ρ-orderings and valuative capacity in ultrametric spaces

Anne Johnson

November 9, 2018

Ultrametric basics

Definition

Let (M, ρ) be a metric space. If ρ satistifies the ultrametric inequality

$$\rho(x, z) \leq max(\rho(x, y), \rho(y, z)), \forall x, y, z \in M$$

then (M, ρ) is an **ultrametric space**.

Definition

Let (V, N) be a normed vector space. Then N satisfies the **strong trianlge inequality** if

$$N(x + y) \leq max(N(x), N(y)), \forall x, y \in V$$

Ultrametric basics

Proposition

[1] All triangles in an ultrametric space (M, ρ) are either equilateral or isocoles, with at most one short side.

Ultrametric basics

Proposition

[1] If (M, ρ) is an ultrametric space and $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are balls in (M, ρ), then either $B_{r_1}(x_0) \cap B_{r_2}(y_0) = \emptyset$, $B_{r_1}(x_0) \subseteq B_{r_2}(y_0)$, or $B_{r_2}(x_0) \subseteq B_{r_1}(x_0)$. That is, in an ultrametric space, all balls are either comparable or disjoint.

Proposition

[1] The distance between any two disjoint balls in an ultrametric is constant. That is, if $B_{r_1}(x_0)$ and $B_{r_2}(y_0)$ are two balls in an ultrametric space (M, ρ), then $\rho(x, y) = c$ for some $c \in \mathbb{R}$ and $\forall x \in B_{r_1}(x_0)$ and $\forall y \in B_{r_2}(y_0)$

Proposition

[1] Every point of a ball in an ultrametric is at its centre. That is, if $B_r(x_0)$ is a ball in an ultrametric space (M, ρ), then $B_r(x) = B_r(x_0)$, $\forall x \in B_r(x_0)$

Proposition

[1] If M is a ultrametric space and Γ_M is the set of all distances occurring between points of M, then Γ_M is a discrete subset of $\mathbb{R}$. In particular if $|\ \Gamma_M\ | = \infty$, then the elements of Γ_M can be indexed by $\mathbb{N}$.

ρ-orderings, ρ-sequences, and valuative capacity

In what follows, let S be a compact subset of an ultrametric space (M, ρ).

Definition

[2] A ρ-**ordering** of S is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ such that $\forall n > 0$, a_n maximizes $\prod_{i=0}^{n-1} \rho(s, a_i)$ over $s \in S$.

Definition-Proposition

[2] The ρ-**sequence** of S is the sequence whose 0^{th}-term is 1 and whose n^{th} term, for $n > 0$, is $\prod_{i=o}^{n-1} \rho(a_n, a_i)$.

ρ-orderings, ρ-sequences, and valuative capacity

Definition-Proposition

[2] Let $\gamma(n)$ be the ρ-sequence of S. The **valuative capacity** of S is

$$\omega(S) := \lim_{n\to\infty} \gamma(n)^{1/n}$$

valuative capacity: quick results

Proposition

(translation invariance) Let (M, ρ) be a compact ultrametric space and suppose M is also a topological group. If ρ is (left) invariant under the group operation, then so is ω. That is, if $\rho(x, y) = \rho(gx, gy)$, $\forall g, x, y \in M$, then $\omega(gS) = \omega(S)$, for $S \subseteq M$.

Proposition

(scaling) Let (V, N) be a normed vector space and suppose N satisfies the strong triangle identity. Then if N is multiplicative, so is ω. That is, if $N(gx) = N(g)N(x)$,$\forall g, x \in V$, then $\omega(gS) = N(g)\omega(S)$, for $g \in V$ and $S \subseteq M$.

valuative capacity: subadditivity

Proposition

[2](subadditivity) If $diam(S) := \max_{x,y \in S} \rho(x,y) = d$ and $S = \cup_i^n A_i$ for A_i compact subsets of M with $\rho(x_i, x_j) = d$, $\forall x_i \in A_i$, $\forall x_j \in A_j$ and $\forall i, j$, then

$$\frac{1}{log(\omega(S)/d)} = \sum_{i=1}^{n} \frac{1}{log(\omega(A_i)/d)}$$

Corollary

Suppose $S = \cup_i^n S_i$ *with* $\rho(S_i, S_j) = d = diam(S)$ *and also* $\omega(S_i) = \omega(S_j)$, $\forall i, j$ *. Let* $r \in \mathbb{R}$ *be such that* $\omega(S_i) = r\omega(S)$, $\forall i$*. Then* $\omega(S) = r^{\frac{1}{n-1}} \cdot d$*. In particular if* $S = \mathbb{Z}$ *and* $(M, \rho) = (\mathbb{Z}, |\cdot|_p)$ *then* $\omega(S) = (\frac{1}{p})^{1/p-1}$ *for any prime* p*.*

Constructing a ρ-ordering

Setup:

- Let $S \subseteq M$ be a compact subset of an ultrametric space (M, ρ).
- Let $\Gamma_S = \{\gamma_0, \gamma_1, \ldots, \gamma_\infty = 0\}$ be the set of distances in S.
- Note that for each $k \in \mathbb{N}$, the closed balls of radius γ_k partition S. That is,

$$S = S_{\gamma_k} := \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$$

where both n and the x_i's depend on k.

Constructing a ρ-ordering

Setup, continuted:

Fix a $k \in \mathbb{N}$.

- Let $S_{\gamma_k} = \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$ be a partition of S, as above.
- Note that we can construct $S_{\gamma_{k+1}}$ by partitioning each of the $\overline{B_{\gamma_k}(x_i)}$, i.e.,

$$S = S_{\gamma_{k+1}} = \cup_{i=1}^{n} \cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})}$$

  where $1 \leq l_i \leq n$ and $\cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})} = \overline{B_{\gamma_k}(x_i)}$, $\forall i$.
- We denote by $x_{i,j}$ the centre of a ball of radius γ_{k+1} partitioning the ball $B_{\gamma_k}(x_i)$.
- Without loss of generality, when $j = 1$, assume $x_{i,j} = x_i$, $\forall i$.

Constructing a ρ-ordering

We now make the following observation due to [3],

Lemma

For each $k \in \mathbb{N}$, the elements of S_{γ_k}, that is, the closed balls of radius γ_k, themselves form an ultrametric space, where

$$\rho_k(\overline{B_{\gamma_k}(x)}, \overline{B_{\gamma_k}(y)}) = \begin{cases} \rho(x, y), & \textit{if } \rho(x, y) > \gamma_k \\ 0, & \textit{if } \rho(x, y) \leq \gamma_k, \textit{ i.e., } \overline{B_{\gamma_k}(x)} = \overline{B_{\gamma_k}(y)} \end{cases}$$

Constructing a ρ-ordering

We make the following observations:

- Since S is compact, S_{γ_k} is a finite metric space $\forall k < \infty$ and $S_{\gamma_\infty} = \cup_{x \in S} \overline{B_0(x)} = \cup_{x \in S} x = S$ and $\rho_\infty = \rho$.
- View S_{γ_k}, for fixed $k < \infty$ as a finite ultrametric space with n elements. Let us denote an element of S_{γ_k}, that is a $\overline{B_{\gamma_k}(x_i)}$, by its centre, x_i.
- Without loss of genearlity, we can reindex the x_i's so that they give the first n terms of a ρ_k-ordering of S_{γ_k}.

Constructing a ρ-ordering

Setup, revisited:

- Let $S_{\gamma_k} = \cup_{i=1}^{n} \overline{B_{\gamma_k}(x_i)}$ be the finite metric space describe above, and suppose the x_i are indexed according to a ρ_k-ordering of S_{γ_k}.
- Let $S_{\gamma_{k+1}}$ be the finite metric space formed by partitioning each of the $B_{\gamma_k}(x_i)$, so that $S_{\gamma_{k+1}} = \cup_{i=1}^{n} \cup_{j=1}^{l_i} \overline{B_{\gamma_{k+1}}(x_{i,j})}$ and $x_{i,j}$ is a point in the ball $B_{\gamma_k}(x_i)$ with the convention that $x_{i,1} = x_i$, $\forall i$.

Constructing a ρ-ordering

Consider the matrix A_k, whose $(i,j)^{th}$-entry is $x_{i,j}$ (or * if $l_i < j$).

$$A_k = \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,l_1} & x_{2,l_2} & \cdots & x_{n,l_n} \end{pmatrix}$$

Constructing a ρ-ordering

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A ρ_{k+1}-ordering of $S_{\gamma_{k+1}}$ can be found by concatenating the rows of A_k (and ignoring *'s).

Some corollaries

Corollary

Interweaving the bottown row of the lattice of closed balls for a set S gives a ρ-ordering of S. In particular, the natural ordering on the integers gives a ρ_p-ordering for every prime p.

Corollary

Suppose S and T are compact subsets of an ultrametric space M with $\Gamma_S = \Gamma_T$ and $|S_{\gamma_k}| = |T_{\gamma_k}|, \forall k$. Then $\omega(S) = \omega(T)$.

references

- Alain M. Robert, A course in p-adic analysis.
- Keith Johnson, P-orderings and Fekete sets
- Nate Ackerman, Completeness in Generalized Ultrametric Spaces