MATH 2030: MORE ON VECTORS

Lines and Planes and Vectors

In $\mathbb{R}^2$, the equation of a line takes a simple form as a linear equation, for example $y = mx + n$ where m can be seen as slope and n is the value at which the line intersects the y-axis. We wish to consider lines in the plane in terms of vectors, this perspective will allow us to generalize the idea of a line and a plane in $\mathbb{R}^3$.

Lines in the plane and in $\mathbb{R}^3$. In the plane, the most general form for a line is of the form $ax + by = c$. Assuming $b \neq 0$ we recover the simpler form with $m = -\frac{a}{b}$ and $n = \frac{c}{b}$. Returning to the general form, the similarity of this linear equation and the dot product of two vectors is hard to ignore.

Example 0.1. The line ℓ with equation $2x + y = 0$ is shown in figure (1). It is a line with slope -2 passing through the origin. The left hand side resembles the dot product of the vectors $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ with $\mathbf{n} \cdot \mathbf{x} = 0$. The vector $\mathbf{n}$ is perpendicular to the line, i.e. orthogonal to any vector $\mathbf{x}$ parallel to the line, we call this the **normal vector** to the line. And we say the equation $\mathbf{n} \cdot \mathbf{x} = 0$ the *normal form* of the equation of ℓ.

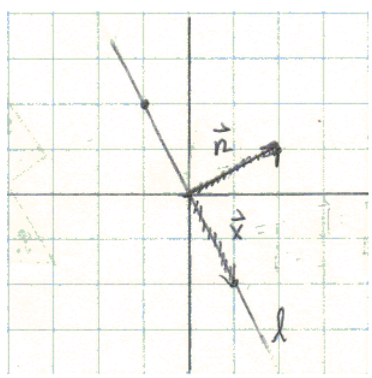

Figure 1. A normal vector and direction vector for the line ℓ

Example 0.2. Alternatively we may imagine a particle traveling along the line ℓ with time as a parameter, as in figure (2). At time $t = 0$, this particle is initially at the origin, and as t varies the particle movies along the line so that the x-coordinate changes 1 unit per unit of t. At $t = 1$ the particle is at $(1, -2)$, at $t = 2$ it will be at $(2, -4)$. Similarly if we allow t to be negative the particle will travel backwards along the line, i.e. at $t = -2$, the particle is at $(-2, 4)$. To summarize these observations we may write this as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We call $\mathbf{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ the **direction vector** for the line ℓ, as it is parallel to the line. The equation of the form $\mathbf{x} = t\mathbf{d}$, called the *vector form* of the line.

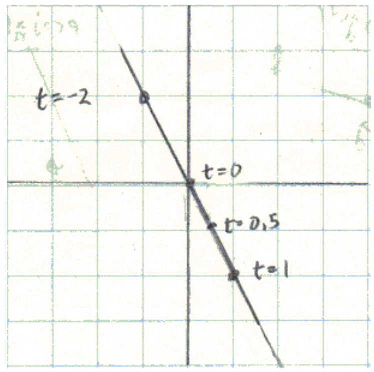

FIGURE 2. The line ℓ seem as a parametrized trajectory of a particle.

If the line does not pass through the origin, the form of these equations must be changed.

Example 0.3. Consider the line ℓ with equation $2x + y = 5$, this is the same line from the previous example except it has been shifted upwards by 5 units. Thus it has slope $m = -2$ and intersects the y-axis at $(0, 5)$. Translating $\mathbf{d}$ and $\mathbf{n}$ it is clear these will still be direction and normal vectors for the new line.

Taking the point $P = (1, 3)$ on ℓ, and noting that $\mathbf{n}$ is orthogonal to any vector parallel to ℓ, if $X = (x, y)$ is any other point on ℓ the vector $\overrightarrow{PX} = \overrightarrow{OX} - \overrightarrow{OP} = \mathbf{x} - \mathbf{p}$ will be parallel to ℓ and hence $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ we find $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$. To verify we calculate

$$\mathbf{n} \cdot \mathbf{x} = 2x + y, \ \ \mathbf{n} \cdot \mathbf{p} = 5.$$

Simplifying the vanishing dot product, we produce the normal form $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, which is a more general form of the equation of the line. What was $\mathbf{p}$ in the previous example?

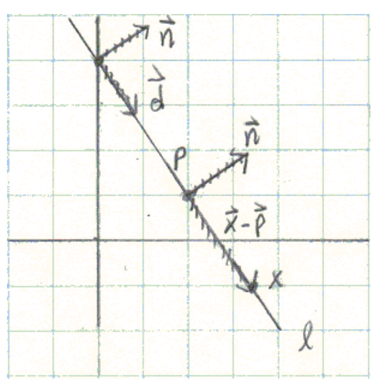

FIGURE 3. The same line translated upwards by five units.

Definition 0.4. The **normal form of the equation of a line** ℓ in $\mathbb{R}^2$ is

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{p})=0,\ \textit{or}\ \mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{n}\neq 0$ is a normal vector for ℓ. The **general form of the equation of** ℓ is $ax+by=c$ where $\mathbf{n}=\begin{bmatrix}a\\b\end{bmatrix}$ is a normal vector for ℓ.

Example 0.5. Let us find the vector form of the previous example. Noting that $\mathbf{x}-\mathbf{p}$ must be a scalar multiple of the direction vector $\mathbf{d}$. We may express this identity as $\mathbf{x}-\mathbf{p}=t\mathbf{d}$ or with components as:

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}+t\begin{bmatrix}1\\-2\end{bmatrix}.$$

In this form we call this the vector form of the equation of ℓ. This may be written as two linear equations, $x=1+t,\ y=3-2t$, which is called the *parametric equations* of the line, where t is the *parameter.*

The idea of a line may be easily generalized to $\mathbb{R}^3$, to do so we generalize the idea of slope of a line in $\mathbb{R}^2$ to a particular direction vector.

Definition 0.6. The vector form of the equation of a line ℓ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\mathbf{x}=\mathbf{p}+t\mathbf{d}$$

where $\mathbf{p}$ is the vector in standard position indicating a point on the line, and $\mathbf{d}\neq 0$ is the direction vector of the line. Taking the components of the vector form one recovers the **parametric equations** of ℓ.

Example 0.7. Q:Find the vector and parametric equations of the line in $\mathbb{R}^3$ through the point $P=(1,2,-1)$, parallel to the vector

$$\mathbf{d}=\begin{bmatrix}5\\-1\\3\end{bmatrix}.$$

A: The vector equation $\mathbf{x}=\mathbf{p}+t\mathbf{d}$ is now

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\-1\end{bmatrix}+t\begin{bmatrix}5\\-1\\3\end{bmatrix}.$$

The parametric form is then $x=1+5t$,$y=2-t$ and $z=-1+3t$.

The point and direction vector chosen are not unique, any point on ℓ other than P and any scalar multiple of $\mathbf{d}$ will produce a set of linear equations whose parameter t' is related to the original equations through an **affine** transformation: $t'=at+b$.

Example 0.8. Q: Given the points $P=(-1,5,0)$ and $Q=(2,1,1)$, find the vector equation of the line determined by them
A: Choosing the vector $\mathbf{p}$ with its tail at the origin and its head at P. For a direction vector we choose $\mathbf{d}=\overrightarrow{\mathrm{QP}}=\begin{bmatrix}3\\-4\\1\end{bmatrix}$. Thus the line will have the vector (and parametric) form

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\5\\0\end{bmatrix}+t\begin{bmatrix}3\\-4\\1\end{bmatrix}.$$

Planes in $\mathbb{R}^3$. Can we use the same trick as in the plane to express the general form of an equation of a line in $\mathbb{R}^3$? Since $ax + by = c$ is the general form of a line in $\mathbb{R}^2$, could the equation of a line in $\mathbb{R}^3$ be $ax + by + cz = d$? In normal form this corresponds to $\mathbf{n}\cdot\mathbf{x} = \mathbf{n}\cdot\mathbf{p}$, wher $\mathbf{n}$ is a normal vector and $\mathbf{p}$ corresponds to a point on the line.

To verify this conjecture, we consider the special case of the equation, $ax + by + cz = 0$. In normal form $\mathbf{n}\cdot\mathbf{x} = 0$, with $\mathbf{n}^t = [a, b, c]$. In $\mathbb{R}^3$, there are infinitely many vectors to pick for $\mathbf{x}$, in fact these vectors determine a family of parallel planes. Thus the equation $ax + by + cz = d$ isn't an equation for a line in $\mathbb{R}^3$, but instead, describes a plane in $\mathbb{R}^3$.

More rigorously, every plane $\wp$ in $\mathbb{R}^3$ can be determined by specifying a point P on $\wp$ and a non-zero vector $\mathbf{n}$ normal to $\wp$. And so, if X represents an arbitrary point on $\wp$, we have $\mathbf{n}\cdot\overrightarrow{\mathrm{PX}} = 0$ or $\mathbf{n}\cdot\overrightarrow{\mathrm{OX}} = \mathbf{n}\cdot\overrightarrow{\mathrm{OP}}$. Denoting $\mathbf{n}^t = [a, b, c]$ and $\mathbf{x}^t = \overrightarrow{\mathrm{OX}}^t = [x, y, z]$, the equation becomes $ax + by + cz = \mathbf{n}\cdot\overrightarrow{\mathrm{OP}}$.

Definition 0.9. The **normal form of the equation of a plane** $\wp$ in $\mathbb{R}^3$ is

$$\mathbf{n}\cdot(\mathbf{x} - \mathbf{p}) = 0, \ \textit{or} \ \mathbf{n}\cdot\mathbf{x} = \mathbf{n}\cdot\mathbf{p}$$

where $\mathbf{p} = \overrightarrow{\mathrm{OP}}$, with P a particular point on $\wp$ and $\mathbf{n} \neq 0$ is a normal vector for $\wp$. The **general form of the equations of** $\wp$ is $ax + by + cz = d$, where $\mathbf{n}^t = [a, b, c]$ is the normal vector to the plane.

Example 0.10. Q: Find the normal and general forms of the equation of the plane that contains the point $P = (6, 0, 1)$ and normal vector $\mathbf{n}^t = [1, 2, 3]$.
A: Taking $\mathbf{p} = \overrightarrow{\mathrm{OP}}$, with components $\mathbf{p}^t = [6, 0, 1]$ and $\mathbf{x}^t = [x, y, z]$, we find $\mathbf{n}\cdot\mathbf{p} = 9$ and so invoking the normal equation from the above definition, the general equation is $x + 2y + 3z = 9$

In a geometric sense, any two planes that share the same normal vector must be parallel, and so their the coefficients of x, y, z and 1 in one equation will be all scaled by the same constant, for example $2x + 4y + 6z = 10$ is the general equation of a plane that is parallel to the plane give in the previous example, since we may divide by 2 on both sides and recover the original equation, $x + 2y + 3z = 5$. These planes will not coincide as the right-hand side of each equation differs.

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}.$$

Alternatively this may be seen by noting $\mathbf{x} - \mathbf{p}$ lies on the plane with its tail at P, thus it may be represented in terms of $\mathbf{u}$ and $\mathbf{v}$.

Definition 0.11. The **vector form of the equation of a plane** $\wp$ in $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

where $\mathbf{p}$ is a vector whose head is at a point on $\wp$ and $\mathbf{u}$ and $\mathbf{v}$ are direction vectors for $\wp$ at that point (i.e. non-zero, parallel to $\wp$ and $\mathbf{u} \neq c\mathbf{v}$).
The equation corresponding to the components of the vector form of this equation are called **parametric equations** of $\wp$.

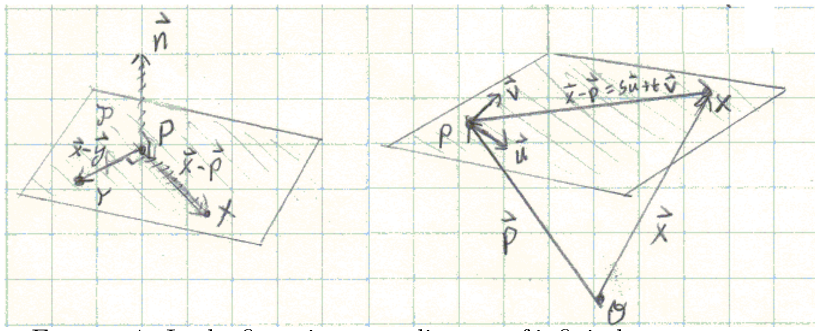

FIGURE 4. In the first picture we list two of infinitely many vectors that are orthogonal to **n**. In the second we show how any vector in standard position whose head lies on a point on the plane $\wp$ may be expressed in terms of P and **u** and **v** on $\wp$.

Example 0.12. Q: Find the vector and parametric equations for the plane $x + 2y + 3z = 9$

A: Knowing that $P = (6, 0, 1)$ lies on the plane, if we find two other points lying on the plane we may be able to build the direction vectors **u** and **v**. With some guesswork, we find $Q = (9, 0, 0)$ and $R = (3, 3, 0)$ both satisfy the general equation, so they lie on the plane. Computing $\mathbf{u} = \overrightarrow{\mathrm{PQ}} = \overrightarrow{\mathrm{OQ}} - \overrightarrow{\mathrm{OP}}$ and $\mathbf{v} = \overrightarrow{\mathrm{PR}}$ we find,

$$\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}.$$

As they are not scalar multiples of each other these will work perfectly find, the vector equation of $\wp$ is then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$$

the corresponding parametric equations are $x = 6+3s-3t$, $y = 3t$ and $z = 1-s-t$.

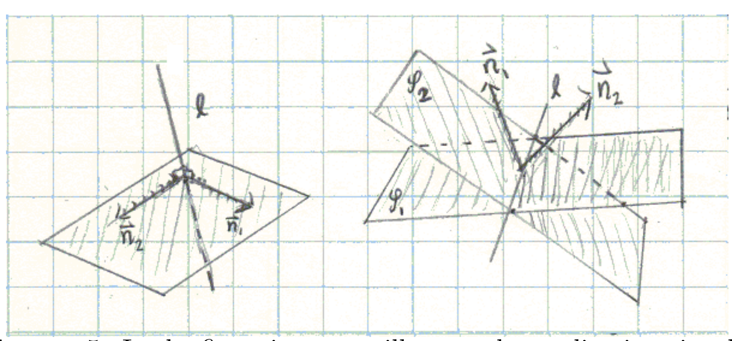

FIGURE 5. In the first picture we illustrate how a line is uniquely determined by two normal vectors. In the second we related the normal vectors to the intersection of planes.

A plane is a two-dimensional object, since its vector or parametric form requires two parameters. To determine a line, which is a one-dimensional object, we pick a point on P and consider the intersection of two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, these uniquely determine a line ℓ as its direction vector must be the normal vector to the plane $\mathbf{p} + s\mathbf{n}_1 + t\mathbf{n}_2$. Therefor any line in $\mathbb{R}^3$ can be specified by a pair of equations

$$a_1x + b_1y + c_1z = d_1, \ a_2x + b_2y + c_2z = d_2$$

corresponding to the two normal vectors. As long as $\mathbf{n}_1 \neq c\mathbf{n}_2$ these planes must be non-parallel, this line is in actuality the intersection of these two planes. Algebraically the line is the set of all points (x, y, z) satisfying both of the general equations.

Normal Form	**General Form**	**Vector Form**	**Parametric Form**
$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$	$ax + by = c$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$x = p_1 + td_1$
			$y = p_2 + td_2$

Table 1. Equations of Lines in $\mathbb{R}^2$

	Normal Form	**General Form**	**Vector Form**	**Parametric Form**
Lines	$\mathbf{n}_1 \cdot \mathbf{x} = \mathbf{n}_1 \cdot \mathbf{p}_1$	$a_1x + b_1y + c_1z = d_1$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$x = p_1 + td_1$
	$\mathbf{n}_2 \cdot \mathbf{x} = \mathbf{n}_2 \cdot \mathbf{p}_2$	$a_2x + b_2y + c_2z = d_2$		$y = p_2 + td_2$
				$z = p_3 + td_3$
Planes	$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$	$ax + by + cz = d$	$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$	$x = p_1 + su_1 + tv_1$
				$y = p_2 + su_2 + tv_2$
				$x = p_2 + su_2 + tv_2$

Table 2. Equations of Lines and Planes in $\mathbb{R}^3$

Notice that a single general equation describe a line in $\mathbb{R}^2$, however in $\mathbb{R}^3$ a lone general equation gives rise to a plane, and that two general equations give rise to a line here. The relationship between the dimension of the object and the number of equations needed is given by

$$(\#\ of\ Parameters) + (\#\ of\ General\ Equations) = \#\ of\ Dimensions$$

With the vector expression for a line determined, we may calculate the distance from a point to a line or a plane using the projection operator.

Example 0.13. Q: Find the distance from the point $B = (1, 0, 2)$ to the line ℓ through the point $A = (3, 1, 1)$ with direction vector $\mathbf{d}^t = [-1, 1, 0]$.
A: We must calculate $\overrightarrow{PB}$, where P is the point on ℓ at the foot of the perpendicular from B. If we label $\mathbf{v} = \overrightarrow{AB}$, then $\overrightarrow{AP} = proj_{\mathbf{d}}(\mathbf{v})$ and $\overrightarrow{PB} = \mathbf{v} - proj_{\mathbf{d}}(\mathbf{v})$ We will break this up into separate steps.

(1) $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 01 \\ 1 \end{bmatrix}$

(2) The projection $\mathbf{v}$ onto $\mathbf{d}$ will be

$$proj_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}}\right) \mathbf{d} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

(3) The vector we want is then

$$\mathbf{v} - proj_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

(4) The distance $d(B, \ell)$ from B to ℓ is

$$||\mathbf{v} - proj_{\mathbf{d}}(\mathbf{v})|| = \frac{1}{2}\sqrt{22}.$$

In $\mathbb{R}^2$, any line ℓ with its equation in general form $ax + by = c$, the distance $d(B, \ell)$ from $B = (x_0, y_0)$ is given by the formula

$$d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

Example 0.14. Q: Find the distance from the point $B = (1, 0, 2)$ to the plane $\wp$ whose general equation is $x + y - z = 1$

A: We must calculate $\overrightarrow{PB}$, where P is the point on $\wp$ at the foot of the perpendicular from B. If A is any point on $\wp$ and we situate the normal vector $\mathbf{n}^t = [1, 1, -1]$ of $\wp$ so that its tail is at A. Calculating the length of the projection of $\overrightarrow{AB}$ onto $\mathbf{n}$, we will break this calculation into steps.

(1) Trying $A = (1, 0, 0)$ we see that this satisfies $x + y - z = 1$, and so it is on the plane.

(2) Set $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

(3) Then the projection of $\mathbf{v}$ onto $\mathbf{n}$ will be

$$proj_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(4) Calculating the distance $d(B, \wp)$ is the norm of this vector,

$$||proj_{\mathbf{n}}(\mathbf{v})|| = \frac{2}{3}\sqrt{3}$$

As in the case of a line in $\mathbb{R}^2$, the distance of a point $B = (x_0, y_0, z_0)$ in $\mathbb{R}^3$ to a plane $ax + by + cz = d$ is given by the formula

$$d(B, \wp) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + d^2}}$$

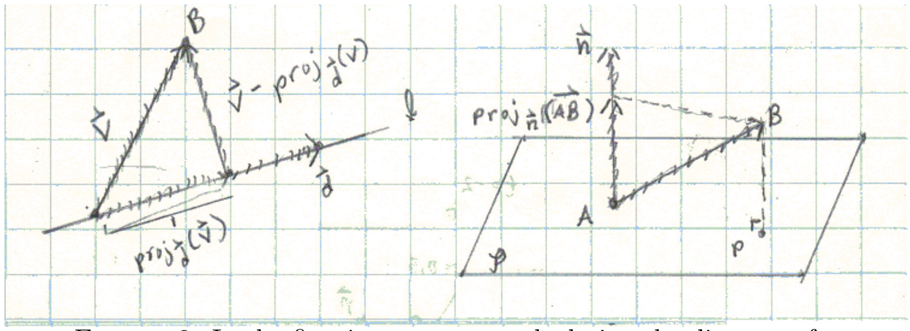

Figure 6. In the first image we are calculating the distance of a point B from the line ℓ. In the second the distance of a point B from the plane $\wp$ is determined.

A Non-geometric Application: Code Vectors

If we want to say something privately, or transmit information as reliably as possible without words, we may want to examine codes. A code can be some one-to-one substitution rule, where each letter is replaced with another - codes like these belong to the domain of *cryptography* and we will not concern ourselves with them.

Instead we are interested in communicating information in an clear and efficient manner, an example of this would be Morse code consisting of dots and dashes meant to communicate information long distances. Alternatively we could talk about our computers, these encode information as 0s and 1s. These codes are essential to the functioning to many of the devices we use everyday. We will explore a simpler application of such codes to universal product codes (UPC) and international standard book numbers (ISBN).

We will work with vectors in $\mathbb{Z}_m^n$ to describe codes that are able to detect errors caused by transmission. As computers represent data in $\mathbb{Z}_m^n$, represented as 1s and 0s we will start by considering binary codes. To do this we take a message we wish to transmit and encode each 'word' of the message into a binary vector.

Definition 0.15. A **binary code** is a set of binary n-vectors, $n \in \mathbb{Z}$ called **code vectors**. The process of converting a message into code vectors is called **encoding** and the reverse process is caleed **decoding**.

This is a good start towards a code, although we will want our codes to do more than just translate our words into computer code. We will want our codes to detect errors that can occur in transmission of a code vector, and - ideally- have the code suggest how to correct the error. For example, suppose we have encoded a message as a set of binary code vectors and we send it across a *channel* (like a a radio transmitter, telephone line, a cd laser, an audio cable, etc,...), if the channel is 'noisy' so that errors are introduced, changing 1s to 0s or vice versa. How can we fix this?

Example 0.16. If we have a simple set of messages to send, say *up,down,left*, or **it right** we may use the four vectors in $\mathbb{Z}_2^2$

Message	up	down	left	right
Code	[0, 0]	[0, 1]	[1, 0]	[1, 1]

This is perfectly fine if the receiver has the same table, we can communicate our commands. However, if a single error occurs, so that only one component has been changed, we cannot detect the change. Consider [0, 1] where transmission error causes this to be received as [1, 1]. Even if the event of an error is known, there is no way to tell if the original message was [0, 1] or [1, 0].

By adding another component we are able to detect an error.

Example 0.17.

Message	up	down	left	right
Code	[0, 0, 0]	[0, 1, 1]	[1, 0, 1]	[1, 1, 0]

By adding the sum of the first two components and recording the result in the third, called the **check digit** we may detect if a single error occurred, these codes are known as a **parity check code** Again if [0, 1, 1] was sent and a single error occurred, the only possibilities are [1, 0, 0], [0, 1, 0] or[1, 1, 1] none of which are code vectors. The receiver would know a mistake had occurred, and could request the original message to be sent again. Notice that one cannot determine *where* the error occurred.

We call such a code an **error-detecting code**, until the 1940s this was the best that could be achieved. Digital computers led to the development of codes that could correct as well as detect errors!

Example 0.18. Consider the message encoded as a binary vector [1, 0, 0, 1, 0, 1] this has an odd number of 1s and so the the check digit will be 1, thus the code vector will be [1, 0, 0, 1, 0, 1, 1] A single error may be detected by this code as the sum of the original binary vector will be even, contradicting the check digit, however if two errors occur the check digit will remain as 1.

To generalize this idea, suppose the message is the binary vector $\mathbf{b} = [b_1, b_2, ..., b_n]$ in $\mathbb{Z}_2^n$ The parity check code vector is $\mathbf{v} = [b_1, ..., b_n, d] \in \mathbb{Z}_2^{n+1}$ where $d \in \mathbb{Z}^2$ is chosen such that

$$\Sigma_{i=0}^{n} b_i + d = 0 mod\ 2$$

. This may be represented as the dot product of the vector with 1 for all components, $\mathbf{1}$ and $\mathbf{v}$, i.e., $\mathbf{v} \cdot \mathbf{1} = 0$. We call $\mathbf{1}$ the **check vector**. If a vector $\mathbf{v'}$ is received such that $\mathbf{v'} \cdot \mathbf{1} = 1$ it is certain that a single error occurred (We are excluding the possibility that more than one error occurring.).

The parity check codes are a special case of a more general class of codes, **check digit codes**, consisting of vectors in $\mathbb{Z}_m^n$

Example 0.19. Let $\mathbf{b} = [b_1, b_2, ..., b_n]$ be a vector in $\mathbb{Z}_3^n$, a check digit code vector may be defined as $[b_1, b_2, ..., b_n, d] \in \mathbb{Z}_3^{n+1}$ so that

$$b_1 + b_2 + ... + b_n + d = 0 mod 3$$

As an example, consider $\mathbf{u} = [2, 2, 0, 1, 2]$, by adding the components we find $2 + 2 + 0 + 1 + 2 = 1 mod 3$, so the check digit must be $d = 2$. Therefore the associated code is $\mathbf{v} = [2, 2, 0, 1, 2, 2]$.

Example 0.20. The Universal Product Code or UPC is associated with the bar codes found on most products. The black and white bars correspond to a 10-ary vector $\mathbf{u} = [u_1, u_2, ..., u_11, d]$ of length 12. The first 11 components are a vector in $\mathbb{Z}_{10}^{11}$ giving the manufacturer and product information. The last component is the check digit chosen so that $\mathbf{c} \cdot u = 0 mod 10$ where the check vector is now $\mathbf{c} = [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$.

Rearranging the sum,

$$3(u_1 + u_3 + u_5 + u_7 + u_9 + u_{11}) + (u_2 + u_4 + u_6 + u_8 + u_{10}) + d = 0 mod 10$$

Supposing we have the vector $\mathbf{u} = [0, 7, 4, 9, 2, 7, 0, 2, 0, 9, 4, 6]$ we find

$$\begin{aligned} \mathbf{c} \cdot \mathbf{u} &= 3(0 + 4 + 2 + 0 + 0 + 4) + (7 + 9 + 7 + 2 + 9) + d mod 10 \\ &= 3(0) + 4 + d mod 10 \\ &= 4 + d mod 10 \end{aligned}$$

Thus the check digit must be $d = 6$ so that the sum is a multiple of 10.

References

[1] D. Poole, Linear Algebra: A modern introduction - 3rd Edition, Brooks/Cole (2012).