A Hessian criterion for totally positive Toeplitz matrices

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Higher Hessians and Toeplitz Matrices

- Homogeneous bivariate real d-form:

$$F = \sum_{k=0}^{d} \binom{d}{k} c_k X^k Y^{d-k} \iff (c_0, \ldots, c_d) \in \mathbb{R}^{d+1}$$

- i^{th} Hessian polynomial ($0 \leq i \leq d$):

$$H_i^F(X,Y) = (-1)^{\left\lfloor \frac{i+1}{2} \right\rfloor} \cdot \det\left(\left(\frac{\partial^{2i} F}{\partial X^{p+q} \partial Y^{2i-p-q}} \right)_{0 \leq p,q \leq i} \right)$$

- i^{th} Toeplitz matrix ($0 \leq i \leq d$): $\phi_d^i(F) = \begin{pmatrix} c_i & c_{i+1} & \cdots & c_d \\ \vdots & \ddots & \ddots & \ddots \\ c_0 & \ddots & \ddots & \ddots \end{pmatrix}$

- the Sperner number:

$$s = s(F) := \min\{i \mid H_i^F(X,Y) \equiv 0\} = \max_i \{\operatorname{rank}\left(\phi_d^i(F)\right)\}$$

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A Hessian Criterion For Total Positivity

Theorem (M. 2025)

For any $F \in \mathbb{R}[X,Y]_d$, with $s = s(F)$, TFAE:

1. $H_i^F(X,Y) > 0, \begin{cases} \forall\ (X,Y) \geq 0,\ (X,Y) \neq (0,0) \\ \forall\ 0 \leq i \leq s-1 \end{cases}$
2. $\phi_d^{s-1}(F)$ *is totally positive.*

- Equivalence is NOT true if $s < s(F)$!

$$H_0^F = F = Y^4 + 12XY^3 + 12X^2Y^2 + 8X^3Y + X^4 > 0$$
$$H_1^F = 144 \cdot \left(7Y^4 + 8XY^3 - X^2Y^2 + 2X^3Y + 2X^4\right) > 0$$
$$H_2^F \equiv -5 < 0$$

but $s = 2 < s(F) = 3$ and $\phi_4^1(F) = \begin{pmatrix} 3 & 2 & 2 & 1 \\ 1 & 3 & 2 & 3 \end{pmatrix}$ not t.p.

- For $s = 2$: $\phi_d^1(F) = \begin{pmatrix} c_1 & c_2 & \cdots & c_d \\ c_0 & c_1 & \cdots & c_{d-1} \end{pmatrix}$ is totally positive

$\Leftrightarrow$ $(c_0, \ldots, c_d)$ is strictly (positive and) log concave

$\Leftrightarrow$ F is strictly Lorentzian in sense of Brändén and Huh

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First Proof: Hodge Theory

- Graded oriented Artinian Gorenstein $\mathbb{R}$-algebra

$$A = \bigoplus_{i=0}^{d} A_i, \; \int_A : A_d \xrightarrow{\cong} \mathbb{R}$$

- Gorenstein = “Poincaré duality”

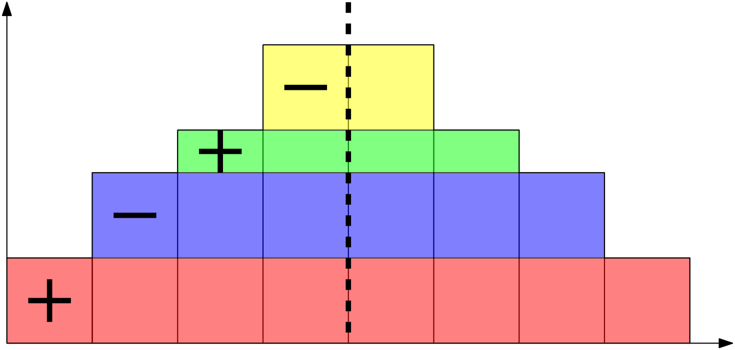
$$\langle -,- \rangle : A_i \times A_{d-i} \to \mathbb{R}, \; \langle \alpha, \beta \rangle = \int_A \alpha \cdot \beta$$

- “Kähler cone” $U \subset A_1$ and Lefschetz multiplication maps

ordinary $\quad \times \ell^{d-2i} : A_i \to A_{d-i}, \quad \begin{cases} \ell \in U \\ 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor \end{cases}$

mixed $\quad \times \ell_1 \cdots \ell_{d-2i} : A_i \to A_{d-i}, \quad \begin{cases} \ell_1, \ldots, \ell_{d-2i} \in U \\ 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor \end{cases}$

Hodge-Riemann Relations (HRR)



- ordinary HRR on U $\Rightarrow$ HR signature on

$$\times\ell^{d-2i}\colon A_i \to A_{d-i}, \quad \begin{cases} \ell \in U \\ 0 \le i \le \left\lfloor \frac{d}{2} \right\rfloor \end{cases}$$

- mixed HRR on U $\Rightarrow$ HR signature on

$$\times\ell_1 \cdots \ell_{d-2i}\colon A_i \to A_{d-i}, \quad \begin{cases} \ell_1, \dots, \ell_{d-2i} \in U \\ 0 \le i \le \left\lfloor \frac{d}{2} \right\rfloor \end{cases}$$

Cattani's Theorem

Theorem (E. Cattani 2008)

If U is a convex cone TFAE:

1. $\left(A, \int_A\right)$ *satisfies ordinary HRR on U*
2. $\left(A, \int_A\right)$ *satisfies mixed HRR on U.*

- $A = \mathbb{R}[x,y]/I$
- $F(X,Y) = \int_A (Xx + Yy)^d$
- $U = \{Xx + Yy \mid (X,Y) \geq 0,\ (X,Y) \neq (0,0)\}$

Theorem (Macias Marques-M.-Seceleanu, 2025)

1. $\left(A, \int_A\right)$ *satisfies ordinary HRR on U* $\Leftrightarrow$

$$H_i^F(X,Y) > 0, \begin{cases} \forall \quad (X,Y) \geq 0,\ (X,Y) \neq (0,0) \\ \forall \quad 0 \leq i \leq s(F) - 1 \end{cases}$$

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Second Proof: Wronskians and Flag Varieties

- full flag variety (Plücker embedding)

$$\mathrm{Fl}(\mathbb{R}[X,Y]_s) \subset \mathrm{Gr}_1(\mathbb{R}[X,Y]_s) \times \cdots \times \mathrm{Gr}_s(\mathbb{R}[X,Y]_s)$$

- Wronski map

$$\mathrm{Wr}_i\colon\ \mathrm{Gr}_i(\mathbb{R}[X,Y]_s) \to \mathbb{P}\Big(\mathbb{R}[X,Y]_{i(s-i+1)}\Big)$$

- totally positive flags $\mathrm{Fl}(\mathbb{R}[X,Y]_s)^{>0} \subset \mathrm{Fl}(\mathbb{R}[X,Y]_s)$

Theorem (S. Karp, 2023)

Fix $V_\bullet = (V_1 \subset \cdots \subset V_s) \in \mathrm{Fl}(\mathbb{R}[X,Y]_s)$. *TFAE:*

1. $\mathrm{Wr}_i(V_i)(X,Y) \neq 0, \begin{cases} \forall \quad (X,Y) \geq 0,\ (X,Y) \neq (0,0) \\ \forall \quad 1 \leq i \leq s \end{cases}$
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2. $\Rightarrow$ 1. easier, 1. $\Rightarrow$ 2. harder

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Step 1: Topology

Fact (Lusztig, 1994)

There exists a distinguished open set in the flag variety

$$\mathrm{Fl}(\mathbb{R}[X,Y]_s)^{>0} \subset \mathcal{O}^F \subset \mathrm{Fl}(\mathbb{R}[X,Y]_s)$$

in which $\mathrm{Fl}(\mathbb{R}[X,Y]_s)^{>0}$ *is a connected component.*

Fact (M. 2025)

There exists a distinguished open set in the Toeplitz space

$$\mathcal{T}(s,n)^{>0} \subset \mathcal{O}^T \subset \mathcal{T}(s,n)$$

in which $\mathcal{T}(s,n)^{>0}$ *is a (union of) connected component(s).*

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Step 2: Membership in the Distinguished Open

Fact (Karp, 2023)

For any $V_\bullet = (V_1 \subset \cdots \subset V_s) \in \mathrm{Fl}(\mathbb{R}[X,Y]_s)$ *we have*

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$$\Downarrow$$

- $V_\bullet \in \mathcal{O}^F$

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For any $F \in \mathbb{R}[X,Y]_d$ *with Sperner number* $s = s(F)$ *we have*

- $H_i^F(X,Y) > 0, \begin{cases} \forall & (X,Y) \geq 0,\ (X,Y) \neq (0,0) \\ \forall & 0 \leq i \leq s-1 \end{cases}$

$$\Downarrow$$

- $\phi_d^{s-1}(F) \in \mathcal{O}^T$

(this is where we need $s = s(F)$!)

Step 2: Membership in the Distinguished Open

Fact (Karp, 2023)

For any $V_\bullet = (V_1 \subset \cdots \subset V_s) \in \mathrm{Fl}(\mathbb{R}[X,Y]_s)$ *we have*

- $\mathrm{Wr}_i(V_i)(X,Y) \neq 0, \begin{cases} \forall \quad (X,Y) \geq 0,\ (X,Y) \neq (0,0) \\ \forall \quad 1 \leq i \leq s \end{cases}$

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Step 3: Deformation Argument

$S_t : \mathbb{R}[X,Y] \to \mathbb{R}[X,Y],\ S_t(F)(X,Y) = F(X+tY, Y)$

Fact (Karp, 2023)

If $V_\bullet = (V_1 \subset \cdots \subset V_s) \in \mathrm{Fl}(\mathbb{R}[X,Y]_s)$ *satisfies*

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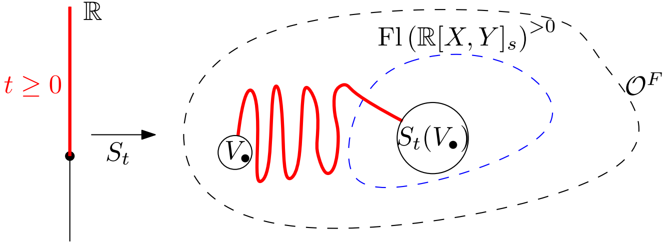
then

- $S_t(V_\bullet) \in \mathcal{O}^F, \quad \forall t \geq 0$
- $S_t(V_\bullet) \in \mathrm{Fl}(\mathbb{R}[X,Y]_s)^{>0}, \quad \forall t >> 0$



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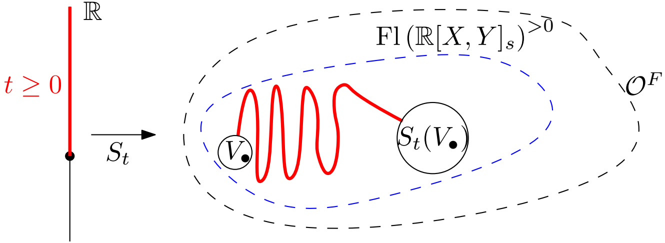

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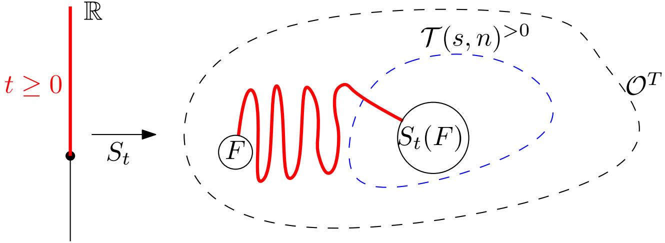

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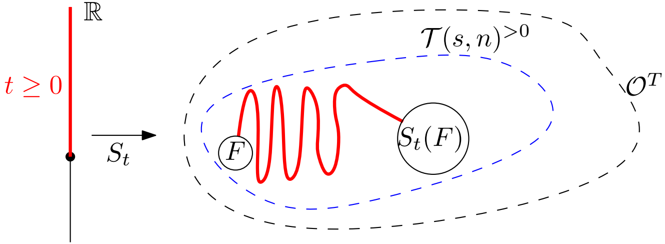

Connections to (real) Schubert Calculus

- moment curve $\gamma\colon \mathbb{C} \to \mathbb{C}[X,Y]_s$, $\gamma(z) = (X + zY)^s$
- codim r osculating spaces to γ at $N = r(s+1-r)$ points

$$W(z_i) = \left\langle \gamma(z_i), \gamma'(z_i), \ldots, \gamma^{(s-r)}(z_i) \right\rangle, \quad z_i \in \mathbb{C},\ 1 \le i \le N$$

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$$V \in \mathrm{Gr}_r(\mathbb{C}[X,Y]_s) \text{ satisfy } V \cap W(z_i) \neq 0,\ \forall i?$$

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