Diagonal Supersymmetry for Coinvariant Rings

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Talk Outline

Background:

- Classical and diagonal coinvariant rings $R_n^{(1,0)}$ & $R_n^{(2,0)}$
- Zabrocki's module for Delta conjecture $R_n^{(2,1)}$
- More generally, $R_n^{(k,j)}$

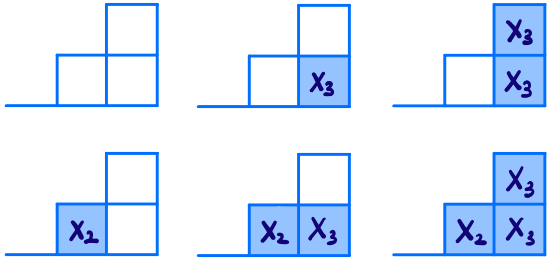
Diagonal supersymmetry (conj. F. Bergeron 2020):

- Theorem statement
- Proof concept

The Classical Coinvariant Ring

- $R_n^{(1,0)} = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\langle \mathbb{C}[x_1,\dots,x_n]^{\mathfrak{S}_n}_{+} \rangle}$

Symmetric group invariants, with no constant term



Artin basis

Schubert polynomials also a basis

- $\mathfrak{S}_n$ acts on $R_n^{(1,0)}$, so it is an $\mathfrak{S}_n$-module.

- **Theorem** (Chevalley, Shephard-Todd)
  $R_n^{(1,0)} \cong$ regular representation of $\mathfrak{S}_n$.

The Classical Coinvariant Ring

- $R_n^{(1,0)}$ is graded by total degree : $R_n^{(1,0)} = \bigoplus_{i \geq 0} (R_n^{(1,0)})_{(i)}$

- **Theorem** (Borel 1953) $$R_n^{(1,0)} \underset{\substack{\text{graded} \\ \mathfrak{S}_n\text{-mod}}}{\cong} H^*(F\ell_n)$$ ↑ flag variety

- $$\text{Frob}(R_n^{(1,0)}; q) = \sum_{i \geq 0} q^i \, F\text{Char}(R_n^{(1,0)})_{(i)}$$

- **Theorem** (Stanley 1979) $$\text{Frob}(R_n^{(1,0)}; q) = \sum_{\lambda \vdash n} \sum_{T \in SYT(\lambda)} q^{\text{maj}(T)} s_\lambda$$

Irreducible representation

$$\begin{tikzcd} V_\lambda \arrow[d, "\text{Char (Character)}"] \\ \chi_\lambda \arrow[d, "\text{Frobenius characteristic map } F", "\wr"'] \\ s_\lambda \end{tikzcd}$$

Irreducible character χ_λ $\quad \chi_\rho : \mathfrak{S}_n \to \mathbb{C}$, $\; g \mapsto \text{tr}(\rho(g))$

Schur function s_λ

The Diagonal Coinvariant Ring

- $$R_n^{(2,0)} = \frac{\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]}{\langle \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}_{+} \rangle}$$

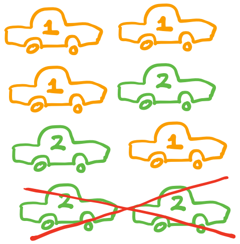
$x_i \to q$ degree

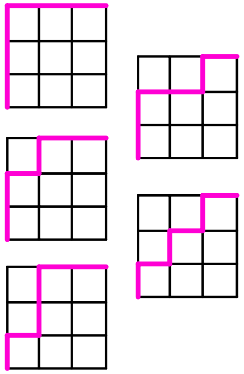
$y_i \to t$ degree

Diagonal action: $\sigma \cdot p(x_1,\dots,x_n,y_1,\dots,y_n) = p(x_{\sigma(1)},\dots,x_{\sigma(n)},y_{\sigma(1)},\dots,y_{\sigma(n)})$

Theorem (Haiman 2002; conj. Haiman 1994)

- $\mathrm{Frob}(R_n^{(2,0)}; q,t) = \nabla(e_n)$
- $\dim(R_n^{(2,0)}) = (n+1)^{n-1}$
- $\langle \mathrm{Frob}(R_n^{(2,0)}; q,t), s_{(1^n)} \rangle = C_n(q,t)$





Zabrocki's Module for the Delta Conjecture

Conjecture (Zabrocki 2019)

$$\mathrm{Frob}(R_n^{(2,1)}; q,t; u) = \sum_{d=0}^{n-1} u^d \overbrace{\Delta'_{e_{n-d-1}}(e_n)}^{\text{operator side of Delta conj.}}.$$

- $$R_n^{(2,1)} = \frac{\mathbb{C}[x_1,\dots,x_n, y_1,\dots,y_n, \theta_1,\dots,\theta_n]}{\langle \mathbb{C}[x_1,\dots,x_n, y_1,\dots,y_n, \theta_1,\dots,\theta_n]^{\mathfrak{S}_n}_+ \rangle}$$

$x_i \to q$ degree
$y_i \to t$ degree
$\theta_i \to u$ degree

- θ_i are fermionic (anticommutative) variables:

$x_i\theta_j = \theta_j x_i$, $y_i\theta_j = \theta_j y_i$, and $x_i y_j = y_j x_i$,

but $\theta_i\theta_j = -\theta_j\theta_i$. Thus $\theta_i^2 = 0$.

General (k,j)-bosonic-fermionic coinvariants

- $$R_n^{(k,j)} = \frac{\mathbb{C}[\overset{\overset{\{x_1^{(1)},\dots,x_n^{(1)}\}}{\uparrow}}{\underline{x}^{(1)}},\dots,\underline{x}^{(k)},\overset{\overset{\{\theta_1^{(1)},\dots,\theta_n^{(1)}\}}{\uparrow}}{\underline{\theta}^{(1)}},\dots,\underline{\theta}^{(j)}] \longleftarrow (\mathrm{Sym}\,\mathbb{C}^n)^{\otimes k}\otimes(\Lambda\mathbb{C}^n)^{\otimes j}}{\left\langle \mathbb{C}[\underline{x}^{(1)},\dots,\underline{x}^{(k)},\underline{\theta}^{(1)},\dots,\underline{\theta}^{(j)}]_+^{\mathfrak{S}_n} \right\rangle}$$

- Bosonic variables commute with all variables

- Fermionic variables anticommute with all fermionic variables

General (k,j)-bosonic-fermionic coinvariants

- $R_n^{(k,j)}$ is multigraded: $R_n^{(k,j)} = \bigoplus_{\substack{r_1,\dots,r_k,\\ s_1,\dots,s_j \geq 0}} \underbrace{\left(R_n^{(k,j)}\right)_{(r_1,\dots,r_k,s_1,\dots,s_j)}}_{\text{each is an } \mathfrak{S}_n\text{-module}}$

r_i = deg in $\underline{x}^{(i)}$

s_i = deg in $\underline{\theta}^{(i)}$

- **Def**: $\mathrm{Frob}\left(R_n^{(k,j)}; \overset{\{q_1,\dots,q_k\}}{\underline{q}}; \underset{\{u_1,\dots,u_j\}}{\underline{u}}\right)$

$$= \sum_{\substack{r_1,\dots,r_k,\\ s_1,\dots,s_j \geq 0}} F\,\mathrm{char}\left(\left(R_n^{(k,j)}\right)_{(r_1,\dots,r_k,s_1,\dots,s_j)}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j}$$

- **Notation**: $q_1 = q,\ q_2 = t,\ u_1 = u,\ u_2 = v$

Table of $\dim(R_n^{(k,j)})$

red = conjecture only

$k \backslash j$	0	1	2	3
0	1	2^{n-1}	$\binom{2n-1}{n}$ [Kim-Rhoades 2022]	?
1	$n!$	$\sum_{d=1}^{n} d!\,S(n,d)$ [Rhoades-Wilson 2024]	$2^{n-1}n!$ [Zabrocki 2020]	?
2	$(n+1)^{n-1}$ [Haiman 2002]	$\sum_{i=0}^{n+1} \binom{n+1}{i} \frac{i^n}{2(n+1)}$ [Zabrocki 2020]	?	?
3	$2^n(n+1)^{n-2}$ [Haiman 1994]	?	?	?

Table of Frobenius Series of $R_n^{(k,j)}$

red = conjecture only

$k \backslash j$	0	1	2
0	s_n	$\sum_{d=0}^{n-1} u^d s_{(n-d,1^d)}$ [Haglund-Sergel 2020]	$\sum_{0\leq d+\ell<n} u^d v^\ell \left(s_{(n-d,1^d)} * s_{(n-\ell,1^\ell)} - s_{(n-d+1,1^{d-1})} * s_{(n-\ell+1,1^{\ell-1})}\right)$ [Kim-Rhoades 2022]
1	$\sum_{\lambda\vdash n} \sum_{T\in SYT(\lambda)} q^{maj(T)} s_\lambda$ [Stanley 1979]	$\sum_{d=0}^{n-1} u^d \Delta'_{e_{n-d-1}}(e_n)\Big\vert_{t=0}$ [Murai, Rhoades, Wilson 2025]	$\sum_{0\leq d+\ell<n} u^d v^\ell \sum_{\sigma\in SW(1^n,d,\ell)} q^{sminv(\sigma)} Q_{split(\sigma),n}$ [Iraci, Nadeau, Vanden Wyngaerd 2024] [L. 2025]
2	$\nabla(e_n)$ [Haiman 2002]	$\sum_{d=0}^{n-1} u^d \Delta'_{e_{n-d-1}}(e_n)$ [Zabrocki 2019]	$\sum_{0\leq d+\ell<n} u^d v^\ell \Theta_{e_d} \Theta_{e_\ell} \nabla(e_{n-d-\ell})$ [D'Adderio, Iraci, Vanden Wyngaerd 2021]
3	Conj. with 2 out of 3 stats [Bergeron-Préville Ratelle 2012]	?	?

$GL(k) \times GL(j) \times \mathfrak{S}_n$ - module structure

- $R_n^{(k,j)}$ has three actions: $GL(k)$ acts on bosonic variables, $GL(j)$ acts on fermionic variables, $\mathfrak{S}_n$ permutes variables in all sets.

- Therefore $R_n^{(k,j)}$ is a $GL(k) \times GL(j) \times \mathfrak{S}_n$ - module.

$GL(k)$ - character

$GL(j)$ - character

$\mathfrak{S}_n$ - Frobenius character

- $$\mathrm{Frob}\left(R_n^{(k,j)} ; \underline{q} ; \underline{u}\right) = \sum_{\substack{\lambda,\nu \text{ s.t.} \\ \ell(\lambda)\leq k,\ \ell(\nu)\leq j}} \sum_{\mu \vdash n} c_{\lambda\nu\mu}\, s_\lambda\Big(\underset{\substack{\downarrow \\ \{q_1,\ldots,q_k\}}}{\underline{q}}\Big)\, s_\nu\Big(\underset{\substack{\downarrow \\ \{u_1,\ldots,u_j\}}}{\underline{u}}\Big)\, s_\mu$$

Stability of coefficients

- **Theorem** (Bergeron 2020) Fix $n \geq 1$.

  There is a universal series

  $$\mathrm{Frob}(R_n^{(\infty,\infty)}; \underline{q}; \underline{u}) = \sum_{\lambda,\nu} \sum_{\mu \vdash n} C_{\lambda\nu\mu} S_\lambda(\underline{q}) S_\nu(\underline{u}) S_\mu$$

  which determines all $C_{\lambda\nu\mu}$ for any (k,j).

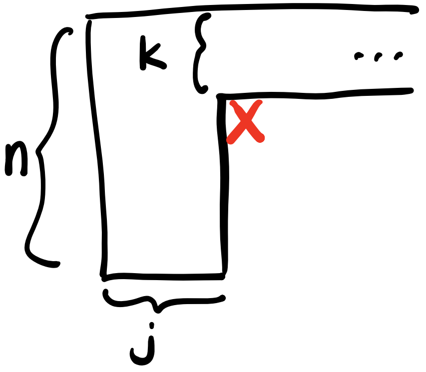
- "Coefficients are stable"

Diagonal Supersymmetry

Diagonal Supersymmetry

- $P(k,j,n) = \{\lambda \in \mathrm{Par} \mid \ell(\lambda) \leq n,\ \lambda_{k+1} \leq j\}.$

$\lambda \subseteq$



- Super Schur function $s_\lambda(\underline{q}/\underline{u}) = \sum_{\nu \subseteq \lambda} s_\nu(\underline{q})\, s_{\lambda'/\nu'}(\underline{u})$

Main Theorem (L. 2025, conj. Bergeron 2020)

Fix $n \geq 1$. For λ s.t. $\ell(\lambda) \leq n$, and $\mu \vdash n$, there exist coefficients $c_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ such that for any (k,j),

$$\mathrm{Frob}\left(R_n^{(k,j)}; \underline{q}; \underline{u}\right) = \sum_{\lambda \in P(k,j,n)} \sum_{\mu \vdash n} c_{\lambda\mu}\, s_\lambda(\underline{q} \backslash \underline{u})\, s_\mu .$$

Diagonal Supersymmetry

$$\operatorname{Frob}(R_n^{(k,j)};\underline{q};\underline{u})=\sum_{\lambda\in P(k,j,n)}\sum_{\mu\vdash n}c_{\lambda\mu}\,s_\lambda(\underline{q}\,\backslash\,\underline{u})\,s_\mu$$

- Knowing $\operatorname{Frob}(R_n^{(\infty,0)};\underline{q})$ means knowing $c_{\lambda\mu}$ for all $\lambda\in P(\infty,0,n)$.
- Knowing $\operatorname{Frob}(R_n^{(\infty,\infty)};\underline{q};\underline{u})$ means knowing $c_{\lambda\mu}$ for all $\lambda\in P(\infty,\infty,n)$.
- Knowing $\operatorname{Frob}(R_n^{(0,\infty)};\underline{u})$ means knowing $c_{\lambda\mu}$ for all $\lambda\in P(0,\infty,n)$.

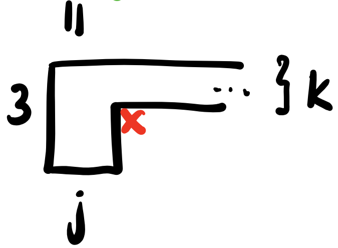
$$P(k,j,n)=\{\lambda\in\mathrm{Par}\mid \ell(\lambda)\leq n,\ \lambda_{k+1}\leq j\}$$

- $P(\infty,0,n)=P(\infty,\infty,n)=P(0,\infty,n)=\{\lambda\in\mathrm{Par}\mid\ell(\lambda)\leq n\}$
- Thus knowing any one is equivalent to knowing all three!

Example at $n=3$

For λ s.t. $\ell(\lambda) \leq 3$ and $\mu \vdash 3$, there exist $c_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ s.t.

$$\mathrm{Frob}\left(R_3^{(k,j)}; \underline{q}; \underline{u}\right) = \sum_{\lambda \in P(k,j,3)} \sum_{\mu \vdash 3} c_{\lambda\mu} s_\lambda(\underline{q} \backslash \underline{u}) s_\mu$$



$$= \left(s_3(\underline{q} \backslash \underline{u}) + s_{11}(\underline{q} \backslash \underline{u})\right) s_{111} + \left(s_2(\underline{q} \backslash \underline{u}) + s_1(\underline{q} \backslash \underline{u})\right) s_{21}$$
$$+ s_\emptyset(\underline{q} \backslash \underline{u}) s_3$$

Equivalently, $c_{(3),(111)}$, $c_{(11),(111)}$, $c_{(2),(21)}$, $c_{(1),(21)}$, $c_{\emptyset,(3)} = 1$ and all other $c_{\lambda,\mu} = 0$.

Example at $n=3$

- $$\mathrm{Frob}\left(R_3^{(k,j)};\underline{q};\underline{u}\right)=\left(s_3(\underline{q}\backslash\underline{u})+s_{11}(\underline{q}\backslash\underline{u})\right)s_{111}$$
$$+\left(s_2(\underline{q}\backslash\underline{u})+s_1(\underline{q}\backslash\underline{u})\right)s_{21}+s_\phi(\underline{q}\backslash\underline{u})s_3$$

- $$\mathrm{Frob}\left(R_3^{(2,0)};q,t\right)=\left(s_3(q,t)+s_{11}(q,t)\right)s_{111}$$
$$+\left(s_2(q,t)+s_1(q,t)\right)s_{21}+s_\phi(q,t)\,s_3$$

- $$\mathrm{Frob}\left(R_3^{(0,2)};u,v\right)=\left(\cancelto{0}{s_{111}(u,v)}+s_2(u,v)\right)s_{111}$$
$$+\left(s_{11}(u,v)+s_1(u,v)\right)s_{21}+s_\phi(u,v)\,s_3$$

Fact:

$$s_\lambda(0\backslash\underline{u})=s_{\lambda'}(\underline{u})$$

so we don't determine $c_{(3),(111)}$ here.

Proof Concept

Preliminary Theorem

general linear Lie superalgebra (not Semisimple!)

universal enveloping algebra

Theorem (L. 2025) Fix $n \geq 1$. As $\mathcal{U}(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[\mathfrak{S}_n]$-modules,

$$R_n^{(k,j)} \cong \bigoplus_{\lambda \in P(k,j,n)} \bigoplus_{\mu \vdash n} \left(U^{\lambda}_{k|j} \otimes V_{\mu} \right)^{\oplus c_{\lambda\mu}}$$

where

- the $U^{\lambda}_{k|j}$ are simple $\mathcal{U}(\mathfrak{gl}(k|j))$-modules
- the V_{μ} are simple $\mathbb{C}[\mathfrak{S}_n]$-modules,

for some nonnegative integer coefficients $c_{\lambda\mu}$.

Also holds for finite groups $G \subset GL(n)$

- Proof uses super Howe duality.

Proof sketch of diagonal supersymmetry

- From preliminary theorem,

$$R_n^{(k,j)} \cong \bigoplus_{\lambda\in P(k,j,n)} \bigoplus_{\mu\vdash n} \left(U^{\lambda}_{k|j} \otimes V_\mu\right)^{\oplus c_{\lambda\mu}}$$

$\mathcal{U}(\mathfrak{gl}(k|j))$-char

Frobenius char

$$\mathrm{Frob}(R_n^{(k,j)}; \underline{z}; \underline{u}) = \sum_{\lambda\in P(k,j,n)} \sum_{\mu\vdash n} c_{\lambda\mu}\, s_\lambda(\underline{z}/\underline{u})\, s_\mu$$

- Check for "coefficient stability".

Thank you!

Main reference:

(L. 2025) Diagonal supersymmetry for coinvariant rings.

arxiv.org/abs/2505.14885

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Applications

Hilbert Series

- **Def**: $\mathrm{Hilb}(R_n^{(k,j)}; \underline{q}; \underline{u})$

$$= \sum_{\substack{r_1,\dots,r_k,\\ s_1,\dots,s_j \geq 0}} \dim\left(\left(R_n^{(k,j)}\right)_{(r_1,\dots,r_k,s_1,\dots,s_j)}\right) q_1^{r_1}\cdots q_k^{r_k} u_1^{s_1}\cdots u_j^{s_j}$$

- $\langle \mathrm{Frob}(R_n^{(k,j)}; \underline{q}; \underline{u}), h_1^n \rangle = \mathrm{Hilb}(R_n^{(k,j)}; \underline{q}; \underline{u}).$

Applications of cancellation

$$s_\lambda(q_1,\dots,q_k/u_1,\dots,u_j)\Big|_{u_j=-q_k} = s_\lambda(q_1,\dots,q_{k-1}/u_1,\dots,u_{j-1})$$

Proposition (L. 2025)

(i) $$\mathrm{Frob}(R_n^{(k,j)};q_1,\dots,q_k;u_1,\dots,u_j)\Big|_{u_j=-q_k}$$
$$=\mathrm{Frob}(R_n^{(k-1,j-1)};q_1,\dots,q_{k-1};u_1,\dots,u_{j-1}).$$

(ii) $$\mathrm{Hilb}(R_n^{(k,j)};q_1,\dots,q_k;u_1,\dots,u_j)\Big|_{u_j=-q_k}$$
$$=\mathrm{Hilb}(R_n^{(k-1,j-1)};q_1,\dots,q_{k-1};u_1,\dots,u_{j-1}).$$

Application #1

- $[d]_q = 1 + q + q^2 + \cdots + q^{d-1}$
- $[d]_q! = [d]_q [d-1]_q \cdots [1]_q$
- $\mathrm{Stir}_q(n,d) = [d]_q \, \mathrm{Stir}_q(n-1,d) + \mathrm{Stir}_q(n-1,d-1)$

  with $\mathrm{Stir}_q(0,d) = \delta_{d,0}$

Proposition (Rhoades-Wilson 2024)

$$\mathrm{Hilb}(R_n^{(1,1)}; q; u) = \sum_{d=0}^{n} [d]_q! \, \mathrm{Stir}_q(n,d) \, u^{n-d}$$

Application #1

Proposition (Sagan-Swanson 2024)

$$\sum_{d=0}^{n} [d]_q! \, \mathrm{Stir}_q(n,d) (-q)^{n-d} = 1.$$

- They proved this with a sign-reversing involution.

- We can also prove it by applying the result on cancellation on the Hilbert series to the result of (Rhoades-Wilson 2024) since

$$\mathrm{Hilb}(R_n^{(1,1)}; q; u)\big|_{q=-u} = \mathrm{Hilb}(R_n^{(0,0)}) = \dim(\mathbb{C}) = 1.$$

Application #2

Theorem (Corteel-L. 2025+)

$$\mathrm{Hilb}(R_n^{(1,1)};q;u)\Big|_{-q^2=u} = 1+\sum_{m=1}^{n} q^m\left(\binom{n}{m}-\binom{n}{m-1}\right)$$

Proof idea $\mathrm{Hilb}(R_n^{(1,1)};q;u)\Big|_{q^2=-u} = \sum_{\lambda} c_\lambda s_\lambda(q\backslash u)\Big|_{q^2=-u}$

$$= 1+\sum_{\substack{a\geq 1,\\ b\geq 0}} c_{(a,1^b)}(-1)^b q^{a+2b}(1-q)$$ and OSP bijections.

Corollary (Corteel-L. 2025+; conj. Sagan-Swanson 2024)

$$\left(\sum_{d=0}^{n}[d]_q!\,\mathrm{Stir}_q(n,d)(-q^2)^{n-d}\right)-1$$ is palindromic up to signs.

Application #3

(A) <u>Conjecture</u> (Zabrocki 2019)

$$\mathrm{Frob}(R_n^{(2,1)}; q, t; u) = \sum_{d=0}^{n-1} u^d \Delta'_{e_{n-d-1}}(e_n).$$

(B) <u>Theorem</u> (D'Adderio, Iraci, Vanden Wyngaerd 2021)

$$\sum_{d=0}^{n-1} (-q)^d \Delta'_{e_{n-d-1}}(e_n) = \nabla(e_n)\big|_{q=0}$$

(C) <u>Theorem</u> (Haiman 2002)

$$\mathrm{Frob}(R_n^{(2,0)}; q, t) = \nabla(e_n) \Rightarrow \mathrm{Frob}(R_n^{(1,0)}; t) = \nabla(e_n)\big|_{q=0}$$

$\boxed{\text{(A)+(C) + Our proposition on cancellation} \Rightarrow \text{(B)}}$ since

$$\mathrm{Frob}(R_n^{(2,1)}; q, t; u)\big|_{q=-u} = \mathrm{Frob}(R_n^{(1,0)}; t) = \nabla(e_n)\big|_{q=0}.$$

Application # 4

(D) Conjecture (D'Addenio, Iraci, Vanden Wyngaerd 2021) "Theta conj."

$$\mathrm{Frob}(R_n^{(2,2)}; q,t; u,v) = \sum_{0 \leq d+\ell < n} u^d v^\ell \, \Theta_{e_d} \Theta_{e_\ell} \nabla(e_{n-d-\ell})$$

$$(D) \Rightarrow \mathrm{Hilb}(R_n^{(2,2)}; q,t; u,v) = \sum_{0 \leq d+\ell < n} u^d v^\ell \left\langle \Theta_{e_d} \Theta_{e_\ell} \nabla(e_{n-d-\ell}), h_1^n \right\rangle$$

(E) Conjecture (Corteel, Josuat-Vergès, Vanden Wyngaerd 2024)

$$\left\langle \Theta_{e_d} \Theta_{e_\ell} \nabla(e_{n-d-\ell}), h_1^n \right\rangle \Big|_{q=-1} \text{ is } t\text{-positive}$$

Application #4

(D) **Conjecture** $\Rightarrow \mathrm{Hilb}(R_n^{(2,2)};q,t;u,v) = \sum_{0\le d+\ell<n} u^d v^\ell \left\langle \Theta_{e_d}\Theta_{e_\ell}\nabla(e_{n-d-\ell}), h_1^n \right\rangle$

(E) **Conjecture** (Corteel, Josuat-Vergès, Vanden Wyngaerd 2024)

$$\left\langle \Theta_{e_d}\Theta_{e_\ell}\nabla(e_{n-d-\ell}), h_1^n \right\rangle\Big|_{q=-1} \text{ is } t\text{-positive}$$

$\boxed{\text{(D)} + \text{Our proposition on cancellation} \Rightarrow \text{(E)}}$ since

- $\mathrm{Hilb}(R_n^{(2,2)};q,t;u,v)\Big|_{q=-1=-v} = \mathrm{Hilb}(R_n^{(1,1)};t;u)$ is t-positive.
- I.e., $\sum_{0\le d+\ell<n} u^d \left\langle \Theta_{e_d}\Theta_{e_\ell}\nabla(e_{n-d-\ell}), h_1^n \right\rangle$ is t-positive.
- Stratifying by u-degree, each $\left\langle \Theta_{e_d}\Theta_{e_\ell}\nabla(e_{n-d-\ell}), h_1^n \right\rangle$ is t-positive.

Super Schur functions

- $$s_\lambda(\underline{q}/\underline{u}) = \sum_{\nu\subseteq\lambda} s_\nu(\underline{q})\, s_{\lambda'/\nu'}(\underline{u}) \left(= s_\lambda[\underline{q} - \varepsilon\, \underline{u}] \right)$$

  $\{q_1,\dots,q_k\}$ $\{u_1,\dots,u_j\}$

  plethysm

- $$s_\lambda(\underline{q}/\underline{u}) = \sum_{T\in \text{Super Tableaux}(\lambda)} \underline{q}^{wt(T)}\, \underline{u}^{wt'(T)}$$

e.g. $\lambda = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$

For any $\nu \subseteq \lambda$, ν is a SSYT with regular numbers.

e.g. $\nu = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$

$\rightsquigarrow q_1^2 q_2 q_3$

Then λ/ν is a Young tableau with primed numbers which strictly increase along rows and weakly increase down columns.

e.g. $\lambda/\nu = \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 3' \\ \cline{2-4} \multicolumn{1}{c|}{} & 1' & 2' & 3' \\ \hline 2' & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$

$\rightsquigarrow u_1 u_2^2 u_3^2$

Some properties of super Schur functions

- **Cancellation:**

$$s_\lambda(q_1,\dots,q_k/u_1,\dots,u_j)\Big|_{u_j=-q_k} = s_\lambda(q_1,\dots,q_{k-1}/u_1,\dots,u_{j-1})$$

- **Restriction:**

$$s_\lambda(q_1,\dots,q_k/u_1,\dots,u_j)\Big|_{q_k=0} = s_\lambda(q_1,\dots,q_{k-1}/u_1,\dots,u_j)$$

$$s_\lambda(q_1,\dots,q_k/u_1,\dots,u_j)\Big|_{u_j=0} = s_\lambda(q_1,\dots,q_k/u_1,\dots,u_{j-1})$$

- Super Schur functions $s_\lambda(\underline{q}/\underline{u})$ are characters of certain Lie superalgebra representations $U^\lambda_{k|j}$ of $gl(k|j)$.

Example $s_{(2,1)}(q_1,q_2/u_1,u_2) = \sum_{\nu\subseteq(2,1)} s_\nu(q_1,q_2)\, s_{(2,1)'/\nu'}(u_1,u_2)$

$$= s_\emptyset(q_1,q_2)\, s_{(2,1)}(u_1,u_2) + s_{(1)}(q_1,q_2)\, s_{(2,1)/(1)}(u_1,u_2)$$

$$+ s_{(2)}(q_1,q_2)\, s_{(1)}(u_1,u_2) + s_{(1,1)}(q_1,q_2)\, s_{(1)}(u_1,u_2)$$

$$+ s_{(2,1)}(q_1,q_2)\, s_\emptyset(u_1,u_2)$$

$\nu=\emptyset$ $\quad\lambda/\nu=(2,1)$

$$\begin{array}{|c|c|}\hline 1' & 2' \\ \hline 1' & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1' & 2' \\ \hline 2' & \\ \cline{1-1}\end{array}$$

→ Transposing gives SSYT of shape λ'/ν'.

$\nu=(1)$ $\quad\lambda/\nu=(2,1)/(1)$

$$\begin{array}{|c|c|}\hline 1 & 1' \\ \hline 1' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2' \\ \hline 1' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2' \\ \hline 2' & \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 1' \\ \hline 1' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2' \\ \hline 1' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1' \\ \hline 2' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2' \\ \hline 2' & \\ \cline{1-1}\end{array}$$

$\nu=(2)$ $\quad\lambda/\nu=(1)$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1' & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2' & \\ \cline{1-1}\end{array}$$

$\nu=(1,1)$ $\quad\lambda/\nu=(1)$

$$\begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2' \\ \hline 2 & \\ \cline{1-1}\end{array}$$

$\nu=(2,1)$ $\quad\lambda/\nu=\emptyset$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

Coefficients $c_{\lambda n}$

What does diagonal supersymmetry tell us?

- Recall: $\mathrm{Frob}(R_n^{(k,j)}; \underline{q}; \underline{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\mu \vdash n} c_{\lambda\mu} s_\lambda(\underline{q} \backslash \underline{u}) s_\mu$.

- Knowing a (multigraded) Frobenius series of $R_n^{(k,j)}$ is equivalent to knowing enough $c_{\lambda\mu}$.

- Since $\mu \vdash n$, $\mathrm{Frob}(R_n^{(k,j)}; \underline{q}; \underline{u})$ and $\mathrm{Frob}(R_{n'}^{(k',j')}; \underline{q}; \underline{u})$ could have overlapping coefficients $c_{\lambda\mu}$ only if $n = n'$.

- If $k \geq k'$ and $j \geq j'$, then $\mathrm{Frob}(R_n^{(k,j)}; \underline{q}; \underline{u})$ determines $\mathrm{Frob}(R_n^{(k',j')}; \underline{q}; \underline{u})$.

Example: $R_n^{(2,0)}$ vs. $R_n^{(0,2)}$

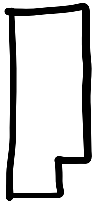
- What about when (k,j) and (k',j') are incomparable?

- Determining $\mathrm{Frob}(R_n^{(0,2)}; u,v)$ (Kim-Rhoades 2022) was "much more straightforward" than $\mathrm{Frob}(R_n^{(2,0)}; q,t)$ (Haiman 2002). Does either imply the other?

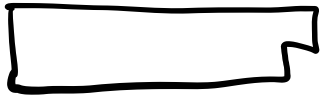
- We will see that the answer is <u>**NO**</u>.

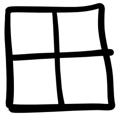
Example: $R_n^{(2,0)}$ vs. $R_n^{(0,2)}$

- $\mathrm{Frob}(R_n^{(0,2)}; u, v)$ determines all $C_{\lambda\mu}$ for $\lambda_1 \leq 2$ and $\mu \vdash n$.



- $\mathrm{Frob}(R_n^{(2,0)}; q, t)$ determines all $C_{\lambda\mu}$ for $\ell(\lambda) \leq 2$ and $\mu \vdash n$.



- They only determine each other if both are inside a 2×2 box , which happens when $n \leq 2$.
- We verify that for $n \gg 2$, there are nonzero coefficients $C_{\lambda\mu}$ for λ outside of $(2,2)$ in both directions.

Proposition (L. 2025) Fix k, j. For fixed $n \leq k+j$, $\mathrm{Frob}(R_n^{(k+j,0)}; \underline{q})$ determines $\mathrm{Frob}(R_n^{(k,j)}; \underline{q}; \underline{u})$, and $\mathrm{Frob}(R_n^{(0,k+j)}; \underline{u})$ determines $\mathrm{Frob}(R_n^{(k,j)}; \underline{q}; \underline{u})$.

Proposition (L. 2025) There are nonzero $c_{\lambda\mu}$ in $\mathrm{Frob}(R_n^{(k',j')}; \underline{q}; \underline{u})$ which are not determined by $\mathrm{Frob}(R_n^{(k,j)}; \underline{q}; \underline{u})$ when

- (k,j) and (k',j') are $(0,1)$ and $(1,0)$
- (k,j) and (k',j') are any two of $(0,2)$, $(1,1)$, $(2,0)$.

Conj Also for any (k,j) & (k',j') s.t. they are incomparable in the componentwise order.

Coefficients for $R_n^{(0,2)}$

Using (Kim-Rhoades 2022), we can show:

Proposition (L. 2025) For $c_{\lambda\mu}$ with $\lambda_1 \leq 2$ and $\mu \vdash n$,

(each part at most 2)

$c_{\lambda\mu} = 1$ if

(i) $\mu = (n)$ and $\lambda = \emptyset$;

(ii) $\mu = (1^n)$ and $\lambda = (1^{n-1})$;

(iii) $\mu = (n-k, 1^k)$ and $\lambda = (1^k)$ or $(2, 1^k)$ for $k \in \{1, \ldots, n-1\}$;

(iv) $\mu = (\mu_1, \mu_1, 2^\ell, 1^{n-2\ell-2\mu_1})$ where $\mu_1 \geq 2$ and

$\lambda = (2^{\ell+\mu_1}, 1^{n-2\ell-2\mu_1-1}), (2^{\ell+\mu_1-1}, 1^{n-2\ell-2\mu_1+1})$, or $(2^{\ell+\mu_1-1}, 1^{n-2\ell-2\mu_1})$;

(v) $\mu = (\mu_1, \mu_2, 2^\ell, 1^{n-2\ell-\mu_1-\mu_2})$ where $\mu_1 > \mu_2 \geq 2$ and

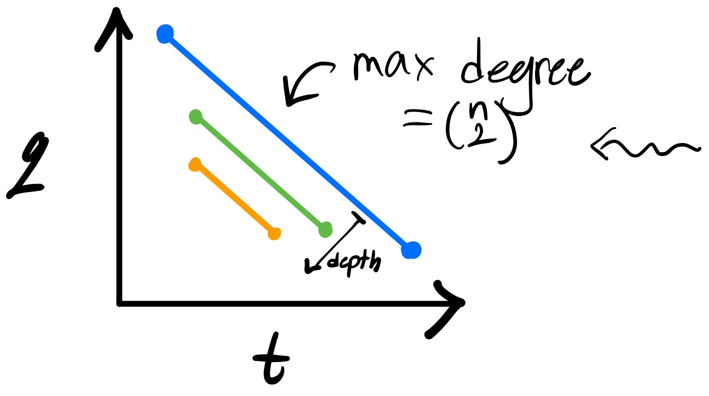
$\lambda = (2^{\ell+\mu_2}, 1^{n-2\ell-\mu_1-\mu_2}), (2^{\ell+\mu_2-1}, 1^{n-2\ell-\mu_1-\mu_2+1})$,

$(2^{\ell+\mu_2}, 1^{n-2\ell-\mu_1-\mu_2-1})$, or $(2^{\ell+\mu_2-1}, 1^{n-2\ell-\mu_1-\mu_2})$,

and $c_{\lambda\mu} = 0$ otherwise.

Coefficients for $R_n^{(2,0)}$

- For $R_n^{(2,0)}$, the situation is far more complicated, and no such characterization of $c_{\lambda\mu}$ is known.
- For example, for just $\mu = (1^n)$, determining $c_{\lambda,(1^n)}$ is the same as knowing all $\mathfrak{sl}_2$-string heads for the q,t-Catalan numbers, which is known only up to depth 9 from the max degree (Lee, Li, Loehr 2018).



example:

$$C_4(q,t) = S_{(6)}(q,t) + S_{(4,1)}(q,t) + S_{(3,1)}(q,t)$$

Ingredients in Proof

Preliminary Theorem

universal enveloping algebra

general linear Lie superalgebra

Theorem (L. 2025) Fix $n \geq 1$. As $\mathcal{U}(\mathfrak{gl}(k|j)) \otimes \mathbb{C}[\mathfrak{S}_n]$-modules,

$$R_n^{(k,j)} \cong \bigoplus_{\lambda \in P(k,j,n)} \bigoplus_{\mu \vdash n} \left(U^{\lambda}_{k|j} \otimes N^{\mu} \right)^{\oplus c_{\lambda\mu}}$$

where

- the $U^{\lambda}_{k|j}$ are simple $\mathcal{U}(\mathfrak{gl}(k|j))$-modules with character $s_\lambda(\underline{g} \backslash \underline{u})$ and
- the N^{μ} are simple $\mathbb{C}[\mathfrak{S}_n]$-modules,

for some nonnegative integer coefficients $c_{\lambda\mu}$.

Proof sketch (main theorem)

- Use preliminary theorem, and take characters of $\mathcal{U}^{\lambda}_{k|j}$ to get $s_\lambda(\underline{z}/\underline{u})$, and Frobenius characters of N^{μ} to get s_μ:

$$R_n^{(k,j)} \cong \bigoplus_{\lambda\in P(k,j,n)} \bigoplus_{\mu\vdash n} \left(\mathcal{U}^{\lambda}_{k|j}\otimes N^{\mu}\right)^{\oplus c_{\lambda\mu}}$$

$\mathcal{U}(\mathfrak{gl}(k|j))$-char

Frobenius char

$$\mathrm{Frob}(R_n^{(k,j)};\underline{z};\underline{u}) = \sum_{\lambda\in P(k,j,n)} \sum_{\mu\vdash n} c_{\lambda\mu}\, s_\lambda(\underline{z}/\underline{u})\, s_\mu$$

- Check for "coefficient stability" using restriction.

Lie superalgebra $\mathfrak{gl}(k|j)$

- $\mathfrak{gl}(k|j)$ consists of block matrices $X = \begin{array}{ccc} & k & j \\ k & A & B \\ j & C & D \end{array}$

- $\mathfrak{gl}(k|j)$ is not semisimple! A $\mathfrak{gl}(k|j)$-representation need not decompose into a direct sum of irreducibles.

- In certain instances, we will get complete reducibility, in our case, via super Howe duality.

Proof sketch (preliminary theorem)

- Start with $(\mathfrak{gl}(k|j), GL(n))$- Howe duality

$$\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) \cong \bigoplus_{\lambda \in P(k,j,n)} U^{\lambda}_{k|j} \otimes U^{\lambda}_{n}.$$

- Since $\mathfrak{S}_n \hookrightarrow GL(n)$ via permutation matrices, restrict U^{λ}_n from $GL(n)$-module to a sum of $\mathfrak{S}_n$-modules.

- Then $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) \cong \bigoplus_{\lambda \in P(k,j,n)} U^{\lambda}_{k|j} \otimes \bigoplus_{\mu} (N^{\mu})^{d_{\lambda\mu}}.$

- $R_n^{(k,j)}$ is a quotient module of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$: $0 \leq c_{\lambda\mu} \leq d_{\lambda\mu}.$

Finite groups G

What about other types?

- Fix $n \geq 1$. For any finite group $G \subseteq GL(n)$, define

$$R_G^{(k,j)} = \frac{\mathbb{C}[\underline{x}^{(1)}, \ldots, \underline{x}^{(k)}, \underline{\theta}^{(1)}, \ldots, \underline{\theta}^{(j)}]}{\left\langle \mathbb{C}[\underline{x}^{(1)}, \ldots, \underline{x}^{(k)}, \underline{\theta}^{(1)}, \ldots, \underline{\theta}^{(j)}]^G_+ \right\rangle} .$$

Theorem (L. 2025) Fix $n \geq 1$ and fix $G \subseteq GL(n)$.
For λ s.t. $\ell(\lambda) \leq n$, and μ indexing irreducible G-characters, there exist nonnegative integer coefficients $c^G_{\lambda\mu}$ s.t. for any (k,j),

$$\mathrm{Char}(R_G^{(k,j)}; \underline{q}; \underline{u}) = \sum_{\lambda \in P(k,j,n)} \sum_{\chi^\mu \in \mathrm{IrrChar}(G)} c^G_{\lambda\mu} \, s_\lambda(\underline{q} \backslash \underline{u}) \, \chi^\mu .$$

multigraded character series

More sets of Commuting Variables

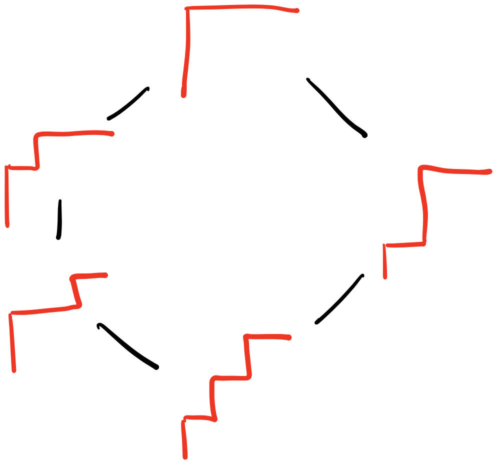
Conjecture (Haiman 1994)

- $\dim R_n^{(3,0)} = 2^n (n+1)^{n-2}$
- $\langle \mathrm{Frob}(R_n^{(3,0)}; 1,1,1), s_{(1^n)} \rangle = \frac{2}{n(n+1)} \binom{4n+1}{n-1}$

↑ number of intervals in Tamari lattice

- Some conjectural work on $R_n^{(3,0)}$ by Bergeron-Préville Ratelle (2012).

- $\dim R_n^{(4,0)} = \{1, 5, 55, 996, 25769, \dots\}$ starts to have large primes.



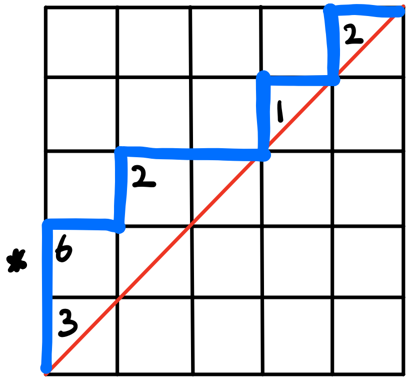
The Delta Conjecture

Theorem "Rise version" (D'Adderio-Mellit 2022; also Blasiak, Haiman, Morse, Pun, Seelinger 2023; conj. Haglund, Remmel, Wilson 2018)

$$\Delta'_{e_{n-d-1}}(e_n(z)) = \sum_{P \in LD(n)^{*d}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} z^P$$

Some Macdonald eigenoperator

Labelled Decorated Dyck paths



Lie superalgebra basics

$\nearrow$ $\mathbb{Z}_2$-graded

- A Lie superalgebra $\mathfrak{g}$ is a superalgebra over $\mathbb{C}$ with Lie superbracket $[\cdot,\cdot]$ satisfying:
  - super skew symmetry $[x,y] = -(-1)^{|x||y|}[y,x]$
  - super Jacobi identity

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$$

- $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the decomposition into even and odd parts. The even part $\mathfrak{g}_0$ is a Lie algebra.

- A $\mathfrak{g}$-representation is a Lie superalgebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}(V),$$

where V is a super vector space.

The actions

- $\mathfrak{gl}(k|j)$ acts via left superderivations on $\mathbb{C}^{k|j}$ and trivially on $V = \mathbb{C}^n$.
- $\mathfrak{S}_n$ acts trivially on $\mathbb{C}^{k|j}$ and diagonally on $V = \mathbb{C}^n$.
- $\mathfrak{gl}(k|j)$ and $\mathfrak{S}_n$-actions commute.
- $\mathbb{C}^{k|j} \otimes V$ is the degree 1 part of $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V)$.
  (total degree)
- Then extend these actions to ($R_n^{(k,j)}$ is a quotient of this)

$$\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) = \mathbb{C}[\underline{x}^{(1)}, \dots, \underline{x}^{(k)}, \underline{\theta}^{(1)}, \dots, \underline{\theta}^{(j)}].$$

Placing Coordinates

- Consider $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) = \mathrm{Sym}(V)^{\otimes k} \otimes \Lambda(V)^{\otimes j}$

- Upon choosing a basis of $V = \mathbb{C}^n$, number coordinates so that $\underline{x}^{(1)}, \ldots, \underline{x}^{(k)}$ correspond to coordinates on k copies of $\mathrm{Sym}(V)$ and $\underline{\theta}^{(1)}, \ldots, \underline{\theta}^{(j)}$ correspond to coordinates on j copies of $\Lambda(V)$.

- $\mathrm{Sym}(\mathbb{C}^{k|j} \otimes V) = \mathbb{C}[\underline{x}^{(1)}, \ldots, \underline{x}^{(k)}, \underline{\theta}^{(1)}, \ldots, \underline{\theta}^{(j)}]$

$R_n^{(k,j)}$ is a quotient of this

The $\mathfrak{gl}(k|j)$ action

- The Lie superalgebra $\mathfrak{gl}(k|j)$ acts naturally on $\mathbb{C}^{k|j}$ by left superderivations.

- For $1 \leq a, b \leq k$ and $1 \leq c, d \leq j$, we have

$$E_{a,b} = \sum_{p=1}^{n} x_p^{(a)} \partial_{x_p^{(b)}}, \quad E_{a,d'} = \sum_{p=1}^{n} x_p^{(a)} \partial_{\theta_p^{(d)}},$$

$$E_{c',b} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{x_p^{(b)}}, \quad E_{c',d'} = \sum_{p=1}^{n} \theta_p^{(c)} \partial_{\theta_p^{(d)}}.$$

where $\partial_{\theta_i}(\overbrace{\theta_g \cdots \theta_h}^{\alpha} \theta_i \theta_\ell \cdots \theta_m) = (-1)^{\alpha} \theta_g \cdots \theta_h \theta_\ell \cdots \theta_m$

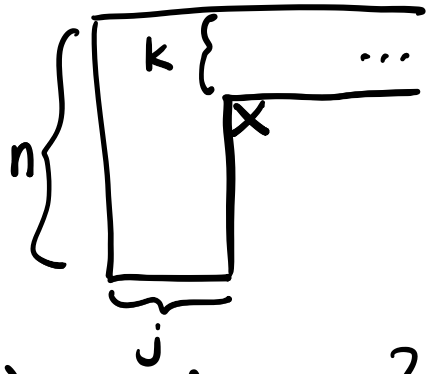
Lie superalgebra $\mathfrak{gl}(k|j)$

- $\mathfrak{gl}(k|j)$ consists of block matrices $X = \begin{array}{c|cc} & \overbrace{}^{k} & \overbrace{}^{j} \\ k\{ & A & B \\ j\{ & C & D \end{array}$
- $\mathfrak{g}_0$ consists of matrices with $B=C=0$ $(\cong \mathfrak{gl}(k) \oplus \mathfrak{gl}(j))$
- $\mathfrak{g}_1$ consists of matrices with $A=D=0$ $(\cong (\mathbb{C}^k \otimes \mathbb{C}^{j\vee}) \oplus (\mathbb{C}^{k\vee} \otimes \mathbb{C}^j))$
- $\mathfrak{gl}(k|j)$ is <u>not semisimple</u>! A $\mathfrak{gl}(k|j)$-representation need not decompose into a direct sum of irreducibles.
- In certain instances, we will get complete reducibility, in our case, via super Howe duality.

$(\mathfrak{gl}(k|j), GL(n))$-Howe Duality

Theorem (Howe 1989) Fix $d \geq 0$. $V = \mathbb{C}^n$.

$$\mathrm{Sym}^d(\mathbb{C}^{k|j} \otimes V) \cong \bigoplus_{\substack{\lambda \in P(k,j,n) \\ \lambda \vdash d}} U^\lambda_{k|j} \otimes U^\lambda_n,$$



where for each $\lambda \in P(k,j,n) = \{\lambda \in \mathrm{Par} \mid \ell(\lambda) \leq n,\ \lambda_{k+1} \leq j\}$,

$U^\lambda_{k|j}$ is a simple $\mathfrak{gl}(k|j)$-module with character $s_\lambda(\underline{z}\backslash\underline{y})$, and

U^λ_n is a simple $GL(n)$-module with character s_λ.

What does "super" mean?

- A super vector space V over $\mathbb{C}$ is a vector space with a $\mathbb{Z}/2\mathbb{Z}$-grading, i.e., a decomposition into even and odd parts $V = V_0 \oplus V_1$.

- $\mathbb{C}^{k|j}$ is a super vector space over $\mathbb{C}$ with dimension $k|j$.

- A superalgebra $\mathcal{A}$ over $\mathbb{C}$ is a super vector space $\mathcal{A}$ with multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ preserving parity.

Lie superalgebra representations

- A $\mathfrak{g}$-representation is a Lie superalgebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}(V),$$

where V is a super vector space.

This means ρ is parity-preserving: even homogeneous elements of $\mathfrak{g}$ are sent to even maps in $\mathfrak{gl}(V)_0$ and odd elements are sent to odd maps in $\mathfrak{gl}(V)_1$.

- For any Lie superalgebra $\mathfrak{g}$, there exists a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, and $\mathfrak{g}$-representations and $\mathcal{U}(\mathfrak{g})$-representations are equivalent.