h*-vectors for matroids

Chris Eur (w/ Alex Fink & Matt Larson)

Let $P \subset \mathbb{R}^n$ be a lattice polytope (i.e. vertices in $\mathbb{Z}^n$) of dim. d.

Ehrhart: the function $k \mapsto \#|kP \cap \mathbb{Z}^n|$ is a deg d polynomial

$$\Rightarrow \text{Ehr}_P(t) := \sum_{k \geq 0} \#|kP \cap \mathbb{Z}^n| t^k = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}}$$

we call $(h_0^*, \ldots, h_d^*)$ the h*-vector of P.

E.g. $P = $ ⊿ , then

•   ⊿   ⊿ $\#|kP \cap \mathbb{Z}^2|$, $\text{Ehr}_P(t) = \frac{1}{(1-t)^3}$

   1   3   6 $\cdots$ $\overset{\shortparallel}{\binom{k+2}{2}}$

Stanley: h*-vectors are nonnegative.

If P is normal (i.e. $kP \cap \mathbb{Z}^n = k(P \cap \mathbb{Z}^n) \ \forall k \geq 0$)
then h*-vector of P is a Macaulay vector.

Assume: ① $\exists a \in \mathbb{Z}$ such that $x_1 + \cdots + x_n = a \quad \forall x \in P$.
② P is normal.

Let k be a field, and define $f_P : \mathbb{P}_k^{n-1} \dashrightarrow \mathbb{P}(k^{\#|P \cap \mathbb{Z}^n|})$ by

$\quad [t_1 : t_2 : \cdots : t_n] \longmapsto [t^m]_{m \in P \cap \mathbb{Z}^n}$, where $t^m := t_1^{m_1} \cdots t_n^{m_n}$.

Let $Y_P :=$ closure of image(f_P).

Hochster: $Y_P \subset \mathbb{P}(k^{|P \cap \mathbb{Z}^n|})$ is projectively normal & arithmetically Cohen-Macaulay
(i.e. its homogeneous coord. ring $S_\bullet(Y_P)$ is normal & Cohen-Macaulay).
$\Rightarrow$ Stanley since Hilbert series of $S_\bullet(Y_P)$ equals $\text{Ehr}_P(t)$
(i.e. numerator of $\text{Hilb}(S_\bullet(Y_P)) \longleftrightarrow$ h*-vector of P).

Braun, Ferroni-Higashitani: surveys on h*-vectors.

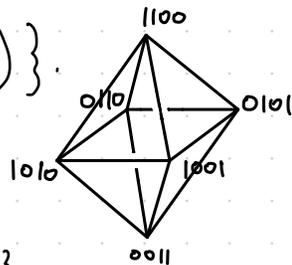Let $L \subseteq k^n$ be a linear subspace of dim r. ($\mathbb{P}L \subseteq \mathbb{P}^{n-1}$ of dim. $r-1$).

Let $Y_{L,P} :=$ closure of image$(f_P|_{\mathbb{P}L})$.

<u>Question</u> Is $Y_{L,P}$ proj. normal & a CM?

No. (Exercise: rat'l twisted quartic in $\mathbb{P}^3$)

<u>Defn</u> A **generalized permutohedron** (GP) is a lattice polytope $P \subset \mathbb{R}^n$ satisfying:

for each edge in P, $\exists\, 1 \leq i < j \leq n$ such that it is parallel to $e_i - e_j$.

<u>E.g.</u> hypersimplices: $\Delta(k,n) := \text{conv}\left\{ \sum_{i \in S} e_i \mid S \in \binom{\{1,\cdots,n\}}{k} \right\}$.

<u>Defn</u> A **matroid** M is a collection $\mathcal{B}$ of subsets of $\{1,\cdots,n\}$

such that $P(M) := \text{conv}\left\{ \sum_{i \in B} e_i \mid B \in \mathcal{B} \right\}$ is a GP.   Serganova '86]

[Gelfand–Goresky–MacPherson'

<u>E.g.</u> $L \subseteq k^n$ defines $M(L)$ by $\mathcal{B} = \left\{ B \subseteq \{1,\cdots,n\} \mid L \hookrightarrow k^n \twoheadrightarrow k^B \text{ isom.} \right\}$.

(i.e. if $L = \text{rowspan}\left[A_{r \times n} \right]$, then $B \longleftrightarrow$ max'l lin. indep. columns).

<u>Thm</u> [E.–Fink–Larson]

(1) For any $\mathbb{P}L \subseteq \mathbb{P}^{n-1}$ and $P \subset \mathbb{R}^n$ a GP, $Y_{L,P}$ is <u>proj. normal & a CM</u>.

(2) The Hilb. series of $S.(Y_{L,P})$ depends only on $M(L)$ and P.

$\hookleftarrow$ call the coefficients of its numerator h^*-vector of $(M(L), P)$

denoted $h^*(M(L), P)$.

(2') One can generalize to define $h^*(M,P)$ for any matroid M & GP P,

and $h^*(M,P)$ is always a Macaulay vector (nonnegative).

Positivity :   Hodge thry of matroids [Adiprasito–Huh–Katz '18] $\Rightarrow$ Chow

Ours $\Rightarrow$ K-theory

<u>Cor</u> ① $P = \Delta(n-1, n)$, $S_\bullet(Y_{L,P})$ is the Orlik–Terao algebra, recovering that it is CM (previously Proudfoot–Speyer).

② $P = \Delta(n-k, n)$ for $k > 1$, the "higher Orlik–Terao algebra" $S_\bullet(Y_{L,P})$ is CM, resolving a conj. of Tohăneanu.

③ $P = -P(M)$, then $h^*_{r-1}(M, -P(M)) \geq 0$ resolves Speyer's f-vec. conj. (for conn. matroids) first resolved in the preprint [Berget–Fink].

④ $L' \subseteq k^{n+1}$ general lift of L, $P' = \sum_{i=1}^{n} \text{conv}\{e_i - e_{n+1}\}$, then $S_\bullet(Y_{L', P'})$ is the homog. coord. ring of Segre embedding of the arrangement Schubert variety.

<u>Proof</u> ⓪ Translate to cohomology vanishing statements on $Y_{L,P}$.

① $Y_{L,P} \hookrightarrow X_P \subset \mathbb{P}(k^{|P \cap \mathbb{Z}^n|})$, where X_P is the toric var. of P.

② Gröbner degenerate $Y_{L,P}$, i.e. for a general $\lambda : k^* \longrightarrow T$ consider the flat limit $\text{in}_\lambda(Y_{L,P}) = \lim_{t \to 0} \lambda(t) \cdot Y_{L,P}$.

Trop. Toolkit ↑

← depends only on M(L). (tropical geom. [Thm 10.1, Katz'09])

※ ③ Show $\underline{\text{in}_\lambda(Y_{L,P})}$ is a reduced & CM union of toric boundaries.

④ Use Frobenius splitting & upper semicontinuity.

<u>Defn</u> kindred subschemes: $Y \subseteq (\mathbb{P}^1)^m$ such that
$$\chi(Y, \mathcal{O}(a_1, \ldots, a_m)) = \sum_{I \in \text{indep}(M)} \prod_{i \in I} a_i$$
for some matroid M on $\{1, \ldots, m\}$.

<u>Thm</u> [E–Fink–Larson] Kindred subschemes are reduced & CM.