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What is a free double category like?

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Abstract

We give a geometric description of the free double category generated by a double reflexive graph. Its cells are homotopy classes of colourings of certain rectangular complexes in the plane. A number of examples illustrate the wide variety of combinatorial properties of the plane this touches. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 18D05

1. Introduction

Double categories were introduced in 1963 by Ehresmann [5]. Since then, considerable work has been done, much of it in the context of homotopy theory (see [1,2] and the references there). Of course, a 2-category is a special kind of double category [8] so that all work done on 2-categories (and bicategories) is also saying something about double categories. In particular, the pasting schemes of [6] and the computads of [11] bear a close relationship to our work.

Double categories can be thought of in various ways, each with its own advantages. The concise definition is that they are category objects in **Cat**; however, this phrase slightly obscures the symmetry between the "category object" and "**Cat**". It can also be useful to think of a double category as a single set of "cells" which, under appropriate

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composition operations obeying the middle-four exchange axiom, are the morphisms of two distinct categories. Traditionally, in the same way that the morphisms of a category are represented as one-dimensional "arrows", the cells of a double category are represented visually as rectangles, pasted along matching edges.

Implicit in the use of concrete diagrams is the fact that "equivalent" concrete diagrams (in a sense widely understood but rarely made explicit) represent "equivalent" abstract diagrams. In the case of ordinary categories, for instance, it is understood that arrows may be stretched, shrunk, or rotated, provided that their domains and codomains move with them; and that a chain of arrows is automatically associative. Diagrams for double categories will follow similar conventions.

Planar diagrams are used elsewhere, as well. In computing science, rooted binary trees are used to represent lists; and even in familiar structured languages such as C and Pascal, the control flow can be represented—and is, in a sense, via indentation—as a binary tree in which control may go "onwards" and/or "inwards".

Such structures as flow charts, circuit diagrams, printed circuits, and entity-relation-attribute diagrams are not intrinsically planar, as connections may cross each other; but the importance of minimizing these bridges where possible is widely understood. This serves the purpose, among others, of localizing structure so that it can be studied and understood a piece at a time. Making a diagram approximately planar is a good heuristic for "chunking" it.

Even the TeX mark-up language in which this paper is typeset can be thought of as a sort of planar diagram, in which boxes are joined horizontally and vertically to make up a two-dimensional structure (see [9], especially Chapters 11, 24, and 25). While it would be overstating the case to say that "TeX is a double category"—in particular, the middle-four axiom is irrelevant outside a `tabular` or similar environment—there is a resemblance.

In this paper, we will first consider the question of what a free double category should be. We will then show that the most promising class of free double categories are complicated enough to warrant explicit study of their nature; and we will see that they are, in fact, represented surprisingly well by a simply defined class of concrete diagrams.

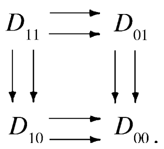
2. Free double categories

In this section we will consider free double categories generated by structures that capture, in various ways, the concept of "double graph". These free double categories always exist, as the categories involved are locally finitely presentable and the forgetful functors obviously preserve limits. The idea is to describe their elements explicitly in geometric terms; this will provide a quick and easy way of deciding equality of composites in such free double categories.

The notion of free double category we get will, of course, depend on the data we start with. This already appears when constructing free categories. The generating data may be merely a graph or a reflexive graph. In the first case, we are given a set of objects and arrows between them (i.e. domains and codomains). The free category generated is the category of paths, with paths of length zero giving identities. This is a remarkably robust construction: it can be carried out inside any category with countable sums and pullbacks which distribute over them.

A reflexive graph has, apart from objects and arrows as above, a specified loop on each object, which is to be an identity. The free category it generates has as morphisms equivalence classes of paths, two paths being equivalent if they differ by the insertion or deletion of "identity arrows". A category would require well-behaved quotients in addition to sums and pullbacks as above, to support this construction.

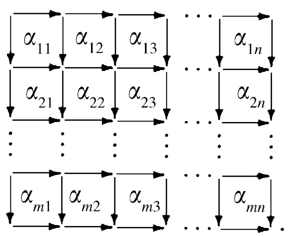
This state of affairs is much more pronounced for double categories. A *double graph* is a graph object in *Graph*, i.e. a diagram of sets



The set D_{00} contains the objects, D_{01} the vertical arrows, D_{10} the horizontal arrows, and D_{11} the cells. The functions between these sets identify the various domains and codomains.

The free double category generated by the above diagram will have as objects the elements of D_{00}. The horizontal and vertical arrow categories will be free on the graphs $D_{10} \rightrightarrows D_{00}$ and $D_{01} \rightrightarrows D_{00}$.

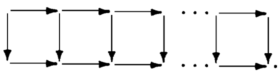
Proposition 1. *A cell in such a free double category is a rectangular grid of elements* $\alpha_{ij} \in D_{11}$ *with compatible domains and codomains*: $\mathrm{cod}_h(\alpha_{ij}) = \mathrm{dom}_h(\alpha_{i\ j+1})$ *and* $\mathrm{cod}_v(\alpha_{ij}) = \mathrm{dom}_v(\alpha_{i+1\ j})$



Proof. This follows from the following result. □

Our discussion of free category objects generated by a graph in a category applies to **Cat** which has well-behaved sums.

Proposition 2. *A cell in the free double category generated by a graph in* **Cat** *is a finite strip of cells whose horizontal domains and codomains match*



Such a double category, however, is not flexible enough to contain many of the diagrams that we might wish to represent. Things become interesting when we consider the free double category generated by a double reflexive graph, i.e. a diagram in **Set** of the form

$$\begin{array}{ccc} D_{11} & \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} & D_{01} \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ D_{10} & \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} & D_{00} \end{array},$$

that is, a reflexive graph object in the category of reflexive graphs. It consists, as the diagram suggests, of a set D_{00} of objects, a set D_{10} of horizontal arrows between objects, a set D_{01} of vertical arrows between objects, and a set D_{11} of 2-cells each bounded by a square

$$\begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & & \downarrow \\ y & \longrightarrow & z \end{array}.$$

Each object x has a distinguished horizontal arrow $i_x : x \to x$ and a distinguished vertical arrow $e_x : x \to x$, not in general the same; these are preserved by all functions between double reflexive graphs. (There may be other edges $f : x \to x$ as well.) Moreover, each arrow h or v has a distinguished cell

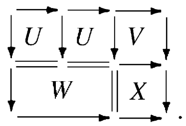
$$\begin{array}{ccc} w & \xrightarrow{h} & x \\ \downarrow{\scriptstyle e_w} & e_h & \downarrow{\scriptstyle e_w} \\ w & \xrightarrow{h} & x \end{array} \quad \text{or} \quad \begin{array}{ccc} w & \xrightarrow{i_w} & w \\ \downarrow{\scriptstyle v} & i_v & \downarrow{\scriptstyle v} \\ x & \xrightarrow{i_x} & x \end{array},$$

which is preserved by all morphisms of double reflexive graphs. For any object, $e_{i_x} = i_{e_x}$. It is important to note, however, that these distinguished arrows can be boundaries of other cells as well. Thus, for instance, a cell may have a distinguished arrow on one edge and "ordinary" arrows on the others.

The free double category on such a double reflexive graph D is the "word" category over D. Firstly, the objects of $F(D)$ are the elements of D_{00}.The horizontal and vertical arrow categories will be free on the graphs $D_{10} \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} D_{00}$ and $D_{01} \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} D_{00}$, respectively.

The cells are defined recursively, elements being the cells of D and formal pastings of them. Two cells are identified only as required by the associative, identity, and middle-four axioms.

Thus, for instance, if U, V, W, and X are appropriate cells of D, an element of $F(D)$ might be:



This represents the words $((U \circ_1 U) \circ_1 V) \circ_2 (W \circ_1 X)$, $(U \circ_1 (U \circ_1 V)) \circ_2 (W \circ_1 X)$, and $((U \circ_1 U) \circ_2 W) \circ_1 (V \circ_2 X)$, all of which are equivalent under the double category axioms.

Our aim is to give a geometric description of the morphisms in such a free double category which allows us to decide whether two words are equal or not at a glance. It will be seen that double categories are intimately related to the combinatorial structure of the plane.

3. Rectangular complexes

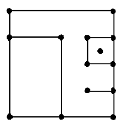
In this section we develop a class of planar geometric objects which will give us a concrete representation of certain free double categories. They will serve as templates for the general free double category developed in the next section.

Definition. A *rectangular complex* K consists of:
- a closed rectangle $R = [a, b] \times [c, d]$ in the plane $\mathbf{R}^2$,
- a finite set P of points in R called *vertices*,
- a finite set H of closed horizontal segments in R called *horizontal edges*,
- a finite set V of closed vertical segments in R called *vertical edges*

satisfying the following conditions:

RC1. The end points of each edge are vertices.
RC2. Every point on the boundary of R lies on some edge.
RC3. Any two edges are disjoint or intersect only at endpoints.
RC4. Any vertex on the horizontal (resp. vertical) boundary of R is the end point of an edge in V (resp. H).

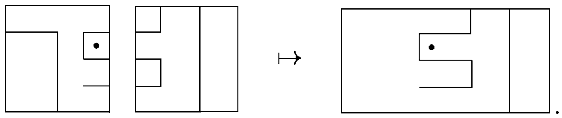
Example 1.



is a rectangular complex with 14 vertices, 7 horizontal edges and 8 vertical ones. In the future, we shall only indicate those vertices whose presence is not clear from RC1–RC4.

Definition. A *horizontal* (resp. *vertical*) *complex* is a horizontal (resp. vertical) segment with a finite set of distinct points on it, called *vertices*. The end points are vertices.

The top and bottom (resp. left and right) boundaries of a rectangular complex K are horizontal (resp. vertical) complexes denoted $\mathrm{dom}_v(K)$ and $\mathrm{cod}_v(K)$ (resp. $\mathrm{dom}_h(K)$ and $\mathrm{cod}_h(K)$). If the right boundary of a rectangular complex K matches the left boundary of another K', we can paste them together horizontally and erase the common boundary to get another complex $K \circ_1 K'$



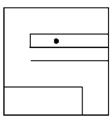
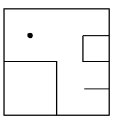
Similarly, we can paste two complexes vertically if the bottom of the first matches the top of the second. We are tacitly assuming that we can translate complexes if we wish, i.e. two complexes that are translates of each other are considered equal.

This defines a double category if we allow degenerate complexes to serve as identities, but it is too rigid for our purposes. We are merely interested in the combinatorial structure given by the horizontal and vertical segments. This leads to the notion of homotopy of complexes.

Two complexes will be homotopic if each can be deformed continuously into the other in a way that respects conditions RC1–RC4. In order to give a precise workable definition of homotopy, note that a complex is determined by its set of vertices and the knowledge of which of them are connected by horizontal or vertical segments. Thus to a complex $K=(R,P,H,V)$ we associate a structure $\mathbf{X}=(X,\prec_H,\prec_V)$ consisting of a finite set X and two binary relations $\prec_H,\prec_V$ on$X$ (not reflexive or transitive as the notation might suggest). We take X to be the set of vertices and define $p \prec_H q$ if p is the left end point of a horizontal edge and q its right end point. A similar definition applies to $\prec_V$. It is possible to give an abstract characterization of these structures (see [4]) but that is not needed here. Two complexes, $K=(R,P,H,V)$ and $K'=(R',P',H',V')$, have the same *type* if the structures $(P,\prec_H,\prec_V)$ and $(P',\prec_{H'},\prec_{V'})$ are isomorphic. All complexes of a given type $\mathbf{X}$ determine a subset $\bar{\mathbf{X}} \subseteq (\mathbf{R}^2)^n$, where n is the number of vertices in $\mathbf{X}$.

Definition. Two rectangular complexes are *homotopic* if they have the same type $\mathbf{X}$ and there is a continuous function $\phi : [0,1] \to \bar{\mathbf{X}}$ such that $\phi(0)$ corresponds to the first and $\phi(1)$ the second.

Example 2.

 and 

have the same type as the one in Example 1 but only the first is homotopic to it.

There is a similar notion of homotopy of horizontal or vertical complexes: the points can be transformed continuously while maintaining their distinctness, and so their respective order is preserved.

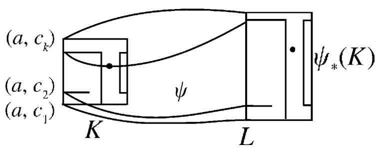
Lemma 1 (Homotopy lifting). *Let K be a rectangular complex, L a vertical complex and ψ: $\mathrm{dom}_h K \to L$ a homotopy. Then there exists a rectangular complex $\psi_* K$ and a homotopy $\phi : K \to \psi_*(K)$ such that*

(1) $\mathrm{dom}_h\, \psi_*(K) = L$, *and*

(2) ϕ *restricted to* $\mathrm{dom}_h K$ *is* ψ.

Homotopy normalizing. *If K' is another rectangular complex homotopic to K and ψ: $\mathrm{dom}_h K \to \mathrm{dom}_h K'$ is a homotopy, then there exists a homotopy $\phi : K \to K'$ such that ϕ restricted to* $\mathrm{dom}_h K$ *is* ψ.

Proof. The idea for the first part is to move the complex K along the homotopy ψ maintaining the distances in the x-direction while scaling in a piecewise linear fashion in the y-direction.



Let the points of $\mathrm{dom}_h(C)$ be (a,c_1), $(a,c_2), \ldots, (a,c_k)$ with $c = c_1 < c_2 < \cdots < c_k = d$. $\psi : [0,1] \to (\mathbf{R}^2)^k$ is given by $\psi(t) = \langle(\psi_0(t), \psi_i(t))\rangle_{i=1,\ldots,k}$. If (x, y) is the mth vertex of K, define

$$\phi_m(t) = \left(x - a + \psi_0(t), \psi_i(t) + \frac{y - c_i}{c_{i+1} - c_i}(\psi_{i+1}(t) - \psi_i(t)) \right),$$

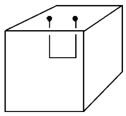
where i is the unique index such that $c_i \leqslant y < c_{i+1}$.

It is readily seen that ϕ_m is continuous. For a fixed t, the first coordinate depends only on x and the second only on y, and both are increasing. Thus any of the horizontality or verticality or distinctness conditions are preserved. So $\{\phi_m(t) | m = 1, \ldots, n\}$ is the set of points of a rectangular complex.

We have $\psi_0(0) = a$ and $\psi_i(0) = c_i$ so $\phi_m(0) = (x, y)$. Thus $\langle \phi_m(0) \rangle$ represents K. On $\mathrm{dom}_h(K)$, when $(x, y) = (a, c_i)$, we get $\phi_i(t) = (\psi_0(t), \psi_i(t))$ so $\phi = \langle \phi_m \rangle$ restricts to ψ.

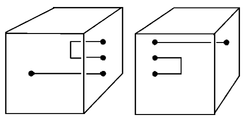
For the second part, let $\theta : K \to K'$ be a homotopy. It gives a path of complexes, each homotopic to K, and ψ gives a path of vertical complexes each homotopic to $\mathrm{dom}_h K$, so we use the first part to transfer $\theta(t)$ to a complex whose horizontal domain is $\psi(t)$, and this in a continuous way. □

We gave this proof in some detail because, although it looks obvious, it is essential for our next theorem, and its generalization to higher dimensions is false. For example, the top face of



consists of a rectangle with two points inside it. The points can be moved around continuously to any other two points but if one is moved in front of the other, the homotopy will not lift.

In fact, even when homotopies do lift things do not work. The problem is that while homotopy is a rather restrictive condition in one dimension, it need not be so in two dimensions. Consider for instance, the following example:



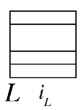
There are several homotopies pairing elements on the boundaries of the left and right complexes. However, some yield complexes with one connected component, while others yield complexes with two. It will be seen that this makes the extension of the main results of this section into higher dimensions rather problematic; once more, triple and higher categories appear to exhibit interesting behaviour above and beyond that suggested by double categories.

Theorem 1. *Homotopy classes of rectangular complexes form a double category*, **RC**.

Proof. There is one object (the homotopy class of all points in R^2), the horizontal (resp. vertical) arrows are homotopy classes of horizontal (resp. vertical) complexes. The horizontal domain of a class of complexes is the homotopy class of the domain of any one of its members: $\mathrm{dom}_h[K]=[\mathrm{dom}_h K]$. Similarly, for the codomain. If K_1 and K_2 are complexes such that $\mathrm{cod}_h[K_1]=\mathrm{dom}_h[K_2]$, then there is a homotopy of vertical complexes $\psi:\ \mathrm{dom}_h K_2 \to \mathrm{cod}_h K_1$. By the homotopy lifting lemma, $\psi_* K_2$ has horizontal domain equal to $\mathrm{cod}_h K_1$. Thus, we define horizontal composition by

$$[K_1]\circ_1[K_2]=[K_1\circ_1\psi_* K_2].$$

This composition is well defined, for suppose $\theta_i : K_i \to K_i'$ are homotopies, $i=1,2$, and $\psi' :\ \mathrm{dom}_h K_2' \to \mathrm{cod}_h K_1'$ is a homotopy of vertical complexes, then $\psi'_* K_2'$ is homotopic to $\psi_* K_2'$. Use homotopy normalizing to choose a homotopy $\phi : \psi_* K_2 \to \psi'_* K_2'$ which on the horizontal domain restricts to θ_1 restricted to the codomain. Then θ_1 and ϕ can be pasted together to give a homotopy of $K_1 \circ_1 \psi_* K_2 \to K_1' \circ_1 \psi'_* K_2'$. Identities are cylinders on one-dimensional complexes

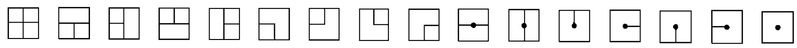


Of course, all that has been said for the horizontal applies equally well to the vertical, giving the definition of $\circ_2$.

As usual, once we have established well definedness, the associative, middle four and identity laws follow immediately. □

Theorem 2. *The double category* **RC** *is freely generated by a reflexive double graph with one object, one non-identity horizontal arrow, one non-identity vertical arrow and* $16 = 2^4$ *non-identity double cells* α_i *whose boundaries correspond to all possible choices of identities or non-identities.*

Proof. The cells α_i are represented, respectively, by the following complexes:

.

The vertex in the centre of the square indicates a non-identity cell, and a vertex in the middle of a side a non-identity arrow. Apart from these 16 non-identity cells, there are 3 identity cells denoted , , and  which we shall call α_{17}, α_{18} and α_{19}, respectively.

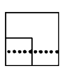
Let **D** be a double category in which we are given an object A, a horizontal arrow $h : A \to A$, and a vertical arrow $v : A \to A$ and 16 cells β_i with domains and codomains h, v or identities like those for α_i. Let $\beta_{17} = i_v$, $\beta_{18} = e_h$ and $\beta_{19} = i_{e_A} = e_{i_A}$. We wish to find a double functor $F : \mathbf{R}C \to \mathbf{D}$ which takes the complexes corresponding to the α_i to β_i. There is no difficulty defining F on objects (there is just one and it goes to A), nor on horizontal or vertical complexes. If a horizontal complex has n interior vertices, it gets sent to h^n. Similarly for vertical complexes.

Let K be a rectangular complex. Subdivide the rectangle $R = [a, b] \times [c, d]$ into smaller subrectangles by subdividing $[a, b]$ into subintervals each containing, as an interior point, at most one x coordinate of any interior vertex of K. Do the same for $[c, d]$. Then each subrectangle has at most one vertex inside it so is a cell of the form of one of the 19 above. Call such a subdivision *admissible*. $[K]$ is the composite in **RC** of these cells $[K_{ij}]$, so if F does exist it is uniquely determined by its values on the 19 cells:

$$F[K] = (F[K_{11}] \circ_1 \cdots \circ_1 F[K_{1n}]) \circ_2 \cdots \circ_2 (F[K_{m1}] \circ_1 \cdots \circ_1 F[K_{mn}]). \qquad (*)$$

Lemma 2. *$F[K]$ as defined by the formula $(*)$ is independent of which admissible subdivision is used.*

Proof. Suppose we have an admissible subdivision of $[a, b] \times [c, d]$ and suppose we add a new horizontal (or vertical) line not passing through a vertex. Then some of the K_{ij} remain the same, and some are divided into two, e.g.

.

One of the two has no interior vertex so is an identity, a vertical one if we added a horizontal line. Then F takes this to an identity and general associativity [3] says that $F[K]$ will have the same value in both cases. An arbitrary refinement can be obtained by adding horizontal or vertical lines, one at a time. Finally any two adequate partitions admit a common refinement. □

Lemma 3. *$F[K]$ as defined by the formula $(*)$ is well defined on homotopy classes of complexes.*

Proof. Let $\phi : [0,1] \to (\mathbf{R}^2)^n$ be a homotopy of rectangular complexes. For a given $t \in [0,1]$, $F(\phi(t))$ is calculated by choosing an adequate partition for $\phi(t)$. Each vertex lies *inside* one rectangle. If we choose $\varepsilon > 0$ small enough, then for each $u \in (t-\varepsilon, t+\varepsilon)$, the corresponding vertex of $\phi(u)$ will lie in the same subrectangle. Thus the formula for $F(\phi(u))$ is the same as that for $F(\phi(t))$. By connectedness of $[0,1]$, $F(\phi(t))$ is constant over all t. □

To finish the proof of our theorem we must show that F preserves horizontal and vertical composition and identities. This follows immediately from the well definedness of F and general associativity. □

4. Coloured complexes

Let $\mathbf{C}$ be

$$\begin{array}{ccc} C_{11} & \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} & C_{01} \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ C_{10} & \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} & C_{00}\,, \end{array}$$

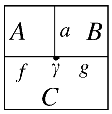
a double reflexive graph. We shall describe the free double category it generates by building on what was done in the previous section.

Let $K = (R, P, H, V)$ be a rectangular complex. A $\mathbf{C}$-*colouring* of K is a function $\chi : R \to C_{11}$ assigning to each point $p \in R$ a "colour", $\chi(p)$, subject to the following restrictions. Every p in the interior of R has a neighbourhood which looks like one of the 19 basic cells in Theorem 2. The colour assigned to that point must be of the same nature as the cell with regard to domains and codomains being identities or not, or the cell itself being an identity. Furthermore, all points of the neighbourhood must be coloured according to the particular domain–codomain structure. Thus if p is not a vertex and does not lie on an edge, it will have a neighbourhood like α_{19}. Then $\chi(p)$ must be a double identity (that is an object) and the whole neighbourhood must have this colour. It follows that the colour is constant on the components of the rectangle minus the edges and vertices. If p lies on a horizontal edge but is not a vertex, then it will have a neighbourhood like α_{17} which is a horizontal identity. Then p and the whole horizontal line through it must be given the colour of a horizontal identity, i.e.

a vertical arrow. The top region is coloured the domain of the arrow, and the bottom region the codomain. If p has a neighbourhood like α_2 say, it can be coloured by a cell of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle f}\downarrow & \gamma & \downarrow{\scriptstyle g} \\ C & = & C \end{array}$$

The rest of the neighbourhood has to be coloured thus



The boundary points of the rectangle take the colour of their neighbours.

In fact, the colouring of a complex is completely determined by the colour of its vertices, but the compatibility conditions are a bit harder to express.

A homotopy of coloured complexes is a homotopy of complexes which preserves the colouring. Minor modifications to the proofs of Theorems 1 and 2 give the following.

Theorem 3. *Given a double reflexive graph* $\mathbf{C}$, *homotopy classes of* $\mathbf{C}$*-coloured rectangular complexes give a double category* **CRC** *which is free on* $\mathbf{C}$.

5. How complex are free double categories?

One heuristic measure of the combinatorial complexity of a category is the difficulty of determining a hom-set. Determining whether $\hom(A,B)$ is empty or not in a finitely generated free category is comparatively straightforward, and amounts to determining whether there is a path from A to B in the underlying graph.

Proposition 1 tells us that, in the free double category over a double graph, morphisms are $m \times n$ arrays of matching generators. A hom-set $\hom(f,g;u,v)$ is therefore always empty if the length of f is not equal to the length of g or the length of u not equal to that of v. If the edge lengths do match, at most $\#(C_{11})^{mn}$ attempts will exhaust all ways to fill in the grid. This upper bound is probably pessimistic; however, it does seem likely that the complexity of this problem is inherently exponential.

The free category over a double reflexive graph is much more interesting; the problem of deciding whether $\hom(f,g;u,v)$ is empty or not can model the word problem for groups, and is hence undecidable!

To see this, let us first consider a finitely generated category, $\mathbf{A}$, generated by a finite graph $\mathbf{G}$ and a finite set of relations $\mathbf{R}$. A relation is a pair of paths

$$r = (A = A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} A_n = B, A = B_0 \xrightarrow{b_1} B_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} B_m = B)$$

with common starting point and common endpoint. We may assume that $\mathbf{R}$ is symmetric, that is, if $r = (\langle a_i \rangle, \langle b_j \rangle) \in \mathbf{R}$ then so is $r^\circ = (\langle b_j \rangle, \langle a_i \rangle)$.

Out of **G** and **R** we construct a double reflexive graph **D**. Its objects are those of **G** and its horizontal arrows are the arrows of **G** together with identities. For each relation r as above we have vertical arrows $x_{ri}: A_i \to A$, $i = 0, 1, \ldots, n$ and $y_{rj}: B \to B_j$, $j = 0, \ldots, m$, with $x_{r0} = e_A$, $x_{rn} = y_{r0}$, and $y_{rm} = e_B$. Also for each r we have cells:

$$\begin{array}{ccc} A_{i-1} & \xrightarrow{a_i} & A_i \\ {\scriptstyle x_{r,i-1}}\downarrow & \alpha_{ri} & \downarrow{\scriptstyle x_{r,i}} \\ A & = & A \end{array} \quad \text{and} \quad \begin{array}{ccc} B & = & B \\ {\scriptstyle y_{r,j-1}}\downarrow & \beta_{rj} & \downarrow{\scriptstyle y_{r,j}} \\ B_{j-1} & \xrightarrow[b_j]{} & B_j \end{array}$$

The composite $\gamma_r = \alpha_{r1} \circ_1 \cdots \circ_1 \alpha_{rn} \circ_1 \beta_{r1} \cdots \circ_1 \beta_{rm}$ is a cell with boundary

$$\begin{array}{ccccccc} A & \xrightarrow{a_1} & A_1 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_n} & B \\ \| & & & \gamma_r & & & \| \\ A & \xrightarrow[b_1]{} & B_1 & \xrightarrow[b_2]{} & \cdots & \xrightarrow[b_m]{} & B \end{array}$$

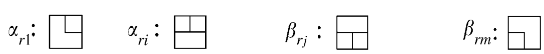
Proposition 3. *Two paths $\langle c_k \rangle$ and $\langle d_l \rangle$ represent equal morphisms in* **A** *if and only if there is a cell δ with boundary*

$$\begin{array}{ccccccc} C & \xrightarrow{c_1} & C_1 & \xrightarrow{c_2} & \cdots & \xrightarrow{c_p} & D \\ \| & & & \delta & & & \| \\ C & \xrightarrow[d_1]{} & D_1 & \xrightarrow[d_2]{} & \cdots & \xrightarrow[d_q]{} & D \end{array}$$

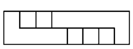
in the free double category generated by **D**.

Proof. The existence of such a cell defines a relation on the set of paths in **G** which is clearly reflexive and transitive. It is also symmetric as **R** was taken to be symmetric. It is a congruence of categories on the free category generated by **G** because of horizontal composition. Finally any pair in **R** is so related, as the γ_r show. We conclude that if $\langle c_k \rangle$ and $\langle d_l \rangle$ represent equal morphisms then there is a cell δ.

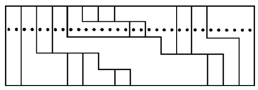
Conversely, assume there is a cell δ. It will be made up of α's and β's which have the shape



Furthermore, because of the domains and codomains of these, an α can only be composed with the next α for the same r (or the last α with the first β) and similarly for the β's. Thus the cell δ will be made up of pieces like γ_r



It might look like (without the dotted line)

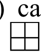
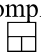


Choose a horizontal segment as high up in the rectangle as possible but not the top. It will be part of some γ_r. Then using a homotopy to lower any other which might have been at the same height but corresponding to a different relation, we can draw a horizontal (dotted) line across the complex, just below these horizontal segments corresponding to γ_r but above any others. Now we see that our rectangle is a vertical composite of the part above the dotted line and the part below. The top part is a γ_r with identities on either side. Thus the morphism of **A** represented by the top of the rectangle is equal to the one represented by the dotted line. It follows by induction that it is also equal to the morphism represented by the bottom line. □

If **G** has a single object our finitely presented category is a monoid, and asking when two words represent the same element is the word problem for monoids of which the word problem for groups is a special case. Thus determining whether there is a cell with a given boundary in a free double category is undecidable. The free double category on a double reflexive graph is very complex indeed.

Remark. It is interesting how relations on categories can be converted to cells in a double category. From this a picture of a proof that words are equal emerges. Then geometric arguments can be used to analyze the proof as we did in the previous proposition. Similar methods are used in combinatorial group theory to great advantage (see [10]). Their notion of "map" is clearly related to our complexes, but our rectangular setting is more rigid and thus easier to use.

6. Examples

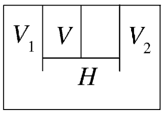
Example. Tileorders (dissections of rectangles into rectangles) can be considered as rectangular complexes with vertex neighbourhoods of the form  where four rectangles meet, or  (which can occur in four orientations) where three rectangles meet. Therefore, as observed in [4], tileorders form a double category freely generated over a reflexive double graph with one object, one horizontal and one vertical non-identity edge, and five non-identity cells.

Example. If one of the four generators corresponding to rotations of  is omitted from the set of five above, the tileorders in the resulting free double category are *binarily composable*—they can be reduced to a single tile by repeatedly merging a pair of tiles with a common edge into a single tile.

Indeed, suppose that the double category included tileorders that were not binarily composable; it would have to include one with a minimal number of tiles. We shall show that this cannot happen.

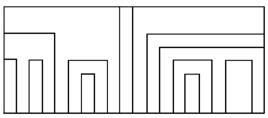
Without loss of generality, assume the omitted tile is the one shown. There must be at least one internal vertical edge V; otherwise the tiling would be a singleton or composable by vertical composition alone. Every vertical edge must extend upwards to the upper edge of the tiling. In a minimal non-binarily composable tiling, such an edge cannot extend to the lower edge of the tiling; for then it would divide the tiling into two subtilings, by assumption each composable.

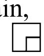
Thus, the vertical edge must end in a node of the form  at an internal horizontal edge H. That edge must end at vertical edges V_1 and V_2, each of which may be either internal or on the boundary of the tiling. In either case, however, V_1 and V_2 must extend to the upper edge of the tiling.



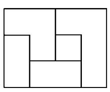
V_1, V_2, and H therefore bound a proper subrectangle of the tiling, which is by hypothesis binarily composable (and, as it is divided by V, contains more than one tile). But composing this yields a tiling with fewer tiles than the original, which is also by hypothesis composable—a contradiction.

Example. The free double category with one object, one horizontal and one vertical non-identity edge, and the 2-cells  and  has elements such as

.

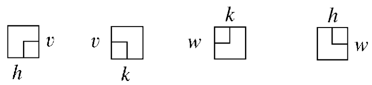
An element can have vertical or horizontal lines (but not both), right angles, and "hoops". In particular, the cells with vertical domain, and horizontal domain and codomain equal to the identity, have every generator  paired with a generator  so that they correspond to nests of "hoops". The number of elements in the hom-set $\hom(i_v, i_v; i_h, h^{2n})$ is therefore given by the nth Catalan number.

Example. A *polyomino* is a connected union of unit squares in the plane joined along common edges. Polyomino tilings of rectangles form a double category, in which tilings are composed by merging along matching edges (so that an $m_1 \times n$ rectangle composed horizontally with a $m_2 \times n$ rectangle would yield an $(m_1 + m_2 - 1) \times n$ rectangle). A typical element of this double category would look like

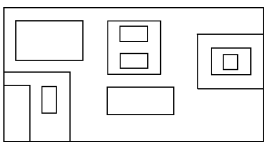
.

Polyomino tilings of rectangles (permitting non-simply connected polyominoes) can be considered as rectangular complexes with vertex neighbourhoods of the form , , , , and , occurring in all possible orientations. However, polyominoes are *not* homotopy classes because the total number of squares counts; so the "missing" edges should be considered not as identities, but as differently colored ("white"?) arrows. Thus the polyomino tilings of rectangles form a double category, freely generated over a double graph with one object, two horizontal and two vertical non-identity edges, and twelve non-identity cells.

Example. A reflexive double graph with one object, two vertical generators v and w, two horizontal generators h and k, and four generating cells



freely generates a double category whose cells correspond to rectangles containing nested or side-by-side rectangles, or 90° or 180° loops going to and from the boundary. The "labelling" of the arrows ensures that the corner generators can only be assembled in one order, so that reflex corners are impossible.



7. Conclusion

Free double categories are of interest both in their own right and because their cells are the "words" used to work with other double categories. We have shown that free double categories are equivalent to double categories of homotopy classes of complexes. The complexes that model cells in free double categories resemble the node-and-string diagrams of Joyal–Street [7] (also see [11]). A deeper study of the relationship should prove fruitful.

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