Extremal Ideals and Bounding Invariants of Square-free Monomial Ideals

Susan Morey, Texas State University

Joint Work with Susan Cooper, Sabine El Khoury, Sara Faridi, Sarah Mayes-Tang, Liana Şega, and Sandra Sprioff denoted (CEFMMSS)

Joint Work with Trung Chau, Art Duval, Sara Faridi, Thiago de Hollenben, Liana Şega denoted(CDFHMS)

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Commutative Algebra meets Algebraic Combinatorics

January 2026

The CAAC Connection

Goal (Combinatorial Commutative Algebra)

Use the interplay between the combinatorics, algebra, and related fields to solve interesting problems.

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Remark (Key Tools)

To do this, we will define extremal ideals, together wtih a family of maps showing how answers for extremal ideals pass to answers for general square-free monomial ideals.

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$J = (xy, yz, xzw)$ is a square-free monomial ideal.
Note that $K = (x + y, x - y)$ is also square-free monomial when
char$(k) \neq 2$ since $(x + y, x - y) = (x, y)$.

We will always focus on the unique monomial generating set.

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- An ideal I is fixed under a "torus" action $\Leftrightarrow I$ is monomial
  G is an invertible diagonal matrix,
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- There are connections between monomial ideals and combinatorially defined objects.

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- Determine how your favorite property/invariant behaves under the map
- Apply the above process to find interesting bounds and algebraic properties.

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Note: To simplify notation, we write y_{123} for $y_{\{1,2,3\}}$, etc.

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Example ($q = 3$)

$$S_{[3]} = k[y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}]$$

- $\epsilon_1 = y_1 y_{12} y_{13} y_{123}$
- $\epsilon_2 = y_2 y_{12} y_{23} y_{123}$
- $\epsilon_3 = y_3 y_{13} y_{23} y_{123}$

$$\mathcal{E}_3 = (\epsilon_1, \epsilon_2, \epsilon_3)$$

Example

Example ($q = 4$)

$$S_{[4]} = k[y_1, y_2, y_3, y_4, y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}, y_{123}, y_{124}, y_{134}, y_{1234}]$$

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Notice the high degree of symmetry.
Given any subset of these generators, there is precisely one variable
that divides them all and no others.

Definition (CEFMMSS)

$I = (m_1, \ldots, m_q)$ square-free monomial ideal. For $\emptyset \neq A \subseteq [q]$:

- $\theta_I(A) = \{x_k \colon x_k \mid m_j \Leftrightarrow j \in A\}$
-
$$\psi_I(y_A) = \prod_{x_k \in \theta_I(A)} x_k$$

The Map

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Example ($I = (x_1 x_2 x_5 x_7, x_2 x_3 x_7, x_3 x_4 x_6)$)

- $m_1 = (x_2 x_7)(x_1 x_5), \quad m_2 = (x_2 x_7)(x_3), \quad m_3 = (x_3)(x_4 x_6).$
- $\psi_I(y_1) = x_1 x_5, \quad \psi_I(y_{12}) = x_2 x_7, \quad \psi_I(y_{23}) = x_3,$
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- The nontrivial images partition the variables appearing in m_i's

- $\psi_I : S_{[q]} \to R$ is a ring map, but it need not be onto R.
- Notation: If J is a monomial ideal of $S_{[q]}$, we will write $\psi_I(J)$ to mean the monomial ideal $\psi_I(J)R = \langle \psi_I(J) \rangle$.

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Question

What can we say about free resolutions and betti numbers of $\mathcal{E}_q^{\,r}$?

Taylor's Resolution If $I = (m_1, \ldots, m_q)$ with m_i monomial, form a simplex on q vertices. Label the vertices with the m_i and label each face with the lcm of its vertices.
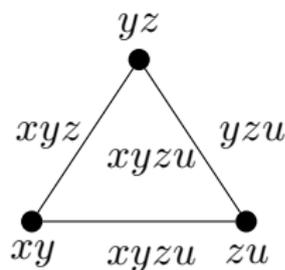
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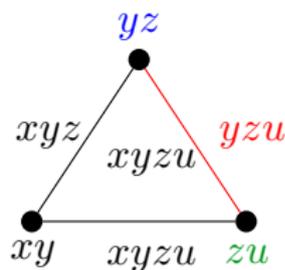
If $I = (xy, yz, zu)$,

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The Beginning: Diana Taylor's Thesis

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If $I = (xy, yz, zu)$,

$$0 \longrightarrow R(xyzu) \longrightarrow \begin{matrix} R(xyzu) \\ \oplus \\ R(xyz) \\ \oplus \\ R(yzu) \end{matrix} \longrightarrow \begin{matrix} R(xy) \\ \oplus \\ R(yz) \\ \oplus \\ R(zu) \end{matrix} \longrightarrow I \longrightarrow 0$$

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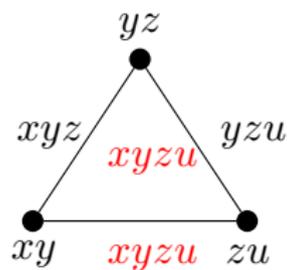
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where for example, $\partial(e_{yzu}) = u e_{yz} - y e_{zu}$ for example.

A Smaller structure supporting the resolution
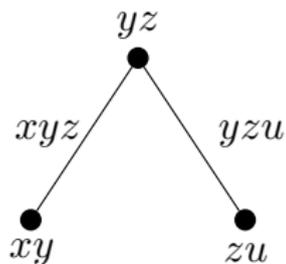

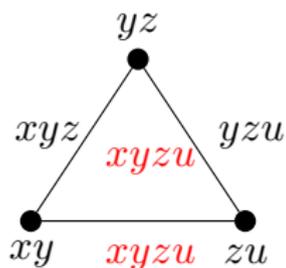Figure: A graph supporting a minimal free resolution of $I = (xy, yz, zu)$

Given a labeled simplicial complex supporting a resolution of I^2:

Lemma (Bayer, Peeva, Sturmfels)

A subcomplex supports a resolution of I^2 if for every monomial M, the subcomplex localized at M is empty or acyclic.

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Lemma (Batzies, Welker)

Pairing embedded matching labels carefully ("acyclic matching") and deleting the matched pairs yields a cell complex supporting a free resolution, which is minimal if all remaining faces have labels distinct from their maximal subfaces.

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- $N_q^2 = \{(a_1, \ldots, a_q) \mid a_i \geq 0, \ a_i \in \mathbb{Z}, \ \sum_{i=1}^q a_i = 2\}$
- N_q^2 is the vertex set of $\mathbb{L}_q^2$.
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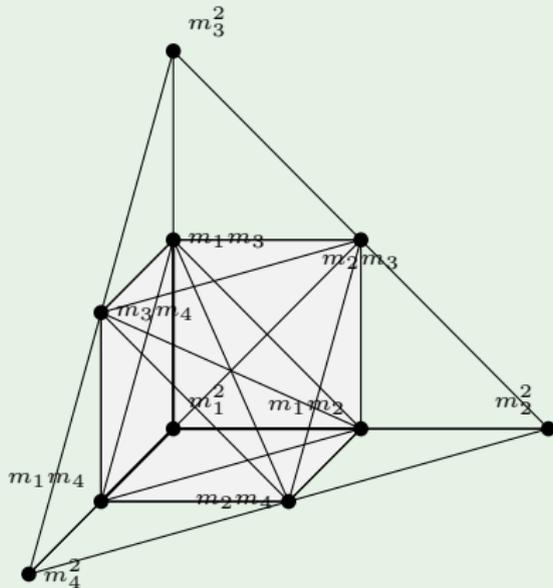
Theorem (CEFMMSS)

$\mathbb{L}_q^2$ *supports a minimal free resolution of* $\mathcal{E}_q^2$.

Example

$\mathbb{L}_4^2$ has five facets; one simplex with six vertices in the center shaded gray and four tetrahedrons with the vertices $m_i^2, m_i m_j, m_i m_k, m_i m_l$ for i, j, k, l distinct elements in $\{1, 2, 3, 4\}$.

Definition

Fix $q, r \in \mathbb{N}$ and $1 \leq i \leq q$. Define:

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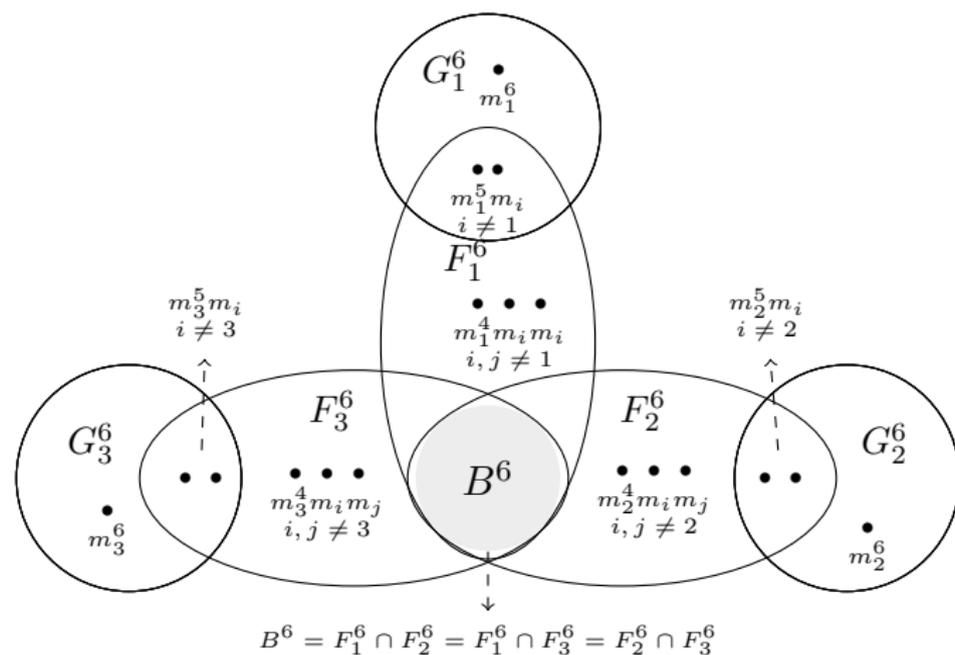- G_i^r are facets of L_q^r as long as $r, q \geq 2$.

Figure: A picture of L_3^6

Bounding Betti Numbers and Projective Dimension

Theorem (CEFMMSS)

$$\beta_i(I^2) \le \beta_i(\mathcal{E}_q^2) \le q\binom{q-1}{i} + \binom{\binom{q}{2}}{i+1}$$

$$\beta_i(I^r) \le \beta_i(\mathcal{E}_q{}^r) \le q\binom{q-1}{i} + \binom{b}{i+1}$$

$$\mathrm{proj.\,dim}(I^r) \le \mathrm{proj.\,dim}(\mathcal{E}_q{}^r) \le \mathit{max}\{q-1, f-1\}$$

Here b is the coefficient of x^r in $(1 + x + x^2 + \cdots + x^{\lceil \frac{r}{2} \rceil})^q$

$$f = \frac{\binom{q+r-1}{r} - b - q}{q} + b$$

> **Conjecture (EFSS, CDFHMS)**
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> $\mathcal{E}_q{}^r$ is Scarf, meaning the Scarf complex supports a resolution.

To get the Scarf complex: Start with labeled Taylor and delete all faces with repeated labels. What remains is $\mathbb{S}_q^r$.

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Theorem (EFSS, CDFHMS)

$\sigma = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\} \in \mathbb{S}_q^r$ where $\mathbf{a}_1 = (a_{11}, \ldots, a_{1q})$ etc., if and only if $\sigma' \in \mathbb{S}_q^r$ for all $\sigma' \subsetneq \sigma$ and $\mathbf{a}_1, \ldots, \mathbf{a}_d$ are the only solutions $\mathbf{w} \in N_q^r$ to the system of inequalities

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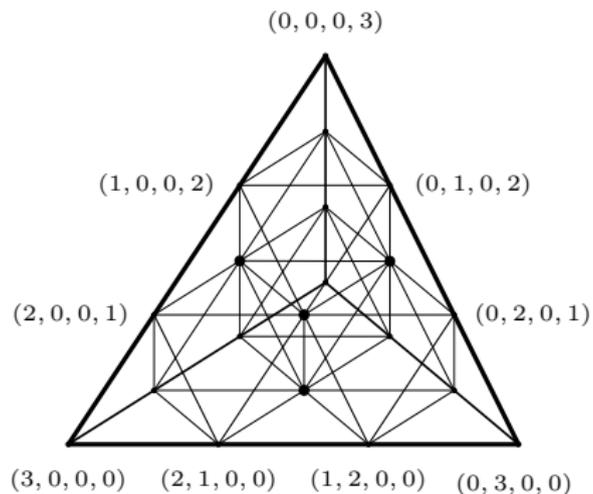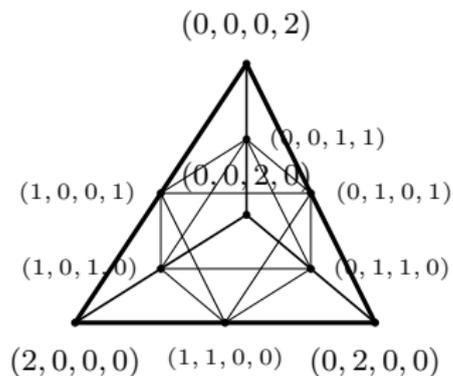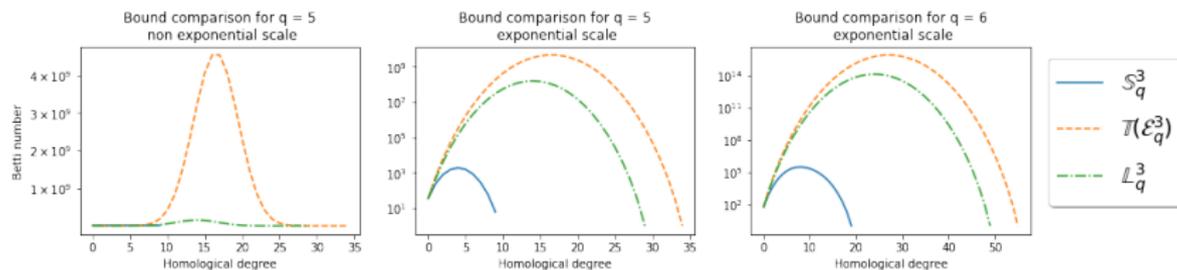These should be precisely the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$

Illustration of Improvements in Bounds

> **Theorem (CDFHMS)**
>
> $$\operatorname{proj.dim}(I^3) \leq \operatorname{proj.dim}(\mathcal{E}_q{}^3) = \begin{cases} \binom{q}{3} - 1 & \text{if} \quad q \geq 5 \\ \binom{q}{2} - 1 & \text{if} \quad 3 \leq q \leq 4 \\ q - 1 & \text{if} \quad 1 \leq q \leq 2 \end{cases}$$

Definition

- f is integral over I ($f \in \overline{I}$) if for some s, $\exists\, a_i \in I^i$ with

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Notation: Write monomials in $S_{[q]}$ as $\mathbf{m} = \prod_{A \subseteq [q]} y_A^{b_A} = \mathbf{y}^{\mathbf{b}}$.

Integral Closure and Linear Programming

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Lemma (CDFHMS)

$\mathbf{y^b} \in \overline{\mathcal{E}_q{}^r} \Leftrightarrow \exists\, \alpha_1, \ldots, \alpha_q \in \mathbb{Q}_{\geq 0}$ *such that*

1. $\alpha_1 + \cdots + \alpha_q = r$ *and*

2. $b_A \geq \sum_{i \in A} \alpha_i$ *for all* $\emptyset \neq A \subseteq [q]$.

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$b_2, b_3 \geq 1,\ b_1 = 0 \Rightarrow \alpha_1 = 0 \Rightarrow \alpha_2 + \alpha_3 = 2 \Rightarrow b_{23} \geq 2$. Then
$\epsilon_2 \epsilon_3 \mid \mathbf{y^b}$ so $\mathbf{y^b} \in \mathcal{E}_3^2$. ETC.

$$\epsilon_2 \epsilon_3 = y_2 y_3 y_{12} y_{23}^2 y_{13} y_{123}^2$$

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This one is small enough to run on Macaulay2.

Integral Closures Continued

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Every square-free monomial ideal with 3 generators is normal!!

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Definition

$$g(r,q) = \prod_{|A| \geq 1} y_A \cdot \prod_{|A| \geq 3} y_A \cdot \epsilon_1^{r-2}$$

Lemma (CDFHMS)

If $q \geq 4$ and $r \geq 2$, then $g(r,q)$ satisfies $z^q - (g(r,q))^q = 0$, hence $g(r,q) \in \overline{\mathcal{E}_q{}^r} \smallsetminus \mathcal{E}_q{}^r$.

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- if $q \geq 4$ and $\psi_I(g(r,q)) \notin I^r$, then $\overline{I^r} \neq I^r$;
- if $q = 4$, then $\overline{I^2} = I^2 + (\psi_I(g(2,4)))$;

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- if $q = 4$, then $\overline{I^2} = I^2 + (\psi_I(g(2, 4)))$;
- if $q = 4$ then I is normal if and only if $\overline{I^r} = I^r$ for $r = 1, 2$
- if $q = 5$ then I is normal if and only if $\overline{I^r} = I^r$ for $r = 1, 2, 3$

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Theorem (CDFHMS)

$$\psi_I(\mathcal{E}_q^{(r)}) = I^{(r)}$$

where $I^{(r)} = P_1^r \cap \cdots \cap P_t^r$ where P_i are the minimal primes of I.

Cont..

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Consequences:

Definition (Resurgance, Asymptotic Resurgance)

$$\rho(I) = \sup\left\{ \frac{s}{r} : s, r \in \mathbb{N} \quad \text{and} \quad I^{(s)} \not\subset I^r \right\}$$

$$\rho_a(I) = \sup\left\{ \frac{s}{r} : s, r \in \mathbb{N} \quad \text{and} \quad I^{(st)} \not\subset I^{rt} \quad \text{for all} \quad t \gg 0 \right\}$$

Corollary

$$\rho(I) \leq \rho(\mathcal{E}_q) \quad \text{and} \quad \rho_a(I) \leq \rho_a(\mathcal{E}_q).$$

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Definition (Symbolic Defect)

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This means it would be very useful to have information about $\mathcal{E}_q^{(r)}$ and related bounds. To see more about what we know, see Trung Chau's talk later today.

Thank You!