INJECTIVES IN TOPOI, I:

REPRESENTING COALGEBRAS AS ALGEBRAS

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Call a cotriple $\mathbb{G}$ on a topos $\mathbb{E}$ with subobject classifier Ω dually algebraic if the composite $(\mathbb{E}_{\mathbb{G}})^{op} \xrightarrow[(V_{\mathbb{G}})^{op}]{} \mathbb{E}^{op} \xrightarrow[\Omega^{(-)}]{} \mathbb{E}$ is tripleable, $V_{\mathbb{G}}$ being the $\mathbb{E}$-valued "underlying" functor on the $\mathbb{G}$-coalgebras (cf. [ZTB]). During what has come to be known as the Lawvere-Tierney topos year 1969-70 at Dalhousie, J. R. Isbell raised the question whether all cotriples on the topos of sets are dually algebraic (see [IGF], p. 588, ℓ. 4*). By April, 1970, we had a composite tripleableness lemma ([LDC], Th. 2; see [MAT], Exer. 3.1.17, for a more polished version) informing us that they are ([LDC], Cor. 5).

In this lemma, as against the earlier ones of Barr and Beck, the crucial ingredient is a requirement (ZHD) on the middle category, roughly, that practically all objects be projective. In 1973, consequently, when it was clear that $\Omega^{(-)}: \mathbb{E}^{op} \longrightarrow \mathbb{E}$ is always tripleable ([PCT], p. 558), the same lemma automatically revealed, for any topos $\mathbb{E}$, that all cotriples on $\mathbb{E}$ must be dually algebraic -- provided (coZHD) each nonzero object of $\mathbb{E}$ is injective.

Here we give the latest full proof of that lemma. Moreover, for those content merely to know all indexed cotriples (à la [RAF]) on $\mathbb{E}$ are dually algebraic, we weaken the seemingly overrestrictive coZHD proviso: it suffices that each object X of $\mathbb{E}$ be internally injective over its own support σX, i.e., as an object of $\mathbb{E}|_{\sigma X}$. It suffices -- but it is also necessary; and, as a corollary, we establish that the coZHD proviso, sufficient for all cotriples on $\mathbb{E}$ to be dually algebraic, is likewise necessary as well. Thus:

Theorem A. *Necessary and sufficient for all cotriples on the topos $\mathbb{E}$ to be dually algebraic is that each nonzero object $X \neq 0$ of $\mathbb{E}$ be injective.*

Theorem B. *Necessary and sufficient for all indexed cotriples on $\mathbb{E}$ to be dually algebraic is that each object X of $\mathbb{E}$ be internally injective in the open subtopos $\mathbb{E}|_{\sigma X}$, where σX is the support of X.*

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1. The ZHD-Lemma and Theorem A (sufficiency proof). The ZHD-Lemma below, of which the sufficiency assertion in Theorem A is a direct consequence, makes use of the following definitions.

(1.1) Definitions. (i) An object Q of a category $\mathcal{B}$ is an artificial terminal object ("Q is AT") if Q is terminal in $\mathcal{B}$ and every $\mathcal{B}$-morphism with domain Q is an isomorphism ("AT" is "isolated" in [MAT], Exer. 3.1.10).

(ii) The category $\mathcal{B}$ is ZHD (think "zero homological dimension") if all objects, save perhaps those that are AT, act projective when tested against coequalizers; if $\mathcal{B}^{op}$ is ZHD, we say $\mathcal{B}$ is coZHD.

[It is clear that a topos is ZHD if and only if it satisfies the axiom of choice (AC). H.-M. Meyer has observed (see Satz 6.4 in [MDT]) that a topos is well-pointed if and only if it is coZHD and Boolean. Sets, pointed sets, modules over a semisimple ring, and their full subcategories, all constitute examples of categories that are both ZHD and coZHD.]

(1.2) ZHD-Lemma. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories, with $\mathcal{C}$ tripleable over $\mathcal{B}$ via $V: \mathcal{C} \longrightarrow \mathcal{B}$, and $\mathcal{B}$ tripleable over $\mathcal{A}$ via $U: \mathcal{B} \longrightarrow \mathcal{A}$. Assume $\mathcal{B}$ is ZHD. Then $\mathcal{C}$ is tripleable over $\mathcal{A}$ via the composite

$$U \circ V: \mathcal{C} \xrightarrow{V} \mathcal{B} \xrightarrow{U} \mathcal{A}.$$

Proof. $U \circ V$ reflects isomorphisms because U and V do. By the absolute version ([PAC], Th. 7.3) of Beck's tripleableness theorem, we need only show that $\mathcal{C}$ has and $U \circ V$ preserves coequalizers of $(U \circ V)$-absolute pairs. If

(1.2.1) $$D \underset{y}{\overset{x}{\rightrightarrows}} W \qquad (\text{in } \mathcal{C})$$

(1.2.2) $$UVD \rightrightarrows UVW \xrightarrow{p} P \qquad (\text{in } \mathcal{A})$$

depict such a pair and accompanying absolute coequalizer data, find a map q in $\mathcal{B}$, with $Uq \cong p$, coequalizing the U-absolute pair (Vx, Vy):

(1.2.3) $$VD \rightrightarrows VW \xrightarrow{q} Q \qquad (\text{in } \mathcal{B}).$$

Where T is (the functor component of) the triple associated with the tripleable functor V, the lemma below will assure that both T and $T \circ T$ preserve the coequalizer diagram (1.2.3). It follows (see [LCA], Prop. 3) that $\mathcal{C}$ will have a map r coequalizing (1.2.1) and satisfying $Vr \cong q$; the inference $UVr \cong p$ then being immediate, the proof will be complete.

(1.3) Lemma. Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on the ZHD category $\mathcal{B}$, and let $U: \mathcal{B} \longrightarrow \mathcal{A}$ be a functor reflecting coequalizers of U-absolute pairs and having a left adjoint $F: \mathcal{A} \longrightarrow \mathcal{B}$, with counit $\epsilon: FU \longrightarrow \mathrm{id}_{\mathcal{B}}$.

Then not only is every diagram

(1.3.1) $$E \rightrightarrows X \longrightarrow Q \quad (\underline{\text{in}}\ \mathcal{B})$$

whose transform under U *is an absolute coequalizer diagram in* $\mathcal{A}$ *already a coequalizer diagram in* $\mathcal{B}$, *but so are its transforms under* T *and* $T \circ T$.

Proof. We distinguish two cases, according as Q is or is not AT. If Q is AT, merely apply the following observation to T, $T \circ T$, and (1.3.1): *If* $D\colon \mathcal{D} \longrightarrow \mathcal{B}$ *is any diagram with AT colimit* Q, *then any functor* $S\colon \mathcal{B} \longrightarrow \mathcal{B}$ *admitting a natural transformation* $\lambda\colon \mathrm{id}_{\mathcal{B}} \longrightarrow S$ *satisfies* $SQ \cong Q$ *and preserves the colimit of* D. Indeed, λ_Q is an isomorphism; moreover, for any cone $S \circ D \circ\!\!\circ\!\!\circ\!\!> B$, the cone $D \circ\!\!\circ\!\!\circ\!\!> B$ induced by composition with λ factors uniquely through Q, whence $B \cong Q$ and $SQ \cong \mathrm{colim}(S \circ D)$.

If instead Q is not AT, neither is X or E, and all three are projective; it will then turn out that (1.3.1) is an *absolute* coequalizer diagram, which amply fulfills our requirement. Writing $G = FU$, consider the beginnings of a G-resolution of (1.3.1):

(1.3.2) $$\begin{array}{ccccc}
GGE & \rightrightarrows & GGX & \longrightarrow & GGQ \\
{\scriptstyle G\epsilon_E}\downarrow\downarrow{\scriptstyle \epsilon_{GE}} & & {\scriptstyle G\epsilon_X}\downarrow\downarrow{\scriptstyle \epsilon_{GX}} & & {\scriptstyle G\epsilon_Q}\downarrow\downarrow{\scriptstyle \epsilon_{GQ}} \\
GE & \rightrightarrows & GX & \longrightarrow & GQ \\
{\scriptstyle \epsilon_E}\downarrow & & {\scriptstyle \epsilon_X}\downarrow & & {\scriptstyle \epsilon_Q}\downarrow \\
E & \rightrightarrows & X & \longrightarrow & Q
\end{array}\quad .$$

Using the unit of adjunction $\mathrm{id}_{\mathcal{A}} \longrightarrow UF$, note that U transforms each column of (1.3.2) into a split coequalizer diagram in $\mathcal{A}$ (compare the discussion around display formula (5) in [McL], Ch. VI, §7). As each column is then a coequalizer diagram with projective coequalizer, it follows that

(1.3.3) the maps ϵ_E, ϵ_X, and ϵ_Q are split epimorphisms.

Moreover, as the transform of (1.3.1) under U is absolute, we know that

(1.3.4) G transforms (1.3.1) into an absolute coequalizer diagram.

The following observation, used again in §2, now concludes the proof:

(1.4) *ABS-Lemma*. *Let* G *be any endofunctor on a category* $\mathcal{B}$, *and* $\epsilon\colon G \longrightarrow \mathrm{id}_{\mathcal{B}}$ *any natural transformation*. *Then* (1.3.1) *is an absolute coequalizer diagram if conditions* (1.3.3) *and* (1.3.4) *hold*.

Proof. Referring to (1.3.2), the upper two rows are obviously absolute coequalizer diagrams by (1.3.4). But the columns are absolute coequalizer diagrams, too -- indeed, they are split: writing B for any one of E, X, or Q, and choosing a section $s\colon B \longrightarrow GB$ for ϵ_B (available by (1.3.3)), we have a split coequalizer diagram

$$GGB \underset{\epsilon_{GB}}{\overset{G\epsilon_B}{\rightrightarrows}} GB \xrightarrow{\epsilon_B} B\,, \qquad Gs\colon GB \to GGB,\quad s\colon B \to GB$$

as is verified by recording the section equation, $\epsilon_B \circ s = id_B$, and applying G to obtain $G\epsilon_B \circ Gs = id_{GB}$; the remaining two splitting equations are but instances of the naturality of ϵ: $\epsilon_{GB} \circ Gs = s \circ \epsilon_B$ and $\epsilon_B \circ G\epsilon_B = \epsilon_B \circ \epsilon_{GB}$. Applying any functor to (1.3.2), therefore, we obtain a similar 3×3 diagram in which all columns and the upper two rows are coequalizers. But then the 3×3 lemma (= Noether Isomorphism Theorem -- the special case of the Fubini Theorem (cf. [M^cL], p. 227) asserting that coequalizers commute with coequalizers) assures that the bottom row is a coequalizer, too, whence the lemma.

(1.5) Corollary (Theorem A -- sufficiency). *If the topos* $\mathbb{E}$ *is coZHD, then every cotriple* $\mathbb{G}$ *on* $\mathbb{E}$ *is dually algebraic.*

Proof. It is obvious that $(V_{\mathbb{G}})^{op}\colon (\mathbb{E}_{\mathbb{G}})^{op} \longrightarrow \mathbb{E}^{op}$ is tripleable, and it is known ([PCT], §2, Th'm) that $\Omega^{(-)}\colon \mathbb{E}^{op} \longrightarrow \mathbb{E}$ is tripleable, too. Now just apply the ZHD-Lemma (1.2).

(1.6) Corollary ([LDC]). *Every category cotripleable over the category* $\mathbb{S}$ *of sets and functions is the dual of a variety.*

Proof. By (1.5), cotriples on the coZHD topos $\mathbb{S}$ are dually algebraic.

2. Internal injectives and Theorem B (sufficiency proof).

Properly to understand the basic facts concerning internal injectives, it helps to bear in mind, by way of comparison, that an object X of a topos $\mathbb{E}$ is injective (in the usual sense, that every extension problem

(2.1.1)
$$\begin{array}{ccc} A & \overset{m}{\rightarrowtail} & B \\ {\scriptstyle\varphi}\downarrow & & \\ X & & \end{array}$$

with m a monomorphism has a solution $\tilde{\varphi}\colon B \longrightarrow X$ extending φ along m, i.e., satisfying $\tilde{\varphi} \circ m = \varphi$) if and only if the functor $X^{(-)}\colon \mathbb{E}^{op} \longrightarrow \mathbb{E}$ converts monomorphisms $m\colon A \rightarrowtail B$ (in $\mathbb{E}$) to split epimorphisms $X^m\colon X^B \longrightarrow X^A$. In fact, because Ω and its powers are injective in any topos, X is injective iff the singleton map $\{\cdot\}_X\colon X \longrightarrow \Omega^X$ has a retraction; but for each such retraction $\rho\colon \Omega^X \longrightarrow X$, the composition

$$X^A \xrightarrow[(\{\cdot\}_X)^A]{} (\Omega^X)^A \cong \Omega^{X\times A} \xrightarrow[\exists_{X\times m}]{} \Omega^{X\times B} \cong (\Omega^X)^B \xrightarrow[\rho^B]{} X^B$$

is easily seen to be a section for X^m when $m: A \rightarrowtail B$ is monic; conversely, if X^m has a section, passage to global elements shows that each extension problem (2.1.1) has a solution.

We say an object X of the topos $\mathbb{E}$ is <u>internally injective</u> if the functor $X^{(-)}: \mathbb{E}^{op} \longrightarrow \mathbb{E}$ merely preserves (that is, converts monomorphisms in $\mathbb{E}$ to) epimorphisms. Observe that the (reversible) "deductions"

$$\frac{\dfrac{I^*B \longrightarrow I^*X \qquad (\text{in } \mathbb{E}|_I)}{1_I = I^*1 \longrightarrow (I^*X)^{I^*B} \cong I^*(X^B) \qquad (\text{in } \mathbb{E}|_I)}}{\Sigma_I 1_I = I \longrightarrow X^B \qquad (\text{in } \mathbb{E})}$$

set up a 1-to-1 correspondence between maps $\tilde{\varphi}: I^*B \longrightarrow I^*X$ in $\mathbb{E}|_I$ <u>extending</u> (as we shall say) φ <u>along</u> m <u>over</u> I (that is, solving the transform under I^* of the extension problem (2.1.1)) and maps $\bar{\varphi}: I \longrightarrow X^B$ rendering

(2.1.2)
$$\begin{array}{ccc} I & \longrightarrow & 1 \\ {\scriptstyle \bar{\varphi}}\downarrow & & \downarrow {\scriptstyle \ulcorner\varphi\urcorner} \\ X^B & \xrightarrow[X^m]{} & X^A \end{array}$$

commutative. When X^m is epic, there are, given $\varphi: A \longrightarrow X$, diagrams (2.1.2) with $I \twoheadrightarrow 1$ epic (take (2.1.2) to be a pullback, for example); so (2.1.1) has solutions <u>locally</u> (that is, over some I with support $\sigma I = 1$) when X^m is epic. In particular, when X is internally injective, $X^{\{\cdot\}_X}$ is epic, and the singleton map $\{\cdot\}_X$ becomes a split monomorphism in $\mathbb{E}|_I$, for some I with $\sigma I = 1$. But for such I, I^*X is then injective in $\mathbb{E}|_I$, i.e., X is <u>locally injective</u>; and it follows, since $I^*(X^m) = I^*X^{I^*m}$, that, for such X and all maps m monic in $\mathbb{E}$, each X^m is a locally split epimorphism. Thus internal injectives and local injectives coincide, and all the maps X^m, for monic maps m, are epic for X internally injective because they are locally split -- indeed, they all split in any $\mathbb{E}|_I$ in which I^*X is injective. To sum up:

(2.1.3) injectives are internally injective;

(2.1.4) internal injectives are injective locally, and conversely.

It then follows easily that

(2.1.5) an object X for which I^*X is internally injective qua object of $\mathbb{E}|_I$, for some I with $\sigma I = 1$, is internally injective in $\mathbb{E}$;

and, as each J^* preserves injectives (because Σ_J preserves monomorphisms),

(2.1.6) J^*X is internally injective if X is, for all J in $\mathbb{E}$.

The proof of Theorem B uses the following amusing characterization of dually algebraic cotriples on topoi.

(2.2) Proposition. A cotriple $\mathbb{G} = (G, \epsilon, \delta)$ on a topos $\mathbb{E}$ is dually algebraic if and only if the functor G preserves equalizers of coreflexive pairs.

Proof. We adopt the terminology of -- and assume known the results in -- §2 of [PCT]. If G preserves such equalizers, $G \circ G$ does too, whence (by the dual of Prop. 3 of [LCA]) $\mathbb{E}_{\mathbb{G}}$ has them and $V_{\mathbb{G}}$ preserves them. It follows that $(V_{\mathbb{G}})^{op}: (\mathbb{E}_{\mathbb{G}})^{op} \longrightarrow \mathbb{E}^{op}$ satisfies the hypotheses of the RTT (cf. [PCT]). But so does the functor $\Omega^{(-)}: \mathbb{E}^{op} \longrightarrow \mathbb{E}$, hence so does their composite. So, applying the RTT, $\mathbb{G}$ is dually algebraic.

Conversely, if $\mathbb{G}$ is dually algebraic, then, because reflexive pairs in $\mathbb{E}^{op}$ are $\Omega^{(-)}$-split (cf. [PCT] again), the tripleable composite $\Omega^{(-)} \circ (V_{\mathbb{G}})^{op}$ is RTT by Beck's theorem; in particular, it preserves coequalizers of reflexive pairs. But $\Omega^{(-)}$ reflects such coequalizers, so $(V_{\mathbb{G}})^{op}$ preserves them, i.e., $V_{\mathbb{G}}$ preserves equalizers of coreflexive pairs. But so does $V_{\mathbb{G}}$'s right adjoint, whose composition with $V_{\mathbb{G}}$ is G, after all; then so does G.

The proof of the sufficiency clause in Theorem B is now at hand. It is convenient to say a topos is σLZ if it satisfies the condition of Theorem B.

(2.3) Lemma (Theorem B -- sufficiency). Every indexed cotriple $\mathbb{G} = (G, \epsilon, \delta)$ on a σLZ topos is dually algebraic.

Proof. Let $\mathbb{G}$ be an indexed cotriple on the σLZ topos $\mathbb{E}$. By (2.2) it suffices to prove G preserves equalizers of coreflexive pairs. So let

$$(\rightarrowtail\!\!\rightrightarrows) \qquad X_1 \longrightarrow X_2 \rightrightarrows X_3$$

be such an equalizer in $\mathbb{E}$. There are four principal steps to take.

Step I. We find an object I having same support $\sigma I = \sigma X_1$ as X_1 for which I^* carries $(\rightarrowtail\!\!\rightrightarrows)$ to an absolute equalizer diagram $I^*(\rightarrowtail\!\!\rightrightarrows)$ in $\mathbb{E}|_I$. To do so, we apply (2.1.4) in each topos $\mathbb{E}|_{\sigma X_i}$ and choose, for each $i = 1, 2, 3$, an object I_i having support $\sigma I_i = \sigma X_i$, for which $I_i{}^* X_i$ is injective in $\mathbb{E}|_{I_i}$. Writing $I = I_1 \times I_2 \times I_3$, it is clear that $\sigma I = \sigma X_1$ and (from the line before (2.1.6)) that $I^* X_i$ is injective in $\mathbb{E}|_I$, for each i. Hence, writing $\eta_i: X_i \longrightarrow \Omega^{(\Omega^{X_i})}$ for (the exponential transposes of) the evaluation maps (which are monic because Ω is an internal cogenerator), the monomorphisms $I^*(\eta_i)$ are split in $\mathbb{E}|_I$ for each i. Thus, we have at least verified the counterpart of condition (1.3.3) for an eventual application of the dual of the ABS-Lemma (1.4) within $\mathbb{E}|_I$ to $\Omega^{\Omega^{(-)}}$ and to the evaluation $\eta\colon \mathrm{id} \longrightarrow \Omega^{\Omega^{(-)}}$ there. For the counterpart of (1.3.4), note that, being a logical functor, $I^*\colon \mathbb{E} \longrightarrow \mathbb{E}|_I$ satisfies $I^*(\Omega^{(\to\rightrightarrows)}) \cong \Omega^{I^*(\to\rightrightarrows)}$. Then, since $\Omega^{(\to\rightrightarrows)}$ is an absolute coequalizer diagram in $\mathbb{E}$, it follows that $\Omega^{I^*(\to\rightrightarrows)}$ is an absolute coequalizer diagram in $\mathbb{E}|_I$, whence $\Omega^{\Omega^{I^*(\to\rightrightarrows)}}$ is an absolute equalizer diagram there, which is (1.3.4). By (1.4), then, $I^*(\to\rightrightarrows)$ is an absolute equalizer diagram in $\mathbb{E}|_I$, as desired.

<u>Step II</u>. Capitalizing on the hypothesis that $\mathbb{G}$ is indexed, we show that I^* carries $G(\to\rightrightarrows)$, the transform of $(\to\rightrightarrows)$ under G, to an equalizer diagram in $\mathbb{E}|_I$. Recall (from [P&S] or [RAF]) that, to be $\mathbb{E}$-indexed, a cotriple $\mathbb{G}$ on $\mathbb{E}$ must, for all I in $\mathbb{E}$, be so accompanied by cotriples $\mathbb{G}_I$ on $\mathbb{E}|_I$ ($\mathbb{G}_1$ being $\mathbb{G}$) that, regardless what the map $j\colon J \longrightarrow I$, each diagram

$$(D(j)) \qquad \begin{array}{ccc} \mathbb{E}|_I & \xrightarrow{G_I} & \mathbb{E}|_I \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \mathbb{E}|_J & \xrightarrow[G_J]{} & \mathbb{E}|_J \end{array}$$

commutes to within a specified equivalence modulo which j^* carries counit to counit and comultiplication to comultiplication. In particular, from the commutativity of $(D(I \longrightarrow 1))$, we see that $I^*(G(\to\rightrightarrows)) \cong G_I(I^*(\to\rightrightarrows))$; hence, recalling Step I, $I^*(G(\to\rightrightarrows))$ is an (absolute!) equalizer diagram in $\mathbb{E}|_I$.

<u>Step III</u>. Since the unique map $I \longrightarrow \sigma I = \sigma X_1$ is epic, pulling back along it gives a functor $\mathbb{E}|_{\sigma X_1} \longrightarrow \mathbb{E}|_I$ that, being a faithful right adjoint, reflects equalizers. In particular, $(\sigma X_1)^*(G(\to\rightrightarrows))$ is an equalizer diagram in $\mathbb{E}|_{\sigma X_1}$ (because $I^*(G(\to\rightrightarrows))$ is, in $\mathbb{E}|_I$, by Step II).

<u>Step IV</u>. We finally show $G(\to\rightrightarrows)$ is an equalizer diagram in $\mathbb{E}$. Suppose $f\colon Y \longrightarrow GX_2$ is a map equalizing the pair $GX_2 \rightrightarrows GX_3$. It follows that the composition $Y \longrightarrow GX_2 \longrightarrow X_2$ of f with the counit ε_{X_2} equalizes the pair $X_2 \rightrightarrows X_3$, hence that there is a map $h\colon Y \longrightarrow X_1$ making the box

$$\begin{array}{ccc} Y & \xrightarrow{f} & GX_2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \varepsilon_{X_2}} \\ X_1 & \longrightarrow & X_2 \end{array}$$

commute. But then $\sigma Y \subseteq \sigma X_1$, and as f factors uniquely through $G(X_1 \to X_2)$ over σX_1, by Step III, it must do so in $\mathbb{E}$ as well, which ends the proof.

3. <u>The necessity arguments</u>. That the condition in Theorem A is necessary will follow from the necessity of the condition in Theorem B. For the latter, however, we need a suitable criterion for a map $m\colon A \longrightarrow B$ in a

topos $\mathbb{E}$ to be locally a split monomorphism; that criterion will center on the following construction. Given the map m, form all the cokernel pairs

$$X^B \xrightarrow{X^m} X^A \underset{y_X}{\overset{x_X}{\rightrightarrows}} H_m(X) \quad (= H(X))$$

of the maps X^m (X in $\mathbb{E}$). Letting H_m ($= H$): $\mathbb{E} \longrightarrow \mathbb{E}$ be the unique functor with these values for which the families $x = \{x_X\}_X$ and $y = \{y_X\}_X$ are natural transformations $(-)^A \longrightarrow H$, it is clear that x and y make H_m the cokernel pair of $(-)^m$ in the category of endofunctors on $\mathbb{E}$. In fact, as pulling back preserves colimits and as $(-)^m$ is an indexed natural transformation, H_m inherits an indexed structure by which it serves (still via x and y) as cokernel pair for $(-)^m$ in the category $\mathcal{E}nd(\mathbb{E})$ of indexed endofunctors on $\mathbb{E}$. [Motivation: the Yoneda Lemma provides a sense in which $(-)^A$ is the indexed endofunctor on $\mathbb{E}$ presented by a single free generator -- id_A -- in the value at A; in the same sense, H_m is the indexed endofunctor presented by two generators in $H_m(A)$ -- the Yoneda correspondents $\bar{x}$ and $\bar{y}$ of x and y -- subject to the defining relation $\{H_m(m)\}(\bar{x}) = \{H_m(m)\}(\bar{y})$ in $H_m(B)$. Thus H_m is the generic indexed endofunctor hoping to convert the map m to a non-monomorphism.]

(3.1) *Lemma. The following conditions on a monomorphism* $m: A \rightarrowtail B$ *in a topos* $\mathbb{E}$ *are equivalent.*

(i) m *is locally split.*

(ii) *For each object* X, *the map* $X^m: X^B \longrightarrow X^A$ *is epic.*

(iii) *Where* $(-)^B \longrightarrow (-)^A \underset{y}{\overset{x}{\rightrightarrows}} H_m$ *is the cokernel pair in the category* $\mathcal{E}nd(\mathbb{E})$ (of indexed endofunctors on $\mathbb{E}$) *of* $(-)^m$, *the transition map* $H_m(m): HA \longrightarrow HB$ *is monic.*

(iv) *The map* $A^m: A^B \longrightarrow A^A$ *is epic.*

Proof. (iv) $\Rightarrow$ (i): argue as in the vicinity of (2.1.2).

(i) $\Rightarrow$ (ii): if I^*m is a split monomorphism, then $I^*(X^m) = (I^*X)^{(I^*m)}$ is a split epimorphism. By (i), there is an object I with $\sigma I = 1$, hence with I^* faithful, for which I^*m is a split monomorphism. But then, as I^* preserves epimorphisms, it reflects them, and X^m is epic.

(ii) $\Rightarrow$ (iii): by (ii), $x = y$ and $H_m = (-)^A$; so $H_m(m) = m^A$ is monic.

(iii) $\Rightarrow$ (iv): let $\bar{x} = x_A \circ \ulcorner 1_A \urcorner$ and $\bar{y} = y_A \circ \ulcorner 1_A \urcorner$ be the global elements $1 \longrightarrow H_m A = HA$ corresponding to x and y via Yoneda. By the definition of $H_m = H$, $Hm(\bar{x}) = Hm(\bar{y})$; then by (iii), $\bar{x} = \bar{y}$; so $x = y$, by an indexed Yoneda Lemma. In particular, $x_A = y_A$, and (iv) follows, as the cokernel pair $\underset{y}{\overset{x}{\rightrightarrows}} H$ is computed pointwise.

We are now ready to prove (somewhat more than) the rest of Theorem B.

(3.2) *Proposition*. *For any topos* $\mathbb{E}$, *the following are equivalent*.

(i) $\mathbb{E}$ *is* σ*LZ*.

(ii) *Every indexed cotriple on* $\mathbb{E}$ *is dually algebraic*.

(iii) *For every indexed cotriple* $\mathbb{G} = (G, \epsilon, \delta)$ *on* $\mathbb{E}$, *the functor* G *preserves monomorphisms*.

Proof. (i) $\Rightarrow$ (ii) is Lemma (2.3).

(ii) $\Rightarrow$ (iii): every monomorphism in $\mathbb{E}$ is the equalizer of its own (coreflexive) cokernel pair. Hence any endofunctor on $\mathbb{E}$ preserving equalizers of coreflexive pairs preserves monomorphisms. Now apply (2.2).

(iii) $\Rightarrow$ (i): to prove that an object A of $\mathbb{E}$ is internally injective as an object of $\mathbb{E}|_{\sigma A}$, it suffices, by the considerations of (2.1), to show that each monomorphism $m\colon A \rightarrowtail B$ in $\mathbb{E}$ becomes locally split when pulled back into $\mathbb{E}|_{\sigma A}$; and for this, in turn, it is enough to prove that $(A\times A)^*m$ is locally a split monomorphism in $\mathbb{E}|_{A\times A}$. To this end, let $H = H_m$ be the indexed endofunctor on $\mathbb{E}$ contemplated in (3.1(iii)). As we shall see in a moment, the endofunctor G given by $GX = X\times X\times HX$ is an indexed cotriple on $\mathbb{E}$, so that, by (iii), $Gm = m\times m\times Hm$ is monic. [Were H itself an indexed cotriple, Hm would be monic and, by (3.1), m would be locally split; but this is, in general, too much to hope for.] Now, because

$$\begin{array}{ccc} A\times A\times HA & \overset{m\times m\times Hm}{\rightarrowtail} & B\times B\times HB \\ {\scriptstyle A\times A\times Hm}\searrow & & \nearrow{\scriptstyle m\times m\times HB} \\ & A\times A\times HB & \end{array}$$

commutes, $A\times A\times Hm$ is monic in $\mathbb{E}$, whence $(A\times A)^*(Hm)$ is monic in $\mathbb{E}|_{A\times A}$. Using the indexedness of H , however, it is easy to verify that $(A\times A)^*(Hm) = H^{A\times A}((A\times A)^*m)$, and an application of (3.1) in $\mathbb{E}|_{A\times A}$ shows $(A\times A)^*m$ is locally split, as required.

It remains only to indicate how G is an indexed cotriple. Given any object X of $\mathbb{E}$, freely adjoin a (global) zero element by forming $X+1$; writing $o\colon 1 \longrightarrow X+1$ for the injection, endow $X+1$ with the trivial (constantly zero) semigroup multiplication $(X+1)^2 \longrightarrow 1 \xrightarrow{o} X+1$. Next, adjoining another (global) "unit" element, convert the semigroup $X+1$ into the monoid $X+2 = (X+1)+1$ it freely generates. It is clear that this procedure is perfectly functorial, so that $m\colon A \rightarrowtail B$ induces a monic map of monoids $m+2 : A+2 \rightarrowtail B+2$. Notice that, whatever the map of monoids $\omega\colon M' \longrightarrow M$, the induced functors $(-)^M$ and $(-)^{M'}$ are indexed cotriples, and $(-)^{\omega}\colon (-)^M \longrightarrow (-)^{M'}$ is a map of indexed cotriples. In particular,

$(-)^{m+2}: (-)^{B+2} \longrightarrow (-)^{A+2}$ is such. To obtain the cotriple $\mathbb{G}$, we use:

(3.3) Lemma. The forgetful functor $\mathcal{C}otrip(\mathbb{E}) \longrightarrow \mathcal{E}nd(\mathbb{E})$, from the category of indexed cotriples on $\mathbb{E}$ to the category of indexed endofunctors on $\mathbb{E}$, creates colimits.

Proof. For any monoidal category $\mathcal{V}$, the forgetful functor from the category $\mathcal{M}on(\mathcal{V})$ of monoids in $\mathcal{V}$ to $\mathcal{V}$ itself creates limits. So take $\mathcal{V} = \mathcal{E}nd(\mathbb{E})^{op}$, note that $\mathcal{M}on(\mathcal{V}) = (\mathcal{C}otrip(\mathbb{E}))^{op}$, and dualize for the lemma.

For $\mathbb{G}$, now, take the cokernel pair indicated below:

$$(-)^{B+2} \xrightarrow{(-)^{m+2}} (-)^{A+2} \underset{y'}{\overset{x'}{\rightrightarrows}} G .$$

By the lemma, G "is" an indexed cotriple $\mathbb{G}$; on the other hand, concluding the proof, we have, for every X, a commutative diagram

$$\begin{array}{ccccc} X^{B+2} & \xrightarrow{X^{m+2}} & X^{A+2} & \underset{y'}{\overset{x'}{\rightrightarrows}} & GX \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ X\times X\times X^{B} & \xrightarrow{X\times X\times X^{m}} & X\times X\times X^{A} & \underset{X\times X\times y_X}{\overset{X\times X\times x_X}{\rightrightarrows}} & X\times X\times HX . \end{array}$$

Finally, we settle the necessity of the condition in Theorem A.

(3.4) Lemma. If every cotriple on the topos $\mathbb{E}$ is dually algebraic, then every nonzero object $X \neq 0$ of $\mathbb{E}$ has a global section.

Proof. There is an idempotent cotriple $\mathbb{G}$ on $\mathbb{E}$ defined by

$$GX = \begin{cases} X, & \text{if } X \text{ has a global section } 1 \longrightarrow X ; \\ 0, & \text{if not.} \end{cases}$$

As $\mathbb{G}$ is dually algebraic and as each X of $\mathbb{E}$ appears as the equalizer

(3.4.1)
$$X \xrightarrow{\text{inj.}} X+1 \underset{X+\text{inj.}_2}{\overset{X+\text{inj.}_1}{\rightrightarrows}} X+1+1 \qquad (X+\text{codiag}: X+1+1 \to X+1)$$

of a coreflexive pair, (2.2) assures that the diagram

$$GX \longrightarrow X+1 \rightrightarrows X+1+1 ,$$

obtained by applying G to (3.4.1), remains an equalizer diagram. But then $GX \cong X$ for all X, and the lemma holds.

(3.5) Lemma. A σLZ topos $\mathbb{E}$ in which every nonzero object has a global section is coZHD.

Proof. If I^*X is injective in $\mathbb{E}|_I$ and $\gamma: 1 \longrightarrow I$ is a global section, then $X \cong \gamma^* I^* X$ is injective in $(\mathbb{E}|_I)|_\gamma \cong \mathbb{E}$.

(3.6) _Corollary_ (Theorem A -- necessity). _If every cotriple on_ $\mathbb{E}$ _is dually algebraic, then_ $\mathbb{E}$ _is coZHD_.

Proof. Apply Lemma (3.5) to Theorem B and Lemma (3.4).

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