Random Walk & Identities

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Euler polynomials

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz} \quad \text{and} \quad \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^p e^{xz}.$$

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x)E_{k_2}(0)\cdots E_{k_p}(0).$$

Inverse Formula $\exists ? P \in \mathbb{R}[x_1,\ldots,x_k]$

$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x),\ldots,E_{n_k}^{(p_k)}(x)\right)?$$

Theorem (LJ, V. H. Moll, and C. Vignat). $\forall N \in \mathbb{Z}_+$

$$E_n(x) = \frac{1}{N^n}\sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)}\left(\frac{\ell-N}{2}+Nx\right),$$

where

$$\frac{1}{T_N\left(\frac{1}{z}\right)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \qquad T_N(\cos\theta) = \cos(N\theta).$$

N = 2:

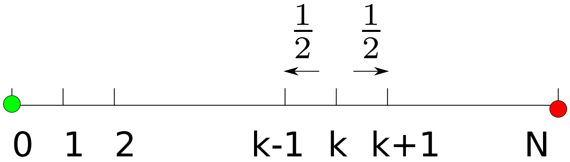
$$\cos(2\theta) = 2\cos^2\theta - 1 \Rightarrow T_2(z) = 2z^2 - 1 \Rightarrow \frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} = \sum_{k=1}^{\infty} \frac{z^{2k}}{2^k}.$$

Recall

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

$$E_n(x) = \frac{1}{2^n} \sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)} \left(\frac{\ell}{2} - 1 + 2x \right), \quad \text{where } p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

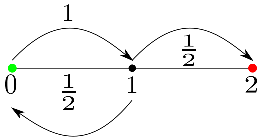
Random Walk



- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk;
- let ν_N be the random number of steps for this process.

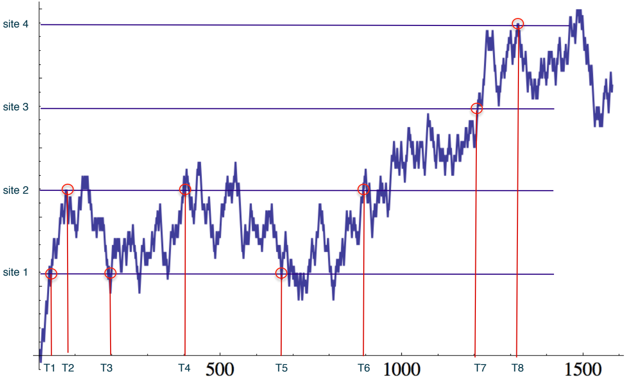
$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

$\underline{N = 2:}$



$$p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

Reflected Brownian Motion



Can This Model Work?

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right)$$

- Is there a stochastic model to interpret this formula?
- Can this method be used to prove/generate identities?

Probabilistic Interpretation:

1. $E_n^{(p)}(x)$: Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

If L_j, $j = 1, 2, \ldots, p$, are independent and identically distributed with $L_j \sim L$,

$$\begin{aligned} E_n^{(p)}(x) &= \mathbb{E}\left[\left(x + \left(iL_1 - \frac{1}{2}\right) + \cdots + \left(iL_p - \frac{1}{2}\right)\right)^n\right] \\ &= \mathbb{E}\left[\left(x + i\sum_{j=1}^{p} L_j - \frac{p}{2}\right)^n\right]. \end{aligned}$$

Random Sum

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right)$$

Probabilistic Interpretation:
2. Proof of the identity. Let L_j's be i. i. d. random variables with $L_j \sim L$.

Theorem. [Klebanov et al.]. Let ν_N be an integer valued random variable independent of the L_j 's, defined by the moment generating function:

$$\mathbb{E}\left[z^{\nu_N}\right] = \frac{1}{T_N\left(\frac{1}{z}\right)}.$$

Then, the random variable

$$Z_N = \frac{1}{N} \sum_{j=1}^{\nu_N} L_j$$

has the same hyperbolic secant distribution (as L_j's).

[Klebanov2012]L. B. Klebanov, A. V. Kakosyan, S. T. Rachev, and G. Temnov. On a class of distributions stable under random summation. *J. Appl. Prob.*, 49:303–318, 2012.

Proof of the identity

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

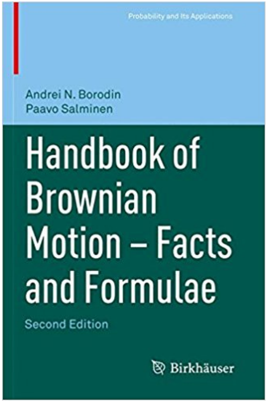
$$L \sim Z_N = \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \quad \Rightarrow \quad x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N} \sum_{j=1}^{\nu_N} iL_j \right) - \frac{1}{2}$$

$$\Rightarrow \quad x + iL - \frac{1}{2} \sim \frac{1}{N} \sum_{j=1}^{\nu_N} \left(iL_j - \frac{\nu_N}{2} + Nx - \frac{N}{2} + \frac{\nu_N}{2} \right)$$

Taking moments:

- LHS: $\mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = E_n(x)$;
- RHS: Each $\nu_N = \ell$, with probability $p_\ell^{(N)}$ and
  $\mathbb{E}\left[\left(i \sum_{j=1}^{\ell} L_j - \frac{\ell}{2} + Nx - \frac{N}{2} + \frac{\ell}{2}\right)^n\right] = E_n^{(\ell)}\left(\frac{\ell - N}{2} + Nx\right).$ □

Hitting Time



- Reflected (Reflecting) Brownian Motion in $\mathbb{R}_+$: W_t =distance to 0 at time t.
- Hitting times: $H_z := \min_t \{W_t = z\}$.
-

$$\mathbb{E}_x\left[e^{-\alpha H_z}\right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \le x \le z; \\ e^{-(x-z)w}, & z \le x. \end{cases}$$

1. $w = \sqrt{2\alpha}$;
2. $\mathbb{E}_x$ means it starts with point x (instead of 0);
3.

$$\mathbb{E}\left[e^{s\left(iL-\frac{1}{2}\right)}\right] = \int_{\mathbb{R}} \frac{e^{s\left(it-\frac{1}{2}\right)}}{\cosh(\pi t)} \mathrm{d}t = \frac{e^{-\frac{s}{2}+sx}}{\cosh\left(\frac{s}{2}\right)} e^{sx}.$$

$$\frac{2}{1+e^s} e^{sx} = \sum_{n=0}^{\infty} E_n(x) \frac{s^n}{n!}$$

Christophe's Idea

Consider a linear Brownian motion W_t starting from 0, with the hitting time T by W_t of level $z = 1$. Define another independent Brownian motion $\omega_t \sim \operatorname{sech}(x)$. Let

$$T_1 < T_2 < \cdots < T_l = T, \quad T_j = \min_s \left\{ W_t = \frac{j}{N} \right\}.$$

This defines a random walk with

$$p_\ell^{(N)} = \mathbb{P}\left\{ W_t \text{ reach the sink in } \ell \text{ steps} \right\}.$$

Now write

$$T = (T - T_{\ell-1}) + (T_{\ell-1} - T_{\ell-2}) + \cdots + (T_1 - 0)$$

and

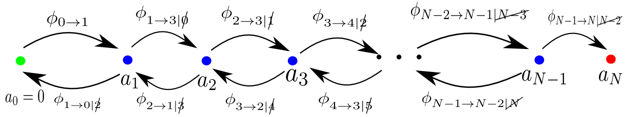
$$\omega_T \sim \omega_{T-T_{\ell-1}} + \omega_{T_{\ell-1}-T_{\ell-2}} + \cdots + \omega_{T_1-0},$$

each term $\sim \operatorname{sech}(x)$. This corresponds Klebanov's random sum decomposition

$$Z_N = \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \sim L.$$

My Goal

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right) \Longleftrightarrow \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \sim L \sim \operatorname{sech}(x)$$



- A reflected Brownian motion model with N equally distributed sites; $a_j = j$ or j/N;
- (modified) hitting times with "step" from one site to a nearby site $\sim \operatorname{sech}(x)$;

$$\mathbb{E}_x \left[e^{-\alpha H_z} \right] = \begin{cases} \frac{\cosh(xw)}{\cosh(zw)}, & 0 \le x \le z; \\ e^{-(x-z)w}, & z \le x. \end{cases}$$

If anyone has an idea, please let me know.

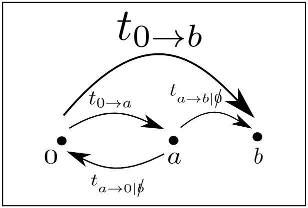
1-dim, 1-loop

With $p \le q \le r$, $w = \sqrt{2\alpha}$

$$\phi_{p\to q} := \mathbb{E}_p\left[e^{-\alpha H_q}\right] = \frac{\cosh(pw)}{\cosh(qw)},$$

$$\phi_{q\to p|\not r} := \mathbb{E}_q\left[e^{-\alpha H_p}|W_t < r\right] = \frac{\sinh((r-q)\,w)}{\sinh((r-p)\,w)},$$

$$\phi_{q\to r|\not p} := \mathbb{E}_q\left[e^{-\alpha H_r}|W_t > p\right] = \frac{\sinh((q-p)\,w)}{\sinh((r-p)\,w)},$$



- The hitting time $t_{0\to b}$ can be decomposed as

$$t_{0\to b} = \underbrace{\left(t_{0\to a} + t_{a\to 0|\not b}\right) + \cdots + \left(t_{0\to a} + t_{a\to 0|\not b}\right)}_{\ell \text{ copies}} + t_{0\to a} + t_{a\to b|\not 0}$$

- Generating functions:

$$\phi_{0\to b} = \phi_{0\to a}\phi_{a\to b|\not 0}\sum_{\ell=0}^{\infty}\left(\phi_{0\to a}\phi_{a\to 0|\not b}\right)^{\ell}$$

$$\begin{aligned}
\phi_{0\to b} &= \operatorname{sech}(bw), \\
\text{RHS} &= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\sum_{\ell=0}^{\infty}\left[\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}\right]^{\ell} \\
&= \operatorname{sech}(aw)\cdot\frac{\sinh(aw)}{\sinh(bw)}\cdot\frac{1}{1-\operatorname{sech}(aw)\cdot\frac{\sinh((b-a)w)}{\sinh(bw)}}
\end{aligned}$$

1-dim, 1-loop

Proposition. [LJ. and C. Vignat]

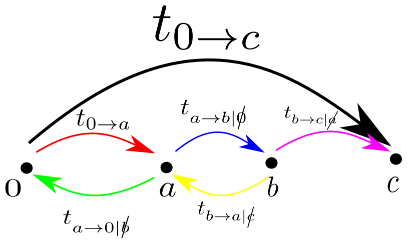
$$E_n\left(\frac{x}{2b}+\frac{3}{2}-2\frac{a}{b}\right)-E_n\left(\frac{x}{b}+\frac{1}{2}\right)=\frac{(n+1)\left(1-2\frac{a}{b}\right)2^n a^n}{b^n}\sum_{\ell=0}^{\infty}\underline{\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}}B_n^{(\ell+1)}\left(\frac{x+b}{4a}+\frac{\ell}{2}\right).$$

- $\frac{a}{b}\left(1-\frac{a}{b}\right)^{\ell}$ are the probability weights of a geometric distribution with parameter a/b.

The case $b=2a$, i.e., equally distributed sites, gives

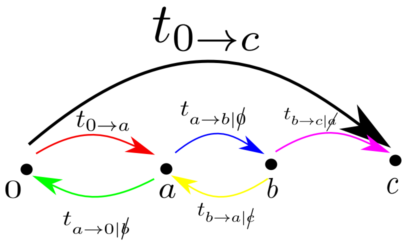
$$0=0.$$

How about 2-loops?



$t,\ t,\ t,\ t,\ t,\ t$

1-dim, 2-loops

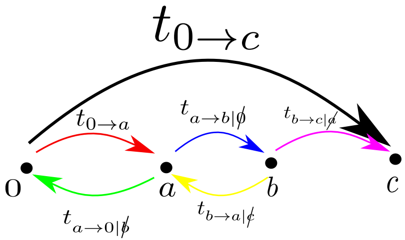


$$\begin{aligned} t &= t + t + t + t + t + t + \cdots + t \\ &= t + t + t + \underbrace{(t + t) + \cdots + (t + t)}_{k \text{ loops}} + \underbrace{(t + t) + \cdots + (t + t)}_{\ell \text{ loops}} \end{aligned}$$

We can generalize it to n-loop model.

Unfortunately, this is WRONG.....

1-dim, 2-loops



$$\phi = \phi \cdot \phi \cdot \phi \cdot \left[\sum_{k=0}^{\infty} (\phi\phi)^k\right] \left[\sum_{\ell=0}^{\infty} (\phi\phi)^\ell\right] = \frac{\phi \cdot \phi \cdot \phi}{(1-\phi\phi)(1-\phi\phi)}$$

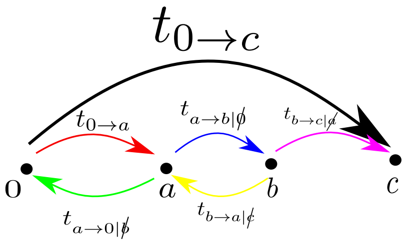
Let $a = 1,\ b = 2$ and $c = 3$.

$$\text{LHS} \quad = \quad \phi = \phi_{0\to 3} = \operatorname{sech}(3w) = \frac{1}{\cosh(3w)}$$

$$\text{RHS} \quad = \quad \frac{\frac{1}{\cosh(w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot \frac{\sinh(w)}{\sinh(2w)}}{\left(1 - \frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}\right)\left(1 - \frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)} = \frac{\frac{1}{4\cosh^3 w}}{\left(1 - \frac{1}{2\cosh^2 w}\right)\left(1 - \frac{1}{4\cosh^2 w}\right)}$$

$$= \quad \frac{2\cosh w}{(2\cosh^2 w - 1)\,(4\cosh^2 w - 1)} \neq \frac{1}{\cosh(3w)} = \frac{1}{4\cosh^3 w - 3\cosh w}$$

Problem



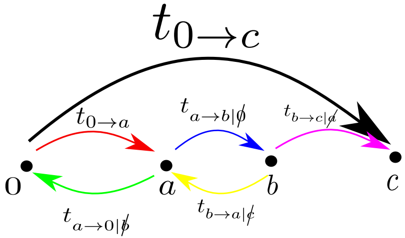
$$\phi = \phi \cdot \phi \cdot \phi \cdot [\text{Loops}]$$

$$\frac{1}{\cosh(3w)} = \frac{1}{\cosh(w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot \frac{\sinh(w)}{\sinh(2w)} \cdot [\text{Loops}] = \frac{1}{4\cosh^2 w} \cdot [\text{Loops}]$$

$$[\text{Loops}] = \frac{4\cosh^2 w}{4\cosh^3 w - 3\cosh w} = \frac{1}{1 - \frac{3}{4\cosh^2 w}} = \sum_{\ell=0}^{\infty} \left(\frac{3}{4\cosh^2 w} \right)^{\ell}.$$

$$\phi\phi + \phi\phi = \frac{1}{\cosh(w)} \frac{\sinh(w)}{\sinh(2w)} + \frac{\sinh(w)}{\sinh(2w)} \frac{\sinh(w)}{\sinh(2w)} = \frac{3}{4\cosh^2 w}$$

Explanation



$$t = t + t + t + \underbrace{(t+t) + \cdots + (t+t)}_{k \text{ loops}} + \underbrace{(t+t) + \cdots + (t+t)}_{\ell \text{ loops}}$$

For instance,

$$t = \underbrace{(t+t) + \cdots + (t+t)}_{k_1 \text{ loops}} + t + \underbrace{(t+t) + \cdots + (t+t)}_{\ell \text{ loops}} + \underbrace{(t+t) + \cdots + (t+t)}_{k_2 \text{ loops}} + t + t$$

Let both k_1 and $k_2 \to \infty$.

$$\phi\phi + \phi\phi = \frac{1}{\cosh(w)} \frac{\sinh(w)}{\sinh(2w)} + \frac{\sinh(w)}{\sinh(2w)} \frac{\sinh(w)}{\sinh(2w)} = \frac{3}{4\cosh^2 w}$$

Two-loops

$$I := \phi_{a\to b|\not A}\,\phi_{b\to a|\not c}, \quad II := \phi_{b\to c|\not a}\,\phi_{c\to b|\not B}$$

- k loops of I followd by l loops of II, with $k, l = 0, 1, \ldots,$ which gives

$$\sum_{k.l} I^k II^l = \frac{1}{1-I} \cdot \frac{1}{1-II};$$

- k_1 loops of I followed by l_1 loops of II, then followed by k_2 loops of I and finally followed by l_2 loops of II, with k_1, l_2 nonnegative and k_2, l_1 positive, which gives

$$\sum_{k_1.l_2=0,k_2,l_1=1}^{\infty} I^{k_1} II^{l_1} I^{k_2} II^{l_2} = \frac{I \cdot II}{(1-I)^2 (1-II)^2};$$

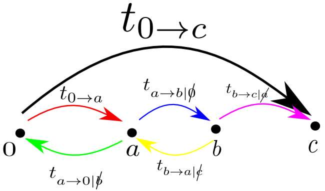
- the general term will be k_1 loops of $I \to l_1$ loops of $II \to \cdots \to k_n$ loops of $I \to l_n$ loops of II, with k_1, l_n nonnegtive and the rest indices positive, which gives

$$\frac{(I \cdot II)^{n-1}}{(1-I)^n (1-II)^n}.$$

Therefore, loops I and II contribute as

$$\sum_{n=1}^{\infty} \frac{(I \cdot II)^{n-1}}{(1-I)^n (1-II)^n} = \frac{1}{1-(I+II)} = \sum_{k=0}^{\infty} (I+II)^k .$$

Two Loops



$$\phi = \frac{\phi\cdot\phi\cdot\phi}{1-\phi\phi-\phi\phi}$$

Proposition. [LJ. and C. Vignat] For any positive integer n,

$$E_n\left(\frac{x}{6}\right) = \sum_{k=0}^{\infty} \frac{3^{k-n}}{4^{k+1}} E_n^{(2k+3)}\left(\frac{x}{2}+k\right).$$

In general

$$(x+2c\mathcal{E}+c)^n = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} q_{k,\ell}\Big[x+2\,(b-a)\,\mathcal{B}+2\,(c-b)\,\mathcal{B}'+a\mathcal{E}^{(\ell)}+2\,(b-a)\,\mathcal{U}^{(l)}$$

$$+\,2a\mathcal{U}'^{(k-\ell)}+2\,(b-a)\,\mathcal{B}'^{(k-\ell)}+q'_{k,\ell}\Big]^n,$$

$$q_{k,\ell} := \binom{k}{\ell}\frac{(b-a)^{\ell+1}\,a^{k-\ell+1}\,(c-b)^{k-\ell}}{b^{k+1}\,(c-a)^{k-\ell+1}} \qquad q'_{k,\ell} = c+(2k-2\ell)\,b+(3\ell-k+1)\,a,$$

where

$$\left(\mathcal{E}^{(p)}+x\right)^n = E_n^{(p)}(x),\quad \left(\mathcal{B}^{(p)}+x\right)^n = B_n^{(p)}(x),\quad \mathcal{U}^n = \frac{1}{n+1},\quad \mathcal{U}^{(p)} = \mathcal{U}_1+\cdots+\mathcal{U}_p.$$

n loops?

Consider consecutive loops $l_1, l_2, \ldots, l_n$, it seems like the contribution is

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=1}^{n} l_\ell\right)^k = \frac{1}{1-(l_1+\cdots+l_n)}. \quad (*)$$

- It feels right.
- I can "prove" it by induction.
- In general sites $0, 1, \ldots, N$:

$$\frac{1}{\cosh(Nw)} \overset{??}{=} \frac{\frac{1}{\cosh w}\cdot\left(\frac{\sinh w}{\sinh(2w)}\right)^N}{1-\left(\frac{1}{\cosh(w)}\frac{\sinh(w)}{\sinh(2w)}+(N-1)\frac{\sinh(w)}{\sinh(2w)}\frac{\sinh(w)}{\sinh(2w)}\right)}$$

$$= \frac{\frac{1}{2^N\cosh^{N+1} w}}{1-\frac{N+3}{4}\cosh^N w}.$$

This shows $(*)$ is not correct.

$$\frac{1}{\cosh(Nw)} = \frac{1}{\cos(Niw)} = \frac{1}{T_N(\cos(iw))} = \frac{1}{T_N(\cosh w)}.$$

Generalization

- Bessel process in $\mathbb{R}^n$:

$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$

- Moment generating functions for hitting times:
  $H_z := \min_s \left\{R_s^{(n)} = z\right\}$.

$$\mathbb{E}_x\left(e^{-\alpha H_z}; \sup_{0\le s\le H_z} R_s^{(n)} < y\right) = \begin{cases} \frac{x^{-\nu} I_\nu(xw)}{z^{-\nu} I_\nu(zw)}, & 0 \le x \le z \le y; \\ \frac{S_\nu(yw, xw)}{S_\nu(yw, zw)}, & z \le x \le y, \end{cases}$$

- $n = 2 + 2\nu$ for $\nu \ge 0$

$$S_\nu(x, y) := (xy)^{-\nu}\left[I_\nu(x)K_\nu(y) - K_\nu(x)I_\nu(y)\right],$$

and

$$I_\nu(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!\Gamma(\ell+\nu+1)}\left(\frac{x}{2}\right)^{2\ell+\nu}, \quad K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

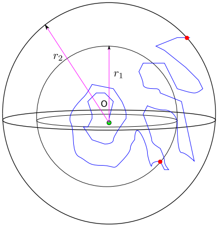
$n = 3 \Leftrightarrow \nu = 1/2$

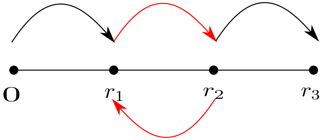
- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\,\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}}\sinh(x)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2}\sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

$$\mathbb{E}_x\left(e^{-\alpha H_z}; \sup_{0\le s\le H_z} R_s^{(3)} < y\right) = \begin{cases} \frac{z\sinh(xw)}{x\sinh(zw)}, & 0 \le x \le z \le y \\ \frac{z\sinh((y-x)w)}{x\sinh((y-z)w)}, & z \le x \le y \end{cases}$$





$n = 3 \Leftrightarrow \nu = 1/2$

Let $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$
Proposition. [LJ. and C. Vignat]

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right)-B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right]=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_n^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \qquad (\circledast)$$

Corollary 1. Take $x = 0$, $n = 2m - 1$ in ($\circledast$). The LHS is

$$\begin{aligned}\frac{3^{2m}}{2m}\left[B_{2m}\left(\frac{5}{6}\right)-B_{2m}\left(\frac{1}{2}\right)\right] &= \frac{3^{2m}}{2m}\left[\frac{1}{2}\left(1-2^{1-2m}\right)\left(1-3^{1-2m}\right)B_{2m}+\left(1-2^{1-2m}\right)B_{2m}\right]\\ &= \frac{3^{2m}}{2m}\left(1-2^{1-2m}\right)B_{2m}\left(\frac{1-3^{1-2m}}{2}+1\right)\\ &= \frac{3}{4m}\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)B_{2m};\end{aligned}$$

while the RHS is

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

Thus,

$$B_{2m}=\frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^k E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

$n = 3 \Leftrightarrow \nu = 1/2$

Proposition. [LJ. and C. Vignat]

$$\frac{3^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x}{6}+\frac{5}{6}\right)-B_{n+1}\left(\frac{x}{6}+\frac{1}{2}\right)\right]=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^{k}E_{n}^{(2k+2)}\left(\frac{x+3+2k}{2}\right). \qquad (\circledast)$$

Corollary 1.

$$B_{2m}=\frac{m}{\left(1-2^{1-2m}\right)\left(3^{2m}-1\right)}\sum_{k\geq 0}\left(\frac{1}{4}\right)^{k}E_{2m-1}^{(2k+2)}\left(k+\frac{3}{2}\right).$$

Corollary 2. Take $n = 1$ in $(\circledast)$.

$$B_2(x)=x^2-x+\frac{1}{6}\Rightarrow \text{LHS}=\frac{x+1}{2},$$

$$\sum_{n=0}^{\infty}E_n^{(p)}(x)\frac{z^n}{n!}=\left(\frac{2}{e^z+1}\right)^{p}e^{xz}\Rightarrow E_1^{(2k+2)}(x)=x-(k+1).$$

$$\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^{k}\left(\frac{x+3+2k}{2}-k-1\right)=\sum_{k\geq 0}\frac{3}{4}\left(\frac{1}{4}\right)^{k}\left(\frac{x+1}{2}\right)=\frac{x+1}{2}.$$

Proposition. [LJ. and C. Vignat] For any positive integer n,

$$3^n B_n\left(\frac{x+4}{6}\right) = \sum_{k=0}^{\infty} \frac{1}{2^k} E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

Several remarks are in order at this point:

- the identities obtained from this approach are not of the usual, convolutional type. Rather, they are connection-type identities between the usual Bernoulli and Euler polynomials and their higher-order counterparts;
- these inherently involve a mixture of higher-order Bernoulli and Euler polynomials;
- the interest of this approach is that each term in such a decomposition can be related to a physical object, namely one loop in a trajectory of a random process;
- this work should be considered as only a first approach to a more general project in which the richness of the possible setups for random walks is expected to generate a number of non-trivial identities about more general special functions.

End

Thank you!

Connection Coefficients for Higher-order Bernoulli and Euler Polynomials:

A Random Walk Approach

https://arxiv.org/abs/1809.04636