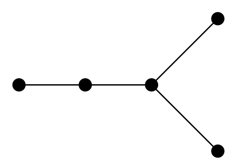
MATH 2113 - Assignment 7 Solutions

Due: Mar 11

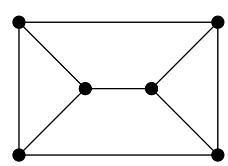
Page 663:

11.1.20 - In a graph with n vertices, the highest degree possible is $n-1$ since there are only $n-1$ edges for any particular vertex to be adjacent to. Therefore, in a graph with 5 vertices, no vertex could have degree 5.

11.1.21 - Here is an example of a graph with degrees 1,1,1,2,3:



11.1.23 - We should begin by noting that if there are 9 edges, the sum of the degrees of all the vertices is 18. Since every vertex has degree 3, we know there must be 6 vertices. Here is an example of such a graph:



11.1.38 - Yes, given integers r and s, we can construct the complete bipartite graph $K_{r,s}$. Each of the vertices in the group of r has degree s and each of the vertices in the group of s has degree r. Since this accounts for all the vertices in the graph, we have satisfied the condition.

11.1.44 - a) As explained in the first problem, a particular vertex can only have an edge to at most $n-1$ vertices of the graph since it cannot be connected to itself.

b) Since all the vertices have different degrees, we would require that they be 0,1,2,3. But then, we find that the vertex of degree 3 has to have an edge to all other vertices and the vertex of degree 0 cannot have any edges. Since this is impossible, there can be no such graph.

c) As in part a) and b), all the vertices have to have degrees less than n. Since there are n vertices, if they all have different degrees, they must be 0,1,2,...,(n-1). But then we have that the vertex of degree (n-1) must have an edge to all other vertices, and the vertex of degree 0 has no edges. This is a contradiction so no such graph can exist.

11.2.7 - a) One of the properties of trees is that there is a unique path from x to y for all pairs of vertices in the tree. This means that if we remove any particular edge (say xy) there will be no path from x to y in the graph. Therefore, the removal of any edge will disconnect the graph. So, any tree on $n+1$ vertices will have n edges and will be disconnected if any edge is removed.

b) Now we will consider the graph C_n which has n vertices and n edges. There is exactly one cycle in the graph and it contains every edge. Therefore, the removal of an edge forces the graph to have no cycles (and therefore a forest). Also, there will be n vertices and $n-1$ edges remaining. This tells us that we must have a tree (and is therefore connected).

11.2.50 - Assume there is a circuit in the graph and the graph is bipartite. If we choose an arbitrary starting point and travel along the circuit, each time we move along an edge we must go to the other partition. If after an odd number of steps (an odd length circuit) we arrive back at the starting point, we find that we are in opposite partition than we had started in. This is a contradiction and so the graph cannot be bipartite.

For the other direction, we assume there are no odd circuits and we would like to prove that the graph is bipartite. We start with any particular vertex v and put it in partition X. We then put all the vertices adjacent to v into partition Y. We repeat this process with all the vertices in Y (their neighbours go into X and so on). Eventually, we will have placed all the vertices in one of the two partitions. The only thing which would prevent this algorithm from working would be if we place two vertices (say a and b) in the same partition which share an edge. In this case, we know we can go backwards through our algorithm to find an even length path from a to b (since they are in the same partition). But since there is an edge between a and b we will have constructed an odd-length cycle which we had assumed does not exist. Therefore, the graph is indeed bipartite.

Prove that any two of the following statements imply the third:

1. G is connected
2. G has no cycles
3. G has n vertices and $n-1$ edges

$(1, 2 \Rightarrow 3)$

We will prove this by mathematical induction. First, if $n = 1$ there are 0 edges, so the statement holds. Now assume that if G is connected and has no cycles with k vertices it has $k-1$ edges. Let's consider a graph which has $k+1$ vertices. We know that if every vertex had degree at least 2, we could find a cycle. Therefore, let us consider a vertex v which has degree 1. If we remove this vertex and the adjacent edge, we have a graph which is connected and has no cycles with k vertices (and therefore $k-1$ edges). Since we removed only one edge, we know our graph with $k+1$ vertices had exactly k edges as desired.

$(1, 3 \Rightarrow 2)$

Assume for a contradiction that there is a cycle of length k in the graph. This accounts for k vertices and k edges. For each vertex we add to the graph, in order to keep the graph connected, we must add at least one edge. By the time we have all n vertices, we will have used at least n edges which gives the contradiction. Therefore, there is no cycle in the graph.

$(2, 3 \Rightarrow 1)$

We have a graph with no cycles, n vertices and $n-1$ edges. We would like to prove that it is connected. That is, for any pair of vertices, there is a path between them. We can again proceed by induction. Clearly, the statement is true for 1 vertex since there are no edges. Now let us assume that the statement holds for all graphs with up to k vertices. Let us now consider a graph with $k+1$ vertices where there are vertices a and b for which there is no a-b path. Then we know that there are at least two components - one of which contains a and another which contains b. Our graph has no cycles and each component must have at most k vertices. Therefore, each component with j vertices will have exactly $j-1$ edges (by the inductive hypothesis). But then if the total number of vertices in the graph is $k+1$ we know that there must be $k+1-$(# of components) edges. Since we assumed that there should be k edges, we conclude that there is only one component in the graph. This contradicts that there is no $a-b$ path. Therefore, the graph is connected.