MATH 2030: EIGENVALUES AND EIGENVECTORS

Determinants

Although we are introducing determinants in the context of matrices, the theory of determinants predates matrices by at least two hundred years. Their study was motivated in the pursuit of solutions to a variety of practical problems that may be seen as independent of matrices as the concept of a matrix had not been fully explored.

In this section we will generalize the definition of the determinant for any 2×2 matrix to an $n \times n$ matrix. The definition of the determinant arose from our study of invertible matrices, we saw that if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has a non-vanishing determinant

$$detA = a_{11}a_{22} - a_{12}a_{21}$$

then it is invertible and the inverse of A may be computed.

At times we will use an alternative notation, we will denote the determinant of a matrix A as $|A|$, or explicitly with the components of the matrix,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

As the determinant can be negative it is very important not to confuse this with the absolute value function $|a|_{abs} = \sqrt{a^2}$, which is always positive. Compare this with the definition of the determinant for a 1×1 matrix $det[a] = |a| = a$.

Notice that we have not given a formula for the determinant in the 1×1 and 2×2 cases we have been given the determinant formula, we will do this once more for the 3×3 to get a flavour of how the general case works. Given a 3×3 matrix A,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

the determinant will be

$$\begin{aligned} |A| &= aei - afh - bdi + bfg + cdh - ceg \\ &= a(ei - fh) - b(di - fg) + c(dh - eg); \end{aligned}$$

these bracketed terms are notable because they look like the determinants of 2×2 matrices, and so we may write this as

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Each of these 2×2 matrices are submatrices of A. We will take this as our definition of determinants as it is *recursive* since we have defined the definition of 3×3 matrices in terms of 2×2 submatrices.

Definition 0.1. Given a 3×3 matrix A, the **determinant** of A is the scalar, det A $= |A|$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{311} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{31} \\ a_{32} & a_{33} \end{vmatrix}.$$

It is important to notice the relationship between the coefficient of the determinant of the 2×2 submatrix and their positions in the original matrix. Each of these 2×2 determinants are obtained by deleting the row and column of A that contain the coefficient of the determinant. For the first term we remove the first column and first row to calculate the determinant multiplied by a_{11}. Furthermore there is an alternating sign in the sum of each term.

If we denote A_{ij} as the submatrix of A made by removing the i-th row and j-th column of A then this equation may be written as

$$\begin{aligned} detA = |A| &= a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| \\ &= \Sigma_{k=1}^{3}(-1)^{1+k}a_{1k}|A_{1k}|. \end{aligned}$$

For any square matrix A, $detA_{ij} = |A_{ij}|$ is called the **(i,j) -minor** of A.

Example 0.2. Q: Compute the determinant of

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

A: Using the formula we compute,

$$\begin{aligned} detA &= 5\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 5(0-(-2)) + 3(3-4) + 2(-1-0) \\ &= 10 - 3 - 2 = 5. \end{aligned}$$

Example 0.3. Q: Compute the determinant of

$$A = \begin{bmatrix} 3 & 0 & -3 \\ 2 & 1 & 4 \\ 3 & 2 & 2 \end{bmatrix}$$

A: Using the formula we compute,

$$\begin{aligned} detA &= 3\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ &= 3(2-8) + 0 + (-3)(4-3) \\ &= -18 - 3 = -21. \end{aligned}$$

Determinants of $n \times n$ matrices. The definition of (i,j)-minor of A is easily extended to square matrices of any size, this allows one to generalize the definition of the determinant to any $n \times n$ matrix

Definition 0.4. Given $A = [a_{ij}]$, a $n \times n$ matrix with $n \geq 2$, the **determinant** of A is the scalar, det A $= |A|$,

$$\begin{aligned} det\ A &= a_{11}det\ A_{11} - a_{12}det\ A_{12} + \cdots + (-1)^{1+n}a_{1n}det\ A_{1n} \\ &= \Sigma_{k=1}^{n}(-1)^{1+k}a_{1k}det\ A_{1k}. \end{aligned}$$

If we combine the minor of a matrix with its plus or minus sign, we produce the **(i,j)-cofactor of A**

$$C_{ij} = (-1)^{1+i} det A_{ij}$$

with this notation, the determinant is very compact

$$det\ A = \Sigma_{k=1}^{n} a_{1k} C_{1k}.$$

The above formula for the determinant will be called the **cofactor expansion along the first row**. If we expand along any other row we will recover the same result, furthermore the same could be done along any column of A as well. We will summarize this fact in a theorem whose proof will wait until later.

Theorem 0.5. *The Laplace Expansion Theorem The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$ may be computed as the sum*

$$\begin{aligned} det\ A = |A| &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ &= \Sigma_{k=1}^{n} a_{ik} C_{ik} \end{aligned} \tag{1}$$

and also as the sum

$$\begin{aligned} det\ A = |A| &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \\ &= \Sigma_{k=1}^{n} a_{kj} C_{kj}. \end{aligned} \tag{2}$$

These are respectively called the **cofactor expansion along the i-th row** *and* **cofactor expansion along the j-th column**.

As a helpful reminder to calculate the sign change between the (i,j)-minor of A and the (i,j)-cofactor of A, remember the checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Example 0.6. Q: Compute the determinant of the matrix

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

by (a) cofactor expansion along the third row and (b) cofactor expansion along the second column.
A: (a) Using the formula we find

$$\begin{aligned} |A| &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= 2\begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & -3 \\ 1 & 0 \end{vmatrix} \\ &= 2(-6) + 8 + 3(3) = 5 \end{aligned}$$

(b) To calculate det A with the cofactor expansion along the second column we use the second formula

$$\begin{aligned} |A| &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= -(-3)\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 3(-1) + 0 + 8 = 5 \end{aligned}$$

This example illustrates the utility of choosing the "right" row or column to expand along in order to compute the determinant quickly; in the second example we did less work as the coefficient of C_{22} is zero and so we need not compute this minor. Thus the Laplace Expansion theorem is relevant when a matrix contains a row or columns with lots of zeroes as the existence of such a row or column simplifies calculations.

Example 0.7. Q: By choosing the proper row or column, compute the determinant of

$$A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ 1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}.$$

A: As the third column has only one non-zero entry we should expand along this column. Reminding ourself of the checkerboard pattern, the entry $a_{23} = 2$ must have a minus sign in the (2,3)-minor:

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\ &= 0C_{13} + 2C_{23} + 0C_{33} + 0C_{43} \\ &= -2\begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 3 \\ -2 & 1 & 0 \end{vmatrix}. \end{aligned}$$

Thus in order to compute the determinant of A we must compute the determinant of the 3×3 submatrix, this is most easily done by expanding along the third row (or third column) as this contains a zero:

$$\begin{aligned} |A| &= -2\left(-2\begin{vmatrix} -3 & 1 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}\right) \\ &= -2(-2(-8) - 5) = -2(11) = -22 \end{aligned}$$

The determinants of upper and lower Triangular matrices are easily determined using the Laplace Expansion theorem

Example 0.8. Q: Compute the determinant of

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 & 4 \\ 0 & 3 & 2 & 5 & 7 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Expanding along the first column we find

$$|A| = 2\begin{vmatrix} 3 & 2 & 5 & 7 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 2*3*\begin{vmatrix} 1 & 6 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & -1 \end{vmatrix} = 2*3*1\begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix}$$

Computing the last determinant we find

$$|A| = 2*3*1(5(-1)-2) = 2*3*1*5*(-1) = -30$$

From this example we suspect that in general the determinant of an upper-triangular matrix will be the product of its diagonal entries.

Proposition 0.9. *The determinant of a triangular matrix is the product of the entries on its main diagonal. That is, if A=$[a_{ij}]$ is an $n \times n$ matrix, then*

$$det\ A = |A| = a_{11} * a_{22} * \cdots * a_{nn}.$$

Properties of Determinants. In general the cofactor expansion of an arbitrary matrix requires an extensive calculation, as n increases the number of computations required to determine the determinant of an $n \times n$ matrix increases at an alarming rate While this approach to computing the determinant does provide the answer for any size matrix, this will be useless for applications because of the number of multiplications, additions and subtractions required will take longer than one could reasonably wait. For this reason we examine the algebraic properties of the determinant to find a better method.

Row reduction will play an important part in the computation of determinants. However some care must be taken as not every elementary row operation leaves the determinant of a matrix unchanged. We summarize these facts in a helpful theorem

Theorem 0.10. *Let $A = [a_{ij}]$ be a square matrix,*

1. *If A has a zero row (or column) then det $A = 0$.*
2. *If B is obtained by interchanging two rows (or columns of A), then det B = - det A.*
3. *If A has two identical rows (or columns) then det $A = 0$.*
4. *If B is obtained by multiplyinga row (or column) of A by k, then det A = k det A.*
5. *If A,B, and C are identical except that the i-th row (or column) of C is the sum of the i-th rows (or columns) of A and B, then det C = det A + det B.*
6. *If B is obtained by adding a multiple of one row (or column) of A to another row (or column), then det B = det A*

Proof. We will save the proof of the second statement in the list until the end of the section. To illustrate how one might prove statements like these we prove (3),(4) and (5) using the rows of A- the proofs for these statements with columns are essentially identical.

- (3) If A has two identical rows, we may interchange them to produce B, with $B = A$, and so det B = det A. However taking (2) we see that this also implies det B = - det A, as this is the case we must have that det A = -det A, implying that det A = 0.

- (4) Suppose the i-th row of A is multiplied by k to produce B, so that $b_{ij} = ka_{ij}$ where $j \in [1, n]$. As the cofactors C_{ij} of the elements of the i-th row of A and B are identical, expanding along the i-th row of B gives

$$|B| = \Sigma_{j=1}^{n} b_{ij} C_{ij} = \Sigma_{j=1}^{n} k a_{ij} C_{ij} = k \Sigma_{j=1}^{n} a_{ij} C_{ij} = k|A|$$

- (5) As in the last example, the cofactors C_{ij} of the elements in the i-th row of A,B and C are identical, as no change has occured outside of the i-th row. Furthermore $c_{ij} = a_{ij} + b_{ij}, j \in [1, n]$ and so expanding along the i-th row of C we find

$$\begin{aligned} |C| &= \Sigma_{j=1}^{n} c_{ij} C_{ij} = \Sigma_{j=1}^{n} (a_{ij} + b_{ij}) C_{ij} \\ &= \Sigma_{j=1}^{n} a_{ij} C_{ij} + \Sigma_{j=1}^{n} b_{ij} C_{ij} \\ &= |A| + |B| \end{aligned}$$

With these statements proven, the proof of (1) and (6) are easily proven. □

Example 0.11. Compute the determinant of the matrix A if

$$a)\ A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ -4 & -6 & 2 \end{bmatrix},\ b)\ A = \begin{bmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}.$$

A: a) Using properties (5) followed by (1) we note that adding twice the first row to the third row, $R_3 + 2R_1$ results in the new third row being all zeroes:

$$\begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ -4 & -6 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

b) To compute the determinant of this matrix we apply row operations and keep track of the sign changes, we apply the row operations $R_1 \leftrightarrow R_2$; $\frac{1}{3}R_1$; $R_3 - 2R_1$, $R_4 - 5R_1$; $R_2 \leftrightarrow R_4$; $R_3 + 4R_2$ and $R_4 + 2R_2$ to get:

$$\begin{aligned} |A| &= \begin{vmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & 0 & -3 & 6 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} = -3 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & \\ 5 & -1 & -3 & 1 \end{vmatrix} \\ &= -3 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 0 & 4 & 7 & 3 \\ 0 & -1 & 2 & -9 \end{vmatrix} = -(-3) \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 4 & 7 & 3 \\ 0 & 2 & -4 & 5 \end{vmatrix} \\ &= 3 \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 0 & 15 & -33 \\ 0 & 0 & 0 & -13 \end{vmatrix} \\ &= 3 * 1 * (-1) * 15 * (-13) = 585 \end{aligned}$$

where semi-colons in the list of row operations is paralleled by equalities in the chain of equalities above.

Determinants of Elementary Matrices. As elementary row operations have come back into our discussion of matrices and their determinants, we will examine the determinants of the elementary matrices arising from these row operations. Setting $A = I_n$ the theorem on the properties of the determinants yields the helpful result

Theorem 0.12. *Let E be an $n \times n$ elementary matrix.*

1. *If E results from interchanging two rows of I_n then det E = -1.*
2. *If E results from multiplying one row of I_n by k, then det E = k.*
3. *If E results from adding a multiple of one row of I_n to another row, then det E = 1.*

Proof. Since det $I_n = 1$, applying the second, fourth and sixth properties of determinants we prove all three results above. □

Multiplying on the left by a matrix B by an elementary matrix is the equivalent of applying the analogous row operation on B, therefore we may rephrase the second, fourth and sixth properties of the determinant in the following way:

Lemma 0.13. *Let B be an $n \times n$ matrix, and let E be an $n \times n$ elementary matrix, then det(EB) = det(E)det(B)*

By applying this lemma we will prove the main result of this section

Theorem 0.14. *A square matrix A is invertible if and only if det A $\neq 0$.*

Proof. Let A be an $n \times n$ matrix and let R be the reduced row echelon form of A. We first show that $det\ A \neq 0$ if and only if $det\ R \neq 0$, by first supposing $E_1, E_2, \cdots, E_r$ be the elementary matrices corresponding to the elementary row operations that reduce A to R, then

$$E_r \cdots E_2 E_1 A = R$$

Taking the determinant of either side we find that

$$(det\ E_r) \cdots (det\ E_2)(det\ E_1)(det\ A) = det\ R.$$

Noting that the determinants of all elementary matrices are non-zero, we conclude that det A vanishes if and only if det R does as well.

To prove the next part, we assume that A is invertible, then by the Fundamental Theorem of Invertible Matrices, $R = I_n$ and so the determinant of R must be 1 which is never zero, thus the determinant of A must be non-zero as well. Alternatively assuming that $det\ A \neq 0$ then det $R \neq 0$, so R cannot contain a zero row, thus R must be I_n and A is invertible. □

Determinants and Matrix Operations. With a whole list of properties for determinants and matrix multiplication with elementary matrices, one wonders how other matrix operations relate to determinants. For example, are there any formulas for det(kA), det(A+B), det(AB), det (A^{-1}) and det(A^t) in terms of det A and det B?

Looking at the fourth property of determinants, we might think $det(kA) = kdet(A)$, however this is not quite right. The correct relationship between scalar multiplication and determinants is

Theorem 0.15. *If A is an $n \times n$ matrix, then*

$$det(kA) = k^n det(A)$$

The proof of this statement uses the fourth property, by examining the case when the n vectors can be scaled by the same value k.

There is no simple formula for de(A+B), and in general we will have $det(A+B) \neq det(A)+det(B)$, despite this there is a surprising connection between matrix multiplication of arbitrary matrices and determinants, due to Cauchy:

Theorem 0.16. *if A and B are $n\times n$ matrices, then*

$$det(AB) = det(A)det(B)$$

Proof. There are two cases to consider, when A is invertible and when it is not.

If A is invertible, then by the Fundamental Theorem of Invertible Matrices it can be written as the product of elementary matrices and the identity matrix, $A = E_1E_2\cdots E_k$, then $AB = E_1E_2\cdots E_kB$ k times and so

$$det(AB) = det(E_k)\cdots det(E_1)det(B) = det(E_k\cdots E_1)det(B) = det(A)det(B).$$

Now supposing that A is not invertible, then neither is AB and so det A = 0 and det(AB) = 0, thus det(AB) = det(A) det(B) as both sides are zero. □

Example 0.17. Q: Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$ we have $AB = \begin{bmatrix} 12 & 3 \\ 16 & 5 \end{bmatrix}$, verify if the formula holds true.
A: Computing the determinants we have det A = 4, det B = 3 and det AB = 12, it follows that det AB = det A det B.

For any invertible matrix, the determinant gives a useful relationship between the determinant of A and its inverse:

Theorem 0.18. *If A is invertible, then*

$$det(A^{-1}) = \frac{1}{det\ A}.$$

Proof. Since A is invertible $AA^{-1} = I$, and so $det(AA^{-1}) = 1$, thus $det(A)det(A^{-1}) = 1$, by solving this equation for det A^{-1} we find the desired identity as $det\ A \neq 0$ □

Example 0.19. Q: Verify this fact for A,

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

A: Computing the inverse we find

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

taking its determinant we find that

$$det\ A^{-1} = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) - \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right) = \frac{3}{8} - \frac{1}{8} = \frac{1}{4}$$

As $det\ A = 4$ we see that this is the case.

As one last application we relate the determinants of a matrix A with its transpose A^t, by noting that the Laplace Expansion theorem allows for an expansion along any row or column and that the rows of A^t are just the columns of A. Combining these two facts we find that

Theorem 0.20. *For any square matrix A,*

$$det\ A = det\ A^t.$$

Cramer's Rule and the Adjoint. We will introduce two useful formulas relating determinants to the solution of a linear system and the inverse of a matrix. The first, **Cramer's Rule** gives a formula for describing the solution of a particular subclass of systems of n linear equations in n variables - entirely in terms of determinants. The second result gives the inverse of a $n \times n$ matrix A.

To introduce Cramer's Rule we must define new notation. For an $n \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$, let $A_i(\mathbf{b})$ denote the matrix obtained by replacing the i-th column of A by $\mathbf{b}$, so that

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1} \mathbf{b} \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

With these new matrices defined we may introduce Cramer's Rule

Theorem 0.21. *Cramer's Rule Let A be an invertible $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\mathbf{x}$of the system $A\mathbf{x} = \mathbf{b}$ is given by*

$$\mathbf{x}_i = \frac{det\ A_i(\mathbf{b})}{det\ A},\ i = 1, ..., n$$

Proof. The columns of the identity matrix $I_n = I$ are the standard unit vectors $\mathbf{e}_1, ..., \mathbf{e}_n$, if $A\mathbf{x} = \mathbf{b}$, then

$$\begin{aligned} AI_i(\mathbf{x}) &= A[\mathbf{e}_1 \cdots \mathbf{x} \cdots \mathbf{e}_n] = [A\mathbf{e}_1 \cdots A\mathbf{x} \cdots A\mathbf{e}_n] \\ &= [\mathbf{a}_1 \cdots \mathbf{b} \cdots \mathbf{a}_n] = A_i(\mathbf{b}). \end{aligned}$$

Thus, using the relationship between matrix multiplication and determinants,

$$(det\ A)(det\ I_i(\mathbf{x})) = det(AI_i(\mathbf{x}) = det(A_i(\mathbf{b})).$$

Finally

$$detI_i(\mathbf{x}) = \begin{vmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & x_2 & \cdots 0 & 0 & \\ \vdots & \vdots & \ddots & \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & x_i & \cdots & 0 & 0 \\ \vdots & \vdots & & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & x_{n-1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & x_n & \cdots & 0 & 1 \end{vmatrix} = x_i$$

which can be seen by expanding along the i-th row. Thus (det A)$x_i = det(A_i(\mathbf{b}))$ and the result follows by dividing by det A. □

Example 0.22. Q: Use Cramer's rule to solve the system

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ -x_1 + 4x_2 &= 1. \end{aligned}$$

A: We compute

$$det\ A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 6,\ det(A_1(\mathbf{b}) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 6,\ det(A_2(\mathbf{b}) = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3.$$

Now using Cramer's rule we find that

$$x_1 = \frac{det(A_1(\mathbf{b}))}{det\ A} = \frac{6}{6} = 1,\ x_2 = \frac{det(A_2(\mathbf{b}))}{det\ A} = \frac{3}{6} = \frac{1}{2}$$

We conclude this section with a formula for the inverse of a matrix in terms of determinants. To start we suppose A is an invertible $n \times n$ matrix, its inverse is the unique matrix X that satisfies $AX = I$. Solve for X one column at a time, let's say $\mathbf{x}_j$ is the j-th column of X,

$$\mathbf{x}_i = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{nj} \end{bmatrix}$$

we consider the problem $A\mathbf{x}_i = \mathbf{e}_i$, and apply Cramer's rule:

$$x_{ij} = \frac{det(A_i(\mathbf{e}_j))}{det\ A}.$$

However,

$$det(A_i(\mathbf{e}_j)) = \begin{vmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} det A_{ji} = C_{ji}$$

this is the (j,i)-cofactor of A.

It follows that $x_{ij} = \frac{1}{det\ A} C_{ji}$ and so $A^{-1} = X = \frac{1}{det\ A}[Cji] = \frac{1}{det\ A}[C_{ij}]^t$. Thus the inverse of A is the *transpose* of the matrix of cofactors of A, divided by the determinant of A. We call this matrix the **adjoint** or **adjugate** of A and is denoted by adj A:

$$adj A = [C_{ij}[^t = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

The result we have proven may be summarized as

Theorem 0.23. *Let A be an invertible $n \times n$ matrix, then*

$$A^{-1} = \frac{1}{det\ A} adj\ A$$

Example 0.24. Q: Use the adjoint method to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

A: Computing the determinant we find det A = -2, and the nine cofactors are

$$C_{11} = \begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} = -18,\ C_{12} = -\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix},\ C_{13} = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = 3,\ C_{22} = \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2,\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = 10,\ C_{23} = -\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = -6,\ C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2.$$

We find that the adjoint of A is then

$$adj\ A = [C_{ij}]^t = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

and hence the inverse is

$$A^{-1} = \frac{1}{det\ A} adj\ A = \begin{bmatrix} -9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & -2 \end{bmatrix}$$

Proof of the Laplace Expansion Theorem. The proof of this theorem is somewhat complicated, to make it simpler to understand we will break it into smaller parts.

Theorem 0.25. *Let A be an $n \times n$ matrix, then*

$$a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = det\ A = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}.$$

Proof. We prove this lemma by induction on n; for $n = 1$ the result is trivial, and so we now assume the result holds for any $(n-1) \times (n-1)$ matrix as our induction hypothesis. By definition of the cofactor, all of the terms containing a_{11} are accounted for by the summand $a_{11}C_{11}$, and so we may ignore terms containing a_{11}

The i-th summand on the right-hand side of equation this equation is $a_{i1}C_{i1} = a_{i1}(-1)^{i+1}detA_{i1}$. Expanding A_{i1} along the first row:

$$\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{(i-1)2} & a_{(i-1)3} & \cdots & a_{(i-1)j} & \cdots & a_{(i-1)n} \\ a_{(i+1)2} & a_{(i+1)3} & \cdots & a_{(i+1)j} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

the j-th term in this expansion of det A_{i1} is $a_{ij}(-1)^{1+j-1}detA_{1i,1j}$ where the notation $A_{1i,1j}$ denotes the submatrix of A obtained by deleting rows k and l and columns r and s. Combining these facts, the term containing $a_{i1}a_{1j}$ on the right hand side of equation is

$$a_{i1}(-1)^{j+1}a_{1j}(-1)^{1+j-1}detA_{1i,1j} = (-1)^{i+j+1}a_{i1}a_{1j}detA_{1i,1j}$$

The term containing $a_{i1}a_{j1}$ on the left-hand side of the equation may be determined by noting that a_{1j} occurs in the j-th summand, $a_{1j}C_{1j} = a_{1j}(-1)^{1+j}det\ A_{1j}$.

By the induction hypothesis we can expand det A_{1j} along its first column,

$$\begin{bmatrix} a_{21} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots a_{2n} \\ a_{31} & \cdots & a_{3(j-1)} & a_{3(j+1)} & \cdots a_{3n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{i(j-1)} & a_{i(j+1)} & \cdots a_{2n} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots a_{nn} \end{vmatrix}.$$

The i-th term in this expansion of $detA_{1j}$ is $a_{i1}(-1)^{(i-1)+1}detA_{1i,1j}$ and so the term containing $a_{i1}a_{1j}$ on the left-hand side of the equation we are proving is

$$a_{1j}(-1)^{1+j}a_{i1}(-1)^{(i-1)+1}detA_{1i,1j} = (-1)^{i+j+1}a_{i1}a_{1j}detA_{1i,1j}$$

proving the equality. □

Next we prove the second property of determinants under row operations

Theorem 0.26. *Let A be an $n \times n$ matrix and let B be obtained by interchanging any two rows (or columns) of A. Then*

$$det\ B = -det\ A.$$

Proof. We use induction on n; the base case for $n = 2$ is easily proven, and so we assume the inductive hypothesis holds: that this is true for any $(n-1) \times (n-1)$ matrix. We now prove the result for $n \times n$ matrices. We will first prove this to be true in the case that two adjacent rows are interchanged, say row r and row r+1 .

We may evaluate det B by cofactor expansion along its first column. The i-th term in this expansion is $(-1)1 + ib_{i1}det\ B_{i1}$. If $i \neq r$ or $r+1$ then $b_{i1} = a_{i1}$ and B_{i1} is an $(n-1) \times (n-1)$ submatrix that is identical to A_{i1} with two adjacent rows interchanged.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{(r+1)1} & a_{(r+1)2} & \cdots & a_{(r+1)n} \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

By the induction hypothesis, det $B_{i1} = -detA_{i1}$ if $i \neq r, r+1$. If $i = r$, then $b_{i1} = a_{(r+1)1}$ and $B_{i1} = A_{(r+1)1}$,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{(r+1)1} & a_{(r+1)2} & \cdots a_{(r+1)n} & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Therefore the r-th summand in det B is

$$\begin{aligned}(-1)^{r+1} b_{r1} det B_{r1} &= (-1)^{r+1} a_{(r+1)1} det A_{(r+1)1} \\ &= -(-1)^{(r+1)+1} a_{(r+1)1} det A_{(r+1)1} \\ &= -(-1)^{(r+1)+1} a_{(r+1)1} det A_{(r+1)1}\end{aligned}$$

Similarly if $i = r+1$ then $b_{i1} = a_{r1}, B_{i1} = A_{r1}$ and the $r+1$-st summand in det B is

$$(-1)^{(r+1)+1} b_{(r+1)1} det B_{(r+1)1} = (-1)^r a_{r1} det A_{r1} = -(-1)^{r+1} a_{r1} det A_{r1}$$

We conclude that the r-th and $r+1$-st terms in the first column cofactor expansion of det B are negatives of the $r+1$-st and r-th terms respectively in the first column cofactor expansion of det A.

Substituting all of these results into det B and using the last result we find

$$\begin{aligned}det B &= \Sigma_{i=1}^{n} (-1)^{i+1} b_{i1} det B_{i1} \\ &= \Sigma_{i=1, i\neq r, r+1}^{n} (-1)^{i+1} b_{i1} det\ B_{i1} + (-1)^{r+1} b_{r1} det B_{r1} + (-1)^{(r+1)+1} b_{(r+1)1} det B_{(r+1)1} \\ &= \Sigma_{i=1}^{n} (-1)^{i+1, i\neq r, r+1} (-1)^{i+1} a_{i1} det A_{i1} - (-1)^{(t+1)+1} a_{(r+1)1} det A_{(r+1)1} - (-1)^{r+1} a_{r1} det A_{r1} \\ &= -\Sigma_{i=1}^{n} (-1)^{i+1} (-1)^{i+1} a_{i1} det A_{1i} = -det\ A.\end{aligned}$$

This proves the result for the interchange of two adjacent rows in an $n \times n$ matrix. To see that this holds for arbtrary row interchanges, we note that the interchange of two rows, say row r and s where $r < s$ we may switch them by performing $2(s-r)-1$ interchanges of adjacent rows. As the number of interchanges is odd and each one changes the sign of the determinant, the net effect is a change of sign as desired. To prove for columns instead of rows, we expand along row 1 instead of column 1. □

We are now able to prove the Laplace Expansion theorem

Proof. Let B be the matrix obtained by moving the i-th row of A to the top, using $i-1$ interchanges of adjacent rows. Thus det B $= (-1)^{i-1}$ det A, but $b_{1j} = a_{ij}$ and $B_{1j} = A_{ij}$ for $j \in [1, n]$ and so

$$det\ B = \begin{vmatrix} a_{i1} & \cdots & a_{ij} & \cdots a_{in} & \\ a_{11} & \cdots & a_{1j} & \cdots a_{1n} & \\ \vdots & & \vdots & & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)j} & \cdots a_{(i-1)n} & \\ a_{(i+1)1} & \cdots a_{(i+1)j} & \cdots & a_{(i+1)n} & \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

Hence

$$\begin{aligned}det\ A &= (-1)^{i-1} det\ B = (-1)^{i-1} \Sigma_{j=1}^{n} (-1)^{1+j} b_{1j} det\ B_{1j} \\ &= (-1)^{i-1} \Sigma_{j=1}^{n} (-1)^{1+j} a_{ij} det\ A_{ij} = \Sigma_{j=1}^{n} (-1)^{1+j} a_{ij} det\ A_{ij}\end{aligned}$$

giving the formula for cofactor expansion along the i-th row. The proof for column expansion is similar. □

References

[1] D. Poole, Linear Algebra: A modern introduction - 3rd Edition, Brooks/Cole (2012).