	35	70	126
s=5	6	21	56	126	252

$$\hat{\mathbf{r}} = (1,1), r = s = n : a_{n,n} = \binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}} + O\left(\frac{4^n}{n^{3/2}}\right)$$

$$\hat{\mathbf{r}} = (2,1), r = 2s = 2n : a_{2n,n} \sim \left(\frac{27}{4}\right)^n \sqrt{\frac{3}{4\pi n}}$$

Saddle point method

We typically analyze the integral $a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}$ using saddle point methods.

The factor $\mathbf{z}^{-\mathbf{r}} = e^{-\mathbf{r}\cdot\log\mathbf{z}}$ controls decay of the integrand, and determines where one can "saddle" the contour to extract dominant contributions.

Define the **height function** $h_{\mathbf{r}}(\mathbf{z}) := \Re(\log \mathbf{z}^{-\mathbf{r}}) = -\sum_{j=1}^{d} r_j \log |z_j|$.

A point $\mathbf{p} \in V(H)$ is called a **critical point in the direction $\mathbf{r}$** if the matrix

$$J(\mathbf{p}, \mathbf{r}) = \begin{bmatrix} z_1 \frac{\partial H}{\partial z_1}(\mathbf{p}) & \cdots & z_d \frac{\partial H}{\partial z_d}(\mathbf{p}) \\ r_1 & \cdots & r_d \end{bmatrix}$$

has rank one, meaning $dh_{\mathbf{r}}$ vanishes on the tangent space of $V(H)$.

Morse theory indicates …

Dominant asymptotics arise from points where $h_{\mathbf{r}}$ has critical points on the variety $V(H)$, i.e.

$$a_{\mathbf{r}} \sim \frac{1}{(2\pi i)^{d-1}} \sum_{i \in I} \sigma_i \int_{\mathscr{C}_i} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}},$$

where I contains all the critical points of $h_{\mathbf{r}}$ on $V(H)$, σ_i are integers, and each integral is in stationary phase over its domain of integration $\mathscr{C}_i$.

Baryshinikov, Melczer, and Pemantle [2023] showed that this method **breaks down at infinity.**

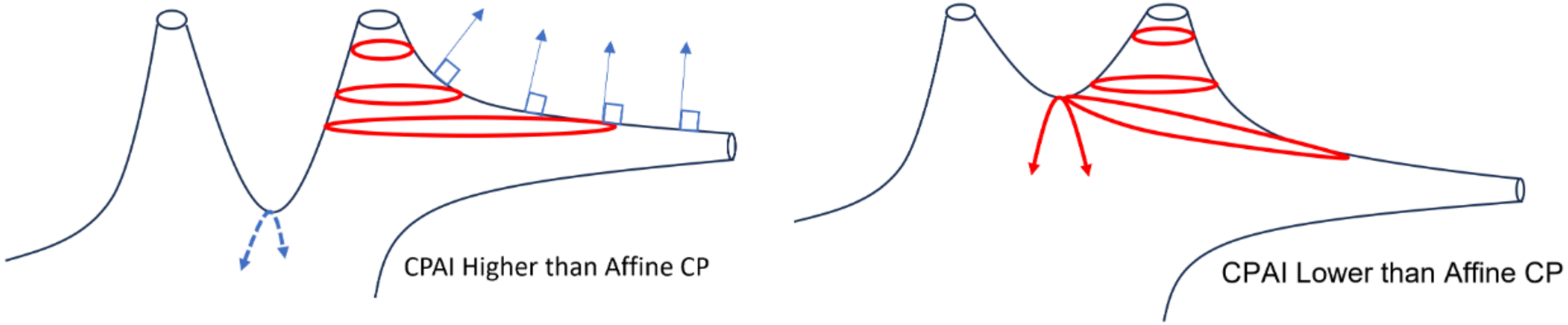
The height function becomes non-proper when approaching boundary faces of $(\mathbb{C}_*)^d$, new critical points at infinity may appear and influence asymptotics.

preimage of a compact set is a compact set

Morse theory breaks down

For Morse theoretic contour deformation to be valid, height function must be proper and well-defined. With our particular height function $h_{\mathbf{r}}$, problems occur at:

1. projective points --- topological obstructions at **infinity**



2. vanishing coordinates -- height function is not well-defined.

Remedy: check of CPAI

This project addresses the central question:

> How can we systematically detect, describe, and compute critical points at infinity?

This gives a principled way to understand obstructions to contour deformation and ultimately determines where asymptotic contributions may arise.

Definition of critical points

A point $\mathbf{p} \in V(H)$ is called **a critical point in the direction r** if $[\nabla_{\log} H(\mathbf{p})] = w_{\mathbf{r}}$,

where $$\nabla_{\log} H(\mathbf{p}) = \sum_{j=1}^{d} \left(z_j \frac{\partial H}{\partial z_j} \right)(\mathbf{p}) \frac{dz_j}{z_j}, \qquad w_{\mathbf{r}} := \left[\sum_{j=1}^{d} r_j \frac{dz_j}{z_j} \right]$$

projectivization of the log gradient $\in \mathbb{P}\left(\mathbb{C}\left\{ \frac{dz_1}{z_1}, \cdots, \frac{dz_d}{z_d} \right\} \right)$

projectivization of one-form $\in \mathbb{P}\left(\mathbb{C}\left\{ \frac{dz_1}{z_1}, \cdots, \frac{dz_d}{z_d} \right\} \right) \simeq \mathbb{CP}^{d-1}$

We define the **critical locus** as

$$\mathfrak{C} := \{(\mathbf{p}, \mathbf{r}) \in V(H) \times \mathbb{CP}^{d-1} : [\nabla_{\log} H(\mathbf{p})] = w_{\mathbf{r}}\} \subseteq \mathbb{C}_*^d \times \mathbb{CP}^{d-1}.$$

To work with critical point at infinity, we embed $(\mathbb{C}_*)^d$ into a compact space Y:

$$\mathfrak{C} \subset (\mathbb{C}_*)^d \times \mathbb{CP}^{d-1} \hookrightarrow Y \times \mathbb{CP}^{d-1}$$
$$(\mathbf{p}, \mathbf{r}) \mapsto (\Phi(\mathbf{p}), \mathbf{r})$$

CPAI with compactification

A point $\mathbf{p}$ is a **critical point at infinity** in direction $\hat{\mathbf{r}}$, if there exists a sequence of critical points $(\mathbf{p}_n, \mathbf{r}_n) \in \mathfrak{C}$ such that

$$(\mathbf{p}_n, \mathbf{r}_n) \to (\mathbf{p}, \hat{\mathbf{r}}) \in \overline{\mathfrak{C}}^Y \text{ and } \mathbf{p} \in \overline{V(H)}^Y \backslash V(H) .$$

We call $\overline{\mathfrak{C}}^Y$ **asymptotic critical locus**, which is the closure of the critical locus $\mathfrak{C}$ in $Y \times \mathbb{CP}^{d-1}$.

$$\begin{array}{ccc} \mathfrak{C} \subseteq Y \times \mathbb{CP}^{d-1} & \xrightarrow{\pi_1} & Y \supseteq \overline{V(H)}^Y \backslash V(H) \\ \downarrow \pi_2 & & \\ \mathbb{CP}^{d-1} & \subseteq & D_\infty := \pi_2(\overline{\mathfrak{C}}^Y \cap \pi_1^{-1}(\overline{V(H)}^Y \backslash V(H))) \end{array}$$

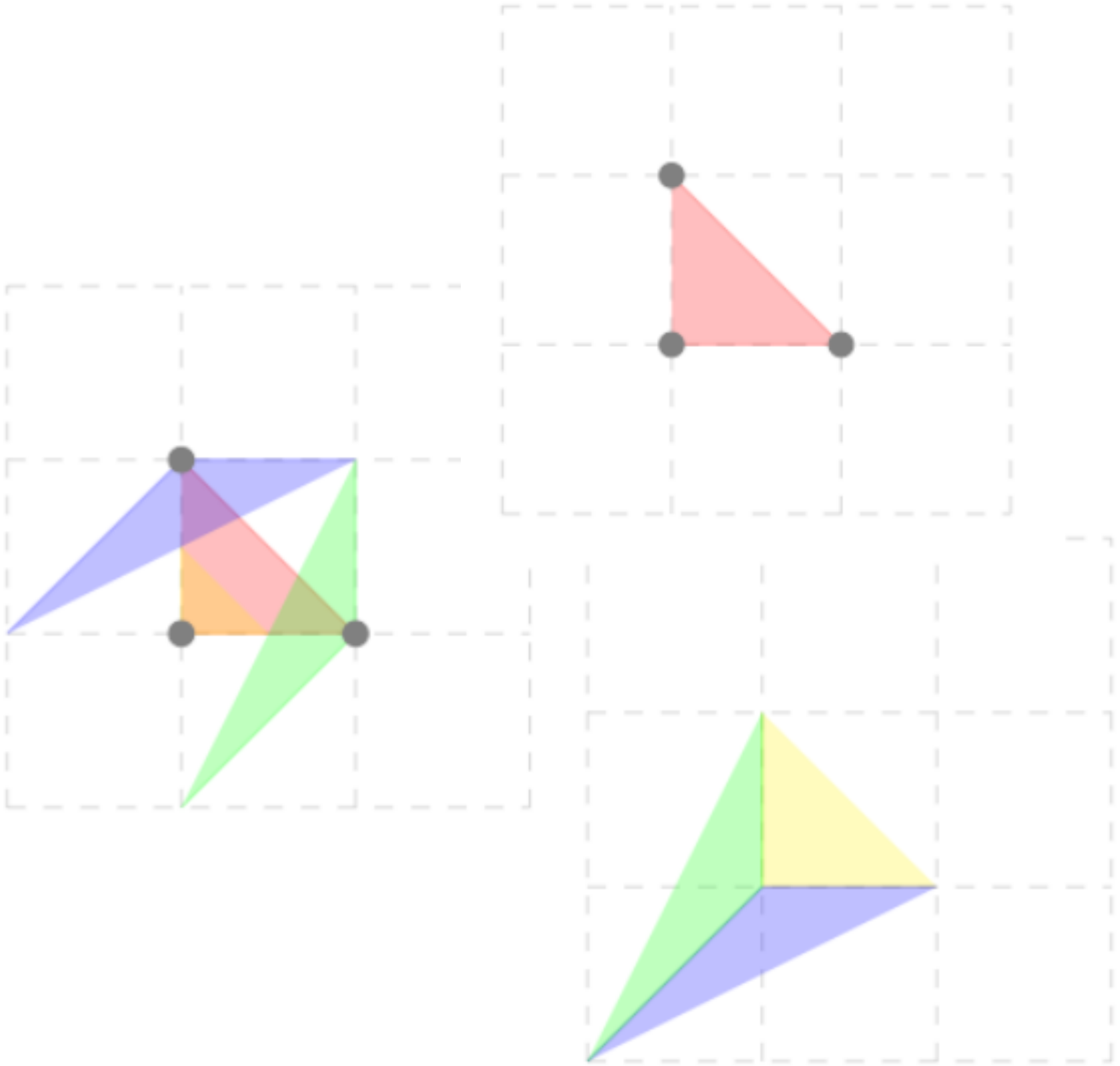
We call D_∞ as the set of **asymptotic critical directions**.

The variety D_∞ does not depend on the compactification Y, which allows us to pick proper toric compactification X_Σ.

Toric compactification

Given a polynomial $H(\mathbf{z}) = \sum_{j=1}^{n} c_j \mathbf{z}^{\mathbf{m}_j}$

For example, $H = 1 - x - y$:



Newton polytope $P = \text{Newt}(H)$ is the lattice polytope defined as the convex hull of the exponent vectors $\mathbf{m}_j \in \mathbb{N}^d_{\geq 0}$.

Normal cone of a face $Q \preccurlyeq P$ is
$\sigma_Q = \{u \in \mathbb{R}^d : \langle m - m', u \rangle \geq 0,\ m \in Q,\ m' \in P\}$.

Normal fan $\Sigma(P)$ consists of all normal cones σ_Q.

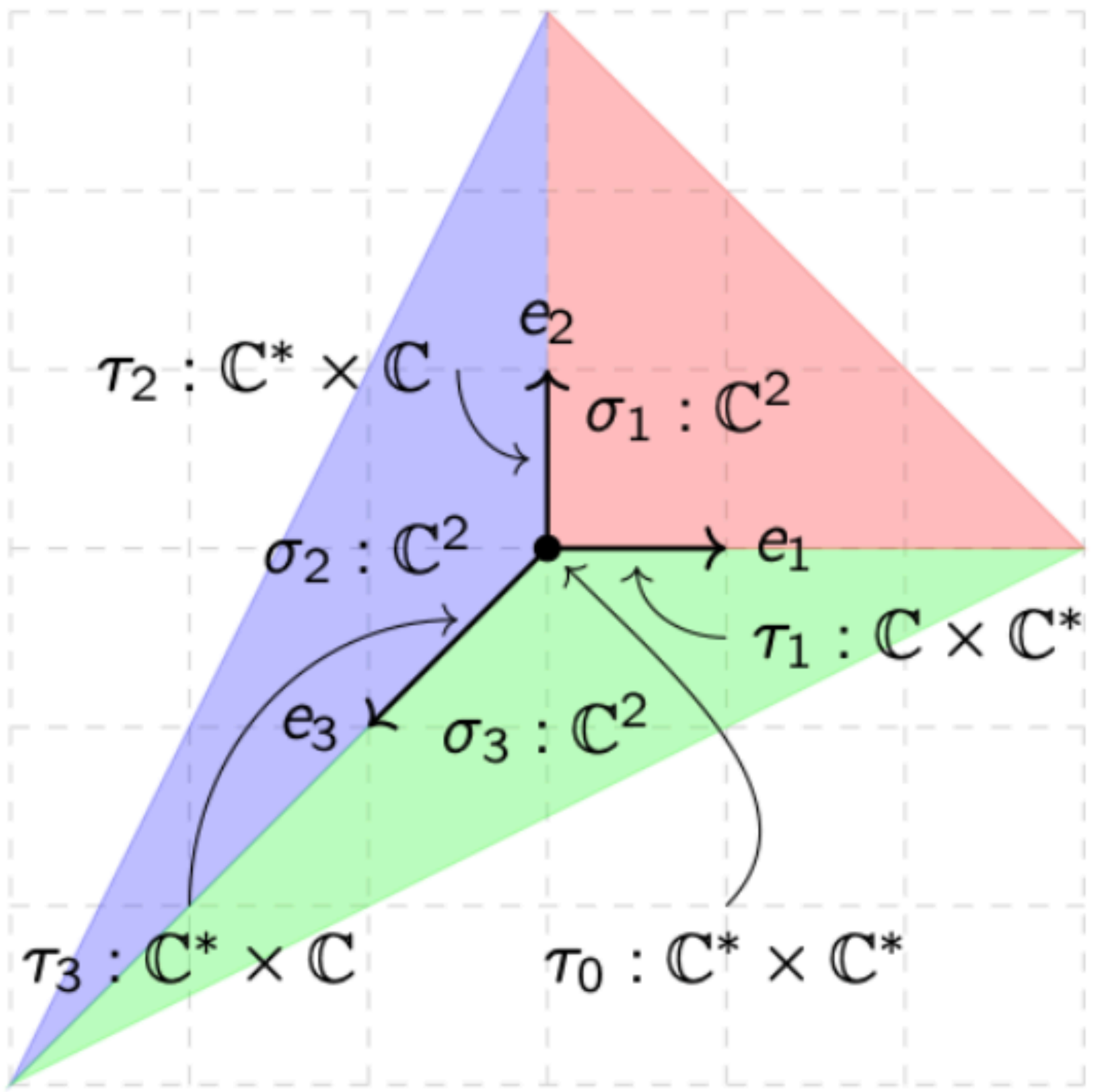
Toric compactification X_Σ

The **toric variety** associated to a fan Σ is

$$X_\Sigma = \bigsqcup_{\sigma \in \Sigma} O(\sigma)$$

where each orbit $O(\sigma)$ is the quotient of $(\mathbb{C}_*)^d$ by the subtorus generated by the cone.

Example continued, $H = 1 - x - y$:



Transversality

Suppose f is a polynomial supported on a set A.

Let $P = \mathrm{Newt}(f)$ and let Σ be a primitive fan subdividing the normal fan $\Sigma(P)$.

If the initial form $E_A(f) \neq 0$, then:

$\overline{V(f)} \subset X_\sigma$ is smooth and intersects the torus orbit $O(\sigma)$ transversely.

Intersect transversely means two varieties meet in the "most generic"/"non-degenerate" way.

Limits of critical points are well-defined!

This indicates that X_Σ is a good space to work with.

Another definition for CPAI

Let X_σ be a chart in X_Σ. $\mathbf{v}$ is a vertex of $\mathrm{Newt}(f)$ and σ is a primitive cone contained in $\sigma_{\mathbf{v}}$.

The **logrithmic critical locus** of f in X_Σ is

$$\mathfrak{C}_{X_\Sigma} := \bigcup_{\sigma \in \Sigma} \{(\mathbf{p}, \mathbf{r}) \in V(g) \times \mathbb{P}\left(\mathbb{C}\left\{\frac{dz_1}{z_1}, \cdots, \frac{dz_d}{z_d}\right\}\right) \mid [\nabla_{\log} g(\mathbf{p})] = w_{\mathbf{r}}\}$$

where $$g := \mathbf{z}^{-\mathbf{v}} f, \quad \nabla_{\log} g(\mathbf{p}) = \sum_{j=1}^{d} \left(z_j \frac{\partial g}{\partial z_j}\right)(\mathbf{p}) \frac{dz_j}{z_j}, \quad w_{\mathbf{r}} := \left[\sum_{j=1}^{d} r_j \frac{dz_j}{z_j}\right]$$

The asymptotic critical locus $\overline{\mathfrak{C}}^{X_\Sigma}$ is based on taking limits of a sequence of critical points.

The logrithmic critical locus $\mathfrak{C}_{X_\Sigma}$ uses the charts in the toric compactification but it is explicit and easy to compute.

Equivalence of two definitions of CPAI

According to Sattelberger and Van Der Veer [2023],

asymptotic critical locus is equal to logrithmic critical locus

$$\overline{\mathfrak{C}}^{X_\Sigma} = \mathfrak{C}_{X_\Sigma}$$

Using sparse resultant to represent the logrithmic critical locus, we obtain:

f has critical point on $O(\sigma)$ in the direction $\mathbf{r}$ if and only if

$$R_{A_u}(g, r) = 0,$$

where u lies in the interior of σ and $g = \mathbf{z}^{-\mathbf{v}} f$ is a localization of f at a vertex $\mathbf{v} \in \sigma$.

Characterizing CPAI

Given a polynomial H, we construct a compact toric variety X_Σ using a primitive fan Σ subdividing the normal fan of Newton polytope of H, then

- Any direction with a CPAI must be perpendicular to some ray of Σ
- H has critical point on $O(\sigma)$ in the direction $\mathbf{r}$ if and only if $R_{A_\mathbf{u}}(g, r) = 0$ where $R_{A_\mathbf{u}}$ is a sparse resultant.

Summarization

- **Problem**: critical points at infinity obstruct standard ACSV methods
- **Method**: Use toric geometry to compactify space and analyze initial form of H
- **Results**: A geometric criterion of CPAI, an explicit algebraic computation using sparse resultants

Thank you!