TAUT FUNCTORS AND THE DIFFERENCE OPERATOR

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Abstract.

We establish a calculus of differences for taut endofunctors of the category of sets, analogous to the classical calculus of finite differences for real valued functions. We study how the difference operator interacts with limits and colimits as categorical versions, of the usual product and sum rules. The first main result is a lax chain rule which has no counterpart for mere functions. We also show that many important classes of functors (polynomials, analytic functors, reduced powers, ...) are taut, and calculate explicit formulas for their differences. Covariant Dirichlet series are introduced and studied. The second main result is a Newton summation formula expressed as an adjoint to the difference operator.

Contents

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Introduction

The category of endofunctors of **Set**, the category of sets, contains many interesting subcategories. Consider, for example, the recent work on polynomial functors, spearheaded by Spivak (see [14] and the references there), building on previous work of Kock chronicled in his arXiv paper [10] (see also [8] which deals with polynomial monads). Before that there were the analytic functors of Joyal [9] stemming from his categorical treatment of combinatorics, and developed extensively since then (see [7] and its extensive bibliography). But the study of endofunctors of **Set** goes back to the early days of category theory. Questions of rank for monads arose, showing the necessity of looking more deeply into the structure of these endofunctors.

This suggests that the study of **Set** endofunctors goes to the very foundation of set theory. For example, a non-principal ultrafilter, whose existence is well-known to require some form of the axiom of choice, produces an interesting endofunctor, the ultra-power functor. It is left exact and preserves finite coproducts, and so it is isomorphic to the identity on finite sets but not infinite ones. In [19] Trnková, and independently Blass [4, 5] showed that the existence of a non-trivial *exact* endofunctor of **Set** is equivalent to the existence of a measurable cardinal. Also see [16] in this connection.

These considerations, among others, indicate that a systematic study of the structure of endofunctors of **Set** might be desirable. And indeed, such a study was initiated in [18, 19] by Trnková, where she made an exhaustive study of their preservation properties.

The present work revisits this project from a different perspective taking into account the many developments of the last century. We develop a difference calculus for a rather large class of functors which parallels the classical finite difference calculus for real valued functions. The class of functors we consider are the taut functors introduced by Manes [12], who was motivated by theoretical computer science considerations. (Taut functors were also considered in [18] under the name "preimage preserving functors".)

Section 1 begins by recalling Manes' definitions of taut functor and taut natural transformation and recording some of their basic properties to be referred to later. Then follows a detailed study of the stability of tautness under limits and colimits. The main result here is a characterization of those colimits that preserve tautness, Theorem 1.3.2, which to our knowledge, has never appeared in print.

In Section 2 we consider some known classes of endofunctors: polynomials, divided power series, analytic functors, reduced powers, all of which are taut. We also consider taut monads. We introduce covariant Dirichlet functors, and what we call sequential Dirichlet functors, which are taut too.

It is in Section 3 that we introduce the difference operator. We study its functorial properties and then how it interacts with limits and colimits. As a special case we get a product (Leibniz) rule. Half of the section is taken up by the proof of one of the main results of the paper, the lax chain rule and its properties.

In Section 4 we return to the special classes considered in Section 2 and obtain explicit formulas for the differences.

We round out the paper with a version of the Newton summation formula, a discrete version of Taylor series. This is meant to recover nice taut functors, analytic say, from their iterated differences at 0. It is given by a left adjoint to the difference functor whose unit is an isomorphism for analytic functors.

Except for a few comments in passing, we mostly ignore questions of size. The "category" of endofunctors is an illegitimate one. Its homs may be proper classes. None of our results depends on this, but the purist will have no difficulty legitimizing things using standard techniques – Grothendieck universes or Gödel-Bernays set theory (with classes), for example.

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1. Taut functors

The notions of taut functor and natural transformation were introduced by Manes [12]. These are precisely what we need to develop our difference calculus of functors.

Everything that follows centres around pullbacks in which one leg is a monomorphism. Following Manes, we call them *inverse image diagrams*.

1.1. Definitions and functorial properties.

1.1.1. Definition. [Manes [12]]

A functor is *taut* if it preserves inverse image diagrams. A natural transformation is *taut* if the naturality squares corresponding to monomorphisms are pullbacks.

We record some of the general properties of tautness.

1.1.2. Proposition.

(1) *Taut functors preserve monos.*

(2) *The composite of taut functors is taut.*

(3) *If $t\colon F \longrightarrow G$ is taut and H is taut, then so is $Ht\colon HF \longrightarrow HG$.*

(4) *If $t\colon F \longrightarrow G$ is taut and K preserves monos, then $tK\colon FK \longrightarrow GK$ is taut.*

(5) *If $t\colon F \longrightarrow G$ and $u\colon G \longrightarrow L$ are taut, then the vertical composite $ut\colon F \longrightarrow L$ is taut.*

(6) *If $t\colon F \longrightarrow G$ is taut and G is taut, then so is F.*

PROOF. Perhaps the only part that is not completely straightforward is (6). Let

$$
\begin{array}{ccc}
A & \overset{m}{\rightarrowtail} & B \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
C & \underset{n}{\rightarrowtail} & D
\end{array}
$$

be an inverse image diagram. Then we have

$$
\begin{array}{ccc}
FA & \xrightarrow{Fm} & FB \\
{\scriptstyle Ff}\downarrow & (1) & \downarrow{\scriptstyle Fg} \\
FC & \xrightarrow[Fn]{} & FD \\
{\scriptstyle tC}\downarrow & (2) & \downarrow{\scriptstyle tD} \\
GC & \underset{Gn}{\rightarrowtail} & GD
\end{array}
\quad = \quad
\begin{array}{ccc}
FA & \xrightarrow{Fm} & FB \\
{\scriptstyle tA}\downarrow & (3) & \downarrow{\scriptstyle tB} \\
GA & \underset{Gm}{\rightarrowtail} & GB \\
{\scriptstyle Gf}\downarrow & (4) & \downarrow{\scriptstyle Gg} \\
GC & \underset{Gn}{\rightarrowtail} & GD
\end{array}
$$

where (2), (3), (4) are pullbacks, so (1) is too. ∎

1.1.3. COROLLARY. *Categories with inverse images, taut functors and taut natural transformations give a sub-2-category $\mathcal{T}aut$ of $\mathcal{C}at$, the 2-category of categories.*

1.2. LIMITS OF TAUT FUNCTORS. Limits of taut functors are again taut. This is just a case of limits commuting with limits but some attention must be paid as to where the diagrams and limits are taken. If $\mathbf{A}$ and $\mathbf{B}$ are categories with inverse images, we have the $\mathcal{C}at$ functor category $\mathcal{C}at(\mathbf{A}, \mathbf{B})$ of all functors from $\mathbf{A}$ to $\mathbf{B}$ and all natural transformations, and the subcategory $\mathcal{T}aut(\mathbf{A}, \mathbf{B})$ of all taut functors and taut natural transformations. And, we also have the full image of $\mathcal{T}aut(\mathbf{A}, \mathbf{B})$ in $\mathcal{C}at(\mathbf{A}, \mathbf{B})$, $\mathcal{T}aut_{full}(\mathbf{A}, \mathbf{B})$, of taut functors and all natural transformations.

1.2.1. PROPOSITION. *Assume that $\mathbf{B}$ has $\mathbf{I}$-limits. Then $\mathcal{T}aut_{full}(\mathbf{A}, \mathbf{B})$ is closed under $\mathbf{I}$-limits in $\mathcal{C}at(\mathbf{A}, \mathbf{B})$. If $t\colon \Phi \longrightarrow \Psi\colon \mathbf{I} \longrightarrow \mathcal{T}aut_{full}(\mathbf{A}, \mathbf{B})$, and tI is taut for every I, then $\varprojlim_I t(I)\colon \varprojlim_I \Phi(I) \longrightarrow \varprojlim_I \Psi(I)$ is also taut.*

PROOF. We use the fact that limits commute with limits, applied to the inverse image diagrams

$$
\begin{array}{ccc}
\Phi(I)(B_0) & \rightarrowtail & \Phi(I)(B) \\
\downarrow & \boxed{\text{Pb}} & \downarrow \\
\Phi(I)(A_0) & \rightarrowtail & \Phi(I)(A)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Phi(I)(A_0) & \rightarrowtail & \Phi(I)(A) \\
{\scriptstyle t(I)(A_0)}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle t(I)(A)} \\
\Psi(I)(A_0) & \rightarrowtail & \Psi(I)(A)
\end{array}
$$

respectively, for the inverse image diagram

$$\begin{array}{ccc} B_0 & \rightarrowtail & B \\ \downarrow & \boxed{\text{Pb}} & \downarrow \\ A_0 & \rightarrowtail & A \end{array} .$$

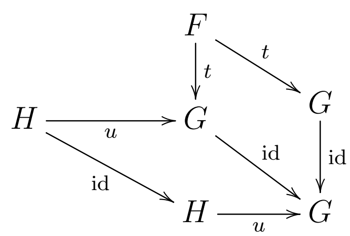
■

1.2.2. Corollary. *If* $\mathbf{B}$ *has pullbacks, then the pullback of a taut transformation along any natural transformation is again taut.*

Proof. Consider the pullback of taut functors

$$\begin{array}{ccc} H \times_G F & \xrightarrow{p_2} & F \\ {\scriptstyle p_1}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle t} \\ H & \xrightarrow[u]{} & G \end{array}$$

with t taut. Apply the previous proposition to the morphism of diagrams



to get that

$$H \times_G F \xrightarrow{\mathrm{id} \times_{\mathrm{id}} t} H \times_G G$$

is taut. But $\mathrm{id} \times_{\mathrm{id}} t$ is p_1 followed by an iso. ■

We emphasize that we are not assuming that the transformations $\Gamma(i)\colon \Gamma(I) \longrightarrow \Gamma(J)$ are taut, but even if we did we still would not get limits in $\mathcal{T}aut(\mathbf{A}, \mathbf{B})$. The projections are not taut. For example, if $\mathbf{B}$ has finite products, then the product of two taut functors is taut but the projection

$$F \times G \longrightarrow F$$

is not. For a mono $m\colon A_0 \rightarrowtail A$, the naturality square

$$\begin{array}{ccc} F(A_0) \times G(A_0) & \xrightarrow{Fm \times Gm} & F(A) \times G(A) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ F(A_0) & \xrightarrow[Fm]{} & F(A) \end{array}$$

is not usually a pullback; the pullback is $F(A_0) \times G(A)$.

A simpler example is that the unique map $F \longrightarrow 1$ is not taut. However this is the only obstruction as we see below (1.2.7).

1.2.3. Remark. We assumed that $\mathbf{B}$ had all $\mathbf{I}$-limits but the proposition holds for any limits that exist in $\mathcal{C}at(\mathbf{A}, \mathbf{B})$ as long as they are pointwise, i.e. calculated in $\mathbf{B}$.

1.2.4. Remark. We can't help pointing out that what we are dealing with are double limits, taken in the double category $\mathbb{T}\mathrm{aut}(\mathbf{A}, \mathbf{B})$ whose objects are taut functors, horizontal arrows arbitrary natural transformations, vertical arrows taut transformations, and commutative squares as cells. This may be worth pursuing but here is not the place.

1.2.5. Proposition. *Let $\mathbf{I}$ be a non-empty category, $\Phi \colon \mathbf{I} \longrightarrow \mathcal{T}aut_{full}(\mathbf{A}, \mathbf{B})$ a diagram of taut functors, $F \colon \mathbf{A} \longrightarrow \mathbf{B}$ a taut functor, and $\gamma \colon F \longrightarrow \Phi$ a cone on Φ with each $\gamma I \colon F \longrightarrow \Phi I$ a taut transformation. Then the induced transformation*

$$\langle \gamma I \rangle \colon F \longrightarrow \varprojlim_I \Phi I$$

is also taut.

Proof. γ is a natural transformation from the constant diagram with value F to Φ, and by Proposition 1.2.1 we get that

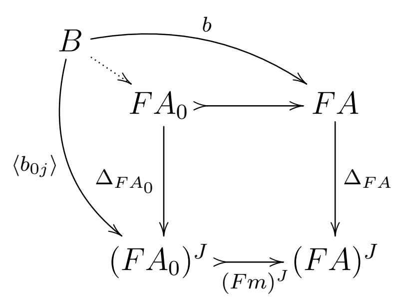
$$\varprojlim_I \gamma I \colon \varprojlim_I F \longrightarrow \varprojlim_I \Phi I$$

is taut. Now, $\varprojlim_I F = F^{\pi_0(\mathbf{I})}$, the product of F's, one for each component of $\mathbf{I}$, and $\langle \gamma I \rangle$ is the composite

$$F \xrightarrow{\ \Delta\ } F^{\pi_0(\mathbf{I})} \xrightarrow{\ \varprojlim_I \gamma I\ } \varprojlim_I \Phi I\ .$$

So we only have to show that Δ is taut.

Let J be a non-empty set and $m \colon A_0 \rightarrowtail A$ a mono in $\mathbf{A}$. Let b and $\langle b_{0j} \rangle$ make



commute, i.e. $F(m)b_{0j} = b$ for every j. Choose any $j \in J$. Then $b_{0j} \colon B \longrightarrow FA_0$ (dotted arrow) makes the top triangle commute. But the left triangle also commutes because $(Fm)^J$ is monic. Thus the square is a pullback and Δ is taut. ∎

1.2.6. Remark. Note that Δ is taut but not cartesian. For example, the pullback of $Ff \times Ff$ along Δ is the kernel pair K of Ff

$$\begin{array}{ccc} K & \longrightarrow & FA \\ \downarrow & \boxed{\text{Pb}} & \downarrow \\ FA_0 \times FA_0 & \xrightarrow[Ff\times Ff]{} & FA \times FA \end{array}$$

which will be FA_0 only if Ff is monic.

So every cone in $\mathcal{T}aut(\mathbf{A}, \mathbf{B})$ gives a unique taut transformation into the limit but this still doesn't make it into a limit in $\mathcal{T}aut(\mathbf{A}, \mathbf{B})$, because, as we said, the projections are not necessarily taut. But for non-empty connected limits, pullbacks or equalizers for example, it does.

1.2.7. Proposition. *Suppose* $\mathbf{B}$ *has* $\mathbf{I}$*-limits and that* $\mathbf{I}$ *is non-empty and connected, then* $\mathcal{T}aut(\mathbf{A}, \mathbf{B})$ *is closed under* $\mathbf{I}$*-limits in* $\mathcal{C}at(\mathbf{A}, \mathbf{B})$.

Proof. Let $\Phi\colon \mathbf{I} \longrightarrow \mathcal{T}aut(\mathbf{A}, \mathbf{B})$ be an $\mathbf{I}$-diagram and $I_0 \in \mathbf{I}$. We want to show that the projection

$$p_0\colon \varprojlim \Phi \longrightarrow \Phi I_0$$

is taut. To this end, let $m\colon A_0 \rightarrowtail A$ be a mono in $\mathbf{A}$ and consider the commutative diagram in $\mathbf{B}$

$$\begin{array}{ccc} B & \xrightarrow{\langle b_I \rangle} & (\varprojlim \Phi)(A) \\ & \searrow^{?} & \\ & (\varprojlim \Phi)(A_0) & \rightarrowtail (\varprojlim \Phi)(A) \\ {\scriptstyle b_0} \searrow & \downarrow {\scriptstyle p_0 A_0} & \downarrow {\scriptstyle p_0 A} \\ & (\Phi I_0)(A_0) & \rightarrowtail (\Phi I_0)(A) \end{array}$$

where $\langle b_I \rangle$ is the morphism induced by a cone for Φ, and which we wish to show factors through $(\varprojlim \Phi)(A_0)$. This will happen if each $b_I\colon B \longrightarrow \Phi(I)(A)$ factors through $\Phi(I)(A_0) \rightarrowtail \Phi(I)(A)$. Let $\mathbf{J} \subseteq \mathbf{I}$ be the full subcategory of $\mathbf{I}$ determined by those J for which we have b'_J such that

$$\begin{array}{ccc} B & & \\ {\scriptstyle b'_J} \downarrow & \searrow^{b_J} & \\ \Phi(J)(A_0) & \xrightarrow[\Phi(J)(m)]{} & (\Phi J)(A) \end{array} .$$

If $J \in \mathbf{J}$ and $j\colon J \longrightarrow K$ then K is in $\mathbf{J}$ because

$$\begin{array}{ccc}
B & & \\
{\scriptstyle b'_J}\big\downarrow & \searrow^{b_J} & \\
\Phi(J)(A_0) & \underset{\Phi(J)(m)}{\rightarrowtail} & \Phi(J)(A) \\
{\scriptstyle \Phi(j)(A_0)}\big\downarrow & & \big\downarrow{\scriptstyle \Phi(j)(A)} \\
\Phi(K)(A_0) & \underset{\Phi(K)(m)}{\rightarrowtail} & \Phi(K)(A)
\end{array}$$

commutes and the composite on the right is b_K, so $b'_K = \Phi(j)(A_0) \cdot b'_J$. Conversely, as $\Phi(j)$ is taut, the above square is a pullback, so if we have a b'_K, then there exists a unique b'_J. This means that if $j\colon J \longrightarrow K$ and $K \in \mathbf{J}$ then so is J. We're given that $I_0 \in \mathbf{J}$ and as anything connected to it will also be in $\mathbf{J}$, we have $\mathbf{J} = \mathbf{I}$. ■

1.3. Colimits of taut functors. We will only be concerned with **Set** valued taut functors, and for these, there are special colimit commutations. The main concept is the following.

1.3.1. Definition. A category $\mathbf{I}$ is *confluent* if for every pair of arrows with the same domain, $\alpha_i\colon I \longrightarrow I_i$ $(i = 1, 2)$ there exist $\beta_i\colon I_i \longrightarrow J$ making $\beta_1\alpha_1 = \beta_2\alpha_2$

$$\begin{array}{ccccc}
 & & I_1 & & \\
 & \nearrow^{\alpha_1} & & \searrow^{\beta_1} & \\
I & & & & J \\
 & \searrow_{\alpha_2} & & \nearrow_{\beta_2} & \\
 & & I_2 & &
\end{array} \quad .$$

1.3.2. Theorem. *$\mathbf{I}$-colimits commute with inverse images in* **Set** *iff* $\mathbf{I}$ *is confluent.*

Some preliminaries before we prove this. Recall that for a diagram $\Gamma\colon \mathbf{I} \longrightarrow \mathbf{Set}$, $\varinjlim \Gamma$ can be computed as the set of equivalence class of pairs $(I, x \in \Gamma I)$ where the equivalence relation is generated by

$$(I, x) \sim (I', \Gamma(\alpha)(x))$$

for $\alpha\colon I \longrightarrow I'$. So $(I, x) \sim (I', x')$ iff there exists a zigzag path

$$I = I_0 \xrightarrow{\alpha_1} I_1 \xleftarrow{\alpha_2} I_2 \xrightarrow{\alpha_3} \cdots \xleftarrow{\alpha_2 n} I_{2n} = I'$$

and $x_i \in \Gamma I_i$ such that

$$x_0 = x,\ x_{2n} = x'$$

$$\Gamma(\alpha_{2i-1})(x_{2i-2}) = x_{2i-1} = \Gamma(\alpha_{2i})(x_{2i}) \quad i = 1, \cdots, n$$

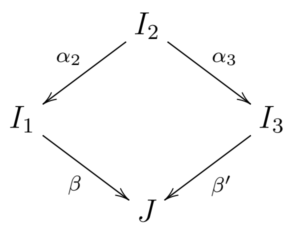
$$\begin{array}{ccccccccccccc}
1 & = & 1 & = & 1 & = & 1 & = & \cdots & = & 1 & = & 1 \\
{\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle x_0} & & \big\downarrow{\scriptstyle x_1} & & \big\downarrow{\scriptstyle x_2} & & & & \big\downarrow{\scriptstyle x_{2n}} & & \big\downarrow{\scriptstyle x'} \\
\Gamma I & = & \Gamma I_0 & \underset{\Gamma\alpha_1}{\longrightarrow} & \Gamma I_1 & \underset{\Gamma\alpha_2}{\longleftarrow} & \Gamma I_2 & \underset{\Gamma\alpha_3}{\longrightarrow} & \cdots & \underset{\Gamma\alpha_{2n}}{\longleftarrow} & \Gamma I_{2n} & = & \Gamma I'
\end{array} \quad .$$

1.3.3. Lemma. *If* **I** *is confluent, then* n *can be taken to be* 1 *in the above description, i.e.* $(I,x) \sim (I',x')$ *iff there exist* $\beta\colon I \longrightarrow J$ *and* $\beta'\colon I' \longrightarrow J$ *with*

$$\Gamma(\beta)(x) = \Gamma(\beta')(x') \ .$$

Proof. The "if" part is obvious.

On the other hand, if the shortest path joining (I,x) to (I',x) had $n > 1$, then we could take β, β' s.t. $\beta\alpha_2 = \beta'\alpha_3$



and replace $I_0 \xrightarrow{\alpha_1} I_1 \xleftarrow{\alpha_2} I_2 \xrightarrow{\alpha_3} I_3 \xleftarrow{\alpha_4} I_4$ by

$$I_0 \xrightarrow{\beta\alpha_1} J \xleftarrow{\beta'\alpha_4} I_4 \ .$$

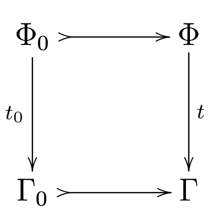
Then a simple calculation shows that $\Gamma(\beta\alpha_1)(x_0) = \Gamma(\beta'\alpha_4)(x_2)$, so we take $y \in \Gamma J$ to be that common value and we get a shorter path. ∎

1.3.4. Lemma. *Let* **I** *be confluent and* Γ_0 *a subdiagram of* Γ*,* $\Gamma_0 \subseteq \Gamma\colon \mathbf{I} \longrightarrow \mathbf{Set}$*. Then limit* $\varinjlim \Gamma_0$ *is isomorphic to the set of elements* $[I,x]$ *in* $\varinjlim \Gamma$ *such that there exists* $\beta\colon I \longrightarrow J$ *with* $\Gamma(\beta)(x) \in \Gamma_0 J$*.*

Proof. An element of $\varinjlim \Gamma_0$ is an equivalence class of pairs (I,x) with $x \in \Gamma_0 I$, which for now we denote $[I,x]_0$. To it there is a well-defined element $[I,x]$ of $\varinjlim \Gamma$, whether **I** is confluent or not, but if **I** is confluent the equivalence relations as described in Lemma 1.3.3 are the same for Γ and Γ_0 (crucial point!), so the function $[I,x]_0 \mapsto [I,x]$ is one-to-one. The classes are not the same – $[I,x]$ may contain more elements – but we can identify $\varinjlim \Gamma_0$ with those elements of $\varinjlim \Gamma$ that have a representative in Γ_0. So $[I,x]$ is in the image of $\varinjlim \Gamma_0$ iff there exist $x' \in \Gamma_0 I'$, $\beta\colon I \longrightarrow J$, $\beta'\colon I' \longrightarrow J$ with $\Gamma(\beta)(x) = \Gamma(\beta')(x') \in \Gamma_0 J$. Then, if x is in the image of Γ_0, there does exist a β with $\Gamma(\beta)(x) \in \Gamma_0 J$, and conversely, if there is such a β we can always take $\beta' = 1_J$. ∎

From now on, we will identify $\varinjlim \Gamma_0$ with its image in $\varinjlim \Gamma$.

Proof of theorem. First of all, let **I** be confluent and



be an inverse image diagram in $\mathbf{Set}^{\mathbf{I}}$, i.e. $\Phi_0(I) = t(I)^{-1}\Gamma_0(I)$, and take the colimits

$$\begin{array}{ccc} \varinjlim \Phi_0 & \rightarrowtail & \varinjlim \Phi \\ {\scriptstyle \varinjlim t_0}\downarrow & & \downarrow{\scriptstyle \varinjlim t} \\ \varinjlim \Gamma_0 & \rightarrowtail & \varinjlim \Gamma \ . \end{array}$$

So $\varinjlim \Phi_0$ is contained in the inverse image of $\varinjlim \Gamma_0$. An element in that inverse image is $[x, I] \in \varinjlim \Phi$ such that $[t(I)(x), I]$ is in $\varinjlim \Gamma_0$, i.e. there is $\beta \colon I \longrightarrow J$ such that $\Gamma(\beta)t(I)(x) \in \Gamma_0 J$. But $\Gamma(\beta)t(I)(x) = t(J)\Phi(\beta)(x)$ and as it is in $\Gamma_0 J$, $\Phi(\beta)(x) \in \Phi_0 J$, which implies $[x, I] \in \varinjlim \Phi_0$. Thus the inverse image of $\varinjlim \Gamma_0$ is equal to $\varinjlim \Phi_0$, and confluent colimits commute with inverse images in **Set**.

Let $\mathbf{I}$ be any small category and

$$\begin{array}{ccc} & & I_1 \\ & \overset{\alpha_1}{\nearrow} & \\ I & & \\ & \underset{\alpha_2}{\searrow} & \\ & & I_2 \end{array}$$

be arrows in $\mathbf{I}$. Consider the following inverse image diagram

$$\begin{array}{ccccc} & & \Phi_0 & \rightarrowtail & \mathbf{I}(I_1, -) \\ & & \downarrow & & \downarrow{\scriptstyle \mathbf{I}(\alpha_1,-)} \\ \mathbf{I}(I_2, -) & \twoheadrightarrow & \Gamma_0 & \rightarrowtail & \mathbf{I}(I, -) \ . \end{array}$$

Here Γ_0 is the image of the natural transformation $\mathbf{I}(\alpha_2, -)$ and Φ_0 the inverse image of Γ_0 under $\mathbf{I}(\alpha_1, -)$. The colimit of a representable is always 1, so taking colimits we get

$$\begin{array}{ccccc} & & \varinjlim \Phi_0 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \\ 1 & \twoheadrightarrow & \varinjlim \Gamma_0 & \longrightarrow & 1 \ . \end{array}$$

Then $\varinjlim \Gamma_0$ must be 1 also, and if $\mathbf{I}$-colimits commute with inverse images, $\varinjlim \Phi_0$ also has to be 1. Now, $\Phi_0(J)$ is the set of all morphisms $\beta_1 \colon I_1 \longrightarrow J$ such that there exists $\beta_2 \colon I_2 \longrightarrow J$ with $\beta_1\alpha_1 = \beta_2\alpha_2$,

$$\begin{array}{ccccc} & & I_1 & & \\ & \overset{\alpha_1}{\nearrow} & & \overset{\beta_1}{\searrow} & \\ I & & & & J \\ & \underset{\alpha_2}{\searrow} & & \underset{\beta_2}{\nearrow} & \\ & & I_2 & & \end{array} .$$

So if $\varinjlim \Phi_0$ is to be 1 there has to exist at least one such pair (β_1, β_2). I.e. $\mathbf{I}$ is confluent. ∎

1.3.5. Remark. Saying that $\mathbf{I}$-colimits commute with inverse images means exactly that the colimit functor

$$\varinjlim\colon \mathbf{Set}^{\mathbf{I}} \longrightarrow \mathbf{Set}$$

is taut.

1.3.6. Proposition. *$\mathcal{T}aut_{full}(\mathbf{A}, \mathbf{Set})$ is closed under confluent colimits. If t is a natural transformation of confluent diagrams in $\mathcal{T}aut_{full}(\mathbf{A}, \mathbf{B})$, and all the values of t are taut, then the induced morphism $\varinjlim t$ is also taut.*

Proof. The proof of Proposition 1.2.1 carries over verbatim, only using that confluent colimits commute with inverse images in $\mathbf{Set}$. ∎

Proposition 1.2.5 also "dualizes" to confluent colimits with some modifications.

1.3.7. Proposition. *Let $\mathbf{I}$ be confluent and $\Phi\colon \mathbf{I} \longrightarrow \mathcal{T}aut_{full}(\mathbf{A}, \mathbf{Set})$ a diagram of taut functors, $F\colon \mathbf{A} \longrightarrow \mathbf{Set}$ a taut functor, $\gamma\colon \Phi \longrightarrow F$ a cocone with each γI taut, then the induced transformation*

$$[\gamma I]\colon \varinjlim_I \Phi I \longrightarrow F$$

is also taut.

Proof. By Proposition 1.3.6 we get that

$$\varinjlim_I \gamma I\colon \varinjlim_I \Phi I \longrightarrow \varinjlim_I F$$

is taut, and $\varinjlim_I F = \pi_0\mathbf{I} \times F$, the coproduct of $\pi_0\mathbf{I}$ copies of F. So it will be sufficient to show that the codiagonal $\nabla\colon \pi_0\mathbf{I} \times F \longrightarrow F$ is taut, which is now clear from the pullback diagram

$$\begin{array}{ccc} \pi_0\mathbf{I} \times F(A_0) & \rightarrowtail & \pi_0\mathbf{I} \times F(A) \\ \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} \\ FA_0 & \rightarrowtail & FA\,. \end{array}$$

∎

1.3.8. Proposition. *$\mathcal{T}aut(\mathbf{A}, \mathbf{Set})$ is closed under confluent colimits in $\mathcal{C}at(\mathbf{A}, \mathbf{Set})$.*

Proof. In view of the previous proposition, we only have to show that the colimit injections

$$j = jI_0\colon \Phi(I_0) \longrightarrow \varinjlim_I \Phi I$$

are taut. Let $A_0 \rightarrowtail A$ be a mono and consider the commutative square, which we want to show is a pullback

$$\begin{array}{ccc} \Phi(I_0)(A_0) & \rightarrowtail & \Phi(I_0)(A) \\ {\scriptstyle jA_0}\downarrow & & \downarrow{\scriptstyle jA} \\ \varinjlim_I \Phi(I)(A_0) & \rightarrowtail & \varinjlim_I \Phi(I)(A) \end{array} .$$

The elements of $\varinjlim_I \Phi(I)(A)$ are equivalence classes $[I, x \in \Phi(I)(A)]$ and from Lemma 1.3.4 we can identify $\varinjlim_I \Phi(I)(A_0)$ with those classes for which there exist J and $\beta\colon I \longrightarrow J$ such that $\Phi(\beta)(A)(x) \in \Phi(J)(A_0)$. Let $x_0 \in \Phi(I_0)(A)$ be such that $j(A)(x_0) \in \varinjlim_I \Phi(I)(A_0)$. $j(A)(x) = [I_0, x_0]$ so there exists $\beta\colon I_0 \longrightarrow J$ with $\Phi(\beta)(A)(x_0) \in \Phi(J)(A_0)$. $\Phi(\beta)$ is taut so the following is a pullback

$$\begin{array}{ccc} \Phi(I_0)(A_0) & \rightarrowtail & \Phi(I_0)(A) \\ {\scriptstyle \Phi(\beta)(A_0)}\downarrow & & \downarrow{\scriptstyle \Phi(\beta)(A)} \\ \Phi(J)(A_0) & \rightarrowtail & \Phi(J)(A) \end{array} .$$

So $\Phi(\beta)(A)(x_0) \in \Phi(J)(A_0)$ implies $x_0 \in \Phi(I_0)(A_0)$, which shows that our original square is a pullback, completing the proof. ∎

1.3.9. Corollary. *Filtered colimits of taut functors into* **Set** *are taut.*

1.3.10. Corollary. *The quotient of a taut functor into* **Set** *by a group action is taut.*

Coproducts of set-valued functors will play a central role in what follows so we end the section on colimits with some results specifically about them.

1.3.11. Proposition. *A coproduct of functors into* **Set** *is taut if and only if each summand is taut.*

Proof. A discrete category is confluent so a coproduct of taut functors is taut, by Proposition 1.3.8 e.g. This is also easy to see directly.

Conversely, if $\sum_{i\in I} F_i$ is taut, then so is each F_i as it is a pullback of taut functors

$$\begin{array}{ccc} F_i & \longrightarrow & \sum_{i\in I} F_i \\ \downarrow & \boxed{\text{Pb}} & \downarrow \\ 1 & \xrightarrow[j_i]{} & \sum_{i\in I} 1 \end{array} .$$

∎

It is well-known that for a small category $\mathbf{A}$, every functor $F\colon \mathbf{A} \longrightarrow \mathbf{Set}$ is a coproduct of indecomposable functors indexed by a set $\pi_0 F$ called the connected components of F (see e.g. [2]). π_0 is left adjoint to the diagonal functor $D\colon \mathbf{Set} \longrightarrow \mathbf{Set}^{\mathbf{A}}$, i.e. it is the colimit functor. F is *connected* if $\pi_0 F = 1$. Here we are concerned with the case where $\mathbf{A} = \mathbf{Set}$, which is, of course, not small. If $\mathbf{A}$ is a large category, then $\pi_0 F$ may well be a proper class, and things fall apart. We don't get the adjointness to D for example. But, if $\mathbf{A}$ has a terminal object, everything is much nicer.

1.3.12. Proposition. *Let $\mathbf{A}$ be a category with terminal object 1. Then for a functor $F\colon \mathbf{A} \longrightarrow \mathbf{Set}$,*

(1) $\pi_0 F \cong F1$

(2) F is connected (indecomposable) if and only if $F1 = 1$

(3) Every F is a coproduct of connected functors.

Proof. The unique morphism $\tau X\colon X \longrightarrow 1$ for every X gives a natural transformation from the identity on $\mathbf{Set}$ to the constant functor 1, $\tau\colon \mathrm{id}_{\mathbf{Set}} \longrightarrow 1$. If we apply F to it we get a natural transformation $F\tau\colon F \longrightarrow F1$, which is easily seen to be a colimit cocone, which gives (1). (2) is a trivial consequence of (1). Finally, $F1 = \sum_{i \in F1} 1$ which gives a decomposition of F into a coproduct $F \cong \sum_{i \in F1} F_i$ where F_i is the pullback along $i\colon 1 \longrightarrow F1$

$$\begin{array}{ccc} F_i & \longrightarrow & F \\ \downarrow & \boxed{\text{Pb}} & \downarrow \\ 1 & \xrightarrow[i]{} & \sum_{i \in F1} 1 \end{array} .$$

As colimits are stable under pullback, we get $\varinjlim F_i \cong 1$. This is the decomposition claimed in (3). ∎

1.3.13. Corollary. *If $\mathbf{A}$ has a terminal object, then every taut functor from $\mathbf{A}$ to $\mathbf{Set}$ is a coproduct of connected taut functors.*

There is a cancellation property for functors into $\mathbf{Set}$ which will be used in the following sections. It is not deep but a little delicate and best stated explicitly. Coproduct of functors into $\mathbf{Set}$ is not cancellative

$$F + G \cong F + H \not\Rightarrow G \cong H \ .$$

If we take $\mathbf{A} = 1$, we have $\mathbb{N} + 1 \cong \mathbb{N} + 2$ but $1 \not\cong 2$, and we can promote this to endofunctors of $\mathbf{Set}$, which is where we will be using it, by taking constant functors with values $\mathbb{N}, 1, 2$. However it is cancellative if we take injections into account.

1.3.14. Proposition. *Let $F, G, H \colon \mathbf{A} \longrightarrow \mathbf{Set}$ be functors and assume we have an isomorphism ϕ commuting with injections*

$$\begin{array}{ccc} F+G & \xrightarrow{\ \phi\ } & F+H \\ & {}_{j}\nwarrow \quad \nearrow_{j} & \\ & F & \end{array},$$

then ϕ restricts to an isomorphism $\psi \colon G \longrightarrow H$.

Proof. For any A, if $x \in GA$ then $\phi(A)(x)$ is in $H(A)$ for if it weren't it would be in FA (on the right) and so $x = \phi^{-1}(A)\phi(A)$ would be in FA on the left which it is not. So ϕ restricts to

$$\begin{array}{ccc} G & \xrightarrow{\ \psi\ } & H \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j'} \\ F+G & \xrightarrow[\ \phi\ ]{} & F+H \end{array}.$$

The same argument applied to ϕ^{-1} gives ψ^{-1}. ■

The proof uses that ϕ is invertible. Clearly an arbitrary natural transformation would not restrict. It is also specific to functors into **Set**. It is false for $\mathbf{Set}^{op}$ for example.

2. Some special classes of taut functors

The class of taut functors is quite large as the following examples will show.

2.1. Polynomials. Classically a polynomial is an expression of the form

$$P(X) = C_0 + C_1X + C_2X^2 + \cdots + C_dX^d = \sum_{n=0}^{d} C_nX^n \tag{*}$$

which, if we want, can be interpreted in any category with finite sums and products, and will produce a polynomial endofunctor on that category. Of course, the quality of these functors will depend on whether the category in question has good properties, e.g. products distribute over sums. We're concerned with the category of sets which has all the properties we want and more.

Given sets $C_0, C_1, \cdots, C_d$, (*) defines a (finitary) polynomial functor

$$P \colon \mathbf{Set} \longrightarrow \mathbf{Set}$$

where for a set X, X^n is the set of n-tuples in X and C_nX^n is the cartesian product $C_n \times X^n$, and $+$ is coproduct (disjoint union). So X represents the identity functor $\mathrm{Id}(X) = X$ and X^n is the product of n-copies of X. As Id is taut and products and sums of taut functors are taut we get the following.

2.1.1. Proposition. *Any finitary polynomial functor is taut.*

Unencumbered by questions of convergence we can define power series to be

$$P(X) = \sum_{n=0}^{\infty} C_n X^n \qquad (**)$$

giving rise to power series functors

$$P\colon \mathbf{Set} \longrightarrow \mathbf{Set}\ .$$

2.1.2. Proposition. *Power series functors are taut.*

While we're at it, we may as well let the powers be any sets and the sum also indexed by a set. This is indeed the natural notion if we're thinking of **Set** as a categorified ring with coproduct as addition and product as multiplication. Now it's more natural not to collect like powers and simply allow repetitions.

2.1.3. Definition. A *polynomial* is an expression of the form

$$P(X) = \sum_{i \in I} X^{A_i}$$

where I is an index set and $\langle A_i \rangle_{i \in I}$ is a family of sets indexed by I. This determines a *polynomial functor*

$$P\colon \mathbf{Set} \longrightarrow \mathbf{Set}$$

given by the above formula.

Again we have:

2.1.4. Proposition. *Polynomial functors are taut.*

There is an extensive body of work on polynomial functors. They have been around for a long time in various settings and at different levels of abstraction and are still an active area of research. It is beyond the scope of this paper, and our competence, to give a comprehensive and accurate history of the subject. We mention only three names – André Joyal, Joachim Kock, and David Spivak – and three references – [11, 8, 14], – with apologies to all those not mentioned. The reader is referred to the historical comments and references in these works.

A particular feature of polynomial functors is that they have morphisms in contrast to polynomials over a field or ring. Morphisms of polynomial functors are simply natural transformations. We've seen that the projection $p_1\colon X^2 \longrightarrow X$ is not taut so not all morphisms are taut.

As a polynomial functor is a sum of powers and powers are just representable functors it is easy to analyze morphisms in terms of the families of powers. If $P(X) = \sum_{i \in I} X^{A_i}$ and $Q(X) = \sum_{j \in J} X^{B_j}$, we have the following bijections

$$\begin{array}{c} t\colon P \longrightarrow Q \\ \hline t\colon \sum_{i\in I} X^{A_i} \longrightarrow \sum_{j\in J} X^{B_j} \\ \hline \langle X^{A_i} \longrightarrow \sum_{j\in J} X^{B_j} \rangle_{i\in I} \\ \hline \alpha\colon I \longrightarrow J \ \&\ \langle X^{A_i} \longrightarrow X^{B_{\alpha(i)}} \rangle_{i\in I} \\ \hline \alpha\colon I \longrightarrow J \ \&\ \langle f_i\colon B_{\alpha(i)} \longrightarrow A_i \rangle. \end{array}$$

The third bijection is because $X^{A_i} = \mathbf{Set}(A_i, -)$ is freely generated by a single element, 1_{A_i}, so the transformation must factor through some injection, the $\alpha(i)^{th}$. The fourth bijection is the Yoneda lemma.

Working back up the bijections we see that given $(\alpha, \langle f_i \rangle)$ as above, the natural transformation

$$t\colon P \longrightarrow G$$

is given as follows. An element of $P(X)$ is a pair (i, ϕ) where $\phi\colon A_i \longrightarrow X$. t then sends (i, α) to $(\alpha(i), \phi f_i)$ in $Q(X)$.

In [14] a morphism $(\alpha, \langle f_i \rangle)$ is called *vertical* if α is an isomorphism, and it is *cartesian* if all f_i are isomorphisms. Here we have something weaker than cartesian.

2.1.5. Proposition. *A morphism $(\alpha, \langle f_i \rangle)$ of polynomials is taut if and only if each of the f_i is an epimorphism.*

Proof. Let $t\colon P \longrightarrow Q$ be given by $(\alpha, \langle f_i \rangle)$ as above and assume each of the f_i is an epimorphism. Let $X_0 \rightarrowtail X$ be a monomorphism and consider the commutative diagram

$$\begin{array}{ccc} \sum_{i\in I} X_0^{A_i} & \rightarrowtail & \sum_{i\in I} X^{A_i} \\ {\scriptstyle t(X_0)}\downarrow & & \downarrow{\scriptstyle t(X)} \\ \sum_{j\in J} X_0^{B_j} & \rightarrowtail & \sum_{j\in J} X^{B_j} \ . \end{array}$$

Let $(i, \phi) \in \sum_{i\in I} X^{A_i}$ be such that $t(X)(i, \phi) \in \sum_{j\in J} X_0^{B_j}$, i.e. ϕf_i factors through X_0, giving ψ and a commutative square

$$\begin{array}{ccc} B_{\alpha(1)} & \overset{f_i}{\twoheadrightarrow} & A_i \\ {\scriptstyle \psi}\downarrow & \swarrow & \downarrow{\scriptstyle \phi} \\ X_0 & \rightarrowtail & X \ . \end{array}$$

By the diagonal fill-in property of factorizations we have the dotted arrow above, i.e. $(i, \phi) \in \sum_{i\in I} X_0^{A_i}$, showing that the original square is a pullback. Thus t is taut.

Conversely, if t is taut, then for every mono $X_0 \rightarrowtail X$ and ϕ, ψ as above we have the fill-in, so by the orthogonality property of factorizations, the f_i are epis. ■

2.2. Divided powers. Divided power series are expressions of the form

$$\sum_{n=0}^{\infty} a_n x^{[n]}$$

where $x^{[n]}$ is to be thought of as $x/n!$. They form a ring with componentwise addition, and multiplication given by the bilinear extension of

$$x^{[n]}x^{[m]} = \binom{n+m}{n} x^{[n+m]}$$

as the intended meaning of $x^{[n]}$ would suggest. Over a field of characteristic 0, we get a ring isomorphic to the ring of formal power series, but over a field of finite characteristic, or a ring, we get something different which often has better properties.

In keeping with our program of extending ring theoretic concepts to **Set** we can define the divided power $X^{[n]}$ as follows. The symmetric group S_n acts on the right on X^n,

$$(x_1 \dots x_n)^\sigma = (x_{\sigma 1}, x_{\sigma 2}, \dots, x_{\sigma n})$$

and we take the quotient by this action

$$X^{[n]} = X^n / S_n \ .$$

If $\mathbf{S}_n$ is the symmetric group considered as a one object category, a right action corresponds to a functor

$$\mathbf{S}_n^{op} \longrightarrow \mathbf{Set}$$

and the quotient $X^{[n]}$ is its colimit. So by Proposition 1.3.8, $F(X) = X^{[n]}$ gives a taut functor $\mathbf{Set} \longrightarrow \mathbf{Set}$.

A *divided power series functor* $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is one given by

$$FX = \sum_{n=0}^{\infty} C_n X^{[n]} \ .$$

From the discussion above we get immediately the following.

2.2.1. Proposition. *Any divided power series is taut.*

Divided power series functors are not polynomial functors except in the affine case

$$C_0 + C_1 X^{[1]} \ .$$

They don't preserve pullbacks. E.g. $FX = X^{[2]}$ takes the pullback below to a commutative diagram which is clearly not a pullback

$$\begin{array}{ccc} 4 & \longrightarrow & 2 \\ \downarrow & \boxed{\text{Pb}} & \downarrow \\ 2 & \longrightarrow & 1 \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} 10 & \longrightarrow & 3 \\ \downarrow & & \downarrow \\ 3 & \longrightarrow & 1 \end{array} \ .$$

But they provide more examples of taut functors.

The analogy with real divided power series is nice but only goes so far. Note, for example, that if X is a finite cardinal k then the cardinality of X^n is k^n but the cardinality of $X^{[n]}$ is not $k^n/n!$, which is not even an integer in general. Instead, its cardinality is

$$\frac{k^{\uparrow n}}{n!} = \frac{k(k+1)\dots(k+n-1)}{n!} \tag{*}$$

which is, of course, an integer, a binomial coefficient in fact.

More importantly, they are not closed under binary products. We definitely don't have

$$X^{[n]} \times X^{[n]} \cong \binom{n+m}{n} X^{[n+m]}$$

which is obvious if we let $X = 1$. Even if we thought it might be a series

$$X^{[n]} \times X^{[n]} \cong \sum_{i=0}^{\infty} C_i X^{[i]} \ ,$$

by letting $X = 1$ we see that all the C_i must be 0 except for one of them, $C_l = 1$, so that we would have

$$X^{[n]} \times X^{[n]} \cong X^{[l]} \ .$$

This is impossible just on the grounds of cardinality: (*) is a polynomial in k with leading term $k^n/n!$ so we would need

$$k^n/n! \cdot k^m/m! = k^l/l!$$

for all k, which we can't have.

What we do have is

$$X^n/S_n \times X^m/S_m \cong X^{n+m}/S_n \times S_m$$

where $S_n \times S_m$ acts on $X^{n+m} \cong X^n \times X^m$ in the obvious way, i.e. componentwise. This leads to taking quotients of X^n by a subgroup G of S_n rather than the full S_n. Then G acts on the left on X^n and we can take the quotient

$$X^n/G \ .$$

Now we can redefine our divided power series as follows.

2.2.2. Definition. An *extended divided power series* is an expression of the form

$$\sum_{n=0}^{\infty} \left(\sum_{i \in I_n} X^n/G_i \right)$$

where, for each n, $\langle G_i \rangle_{i \in I_n}$ is a family of subgroups of S_n.

2.2.3. Proposition. *Extended divided power series define taut functors.*

Dare we write $\sum_{i \in I_n} X^n/G_i$ as $\sum_{i \in I_n} (1/G_i)\, X^n$ and think of $\sum 1/G_i$ as some kind of rational number?

Note that power series are now special cases of extended divided power series, by taking all of the G_i to be trivial.

We can replace finite powers by arbitrary ones. For a set A, S_A is the group of bijections $A \longrightarrow A$. S_A acts on the right on X^A and we take the quotient by that action to get a taut functor

$$FX = X^A/S_A\ .$$

As before, no need to take the full symmetric group. We can restrict to a subgroup $G \leq S_A$ and get

$$FX = X^A/G\ .$$

A bit more canonically, we can take any group G and A a left G-set. Then G will act on the right on X^A and we get again a taut functor.

A *generalized divided power series functor* is one of the form

$$FX = \sum_{i \in I} X^{A_i}/G_i$$

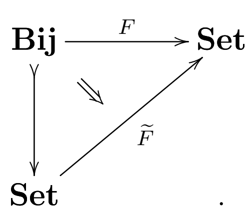
where I is a set, $\langle G_i \rangle$ a family of groups and $\langle A_i \rangle$ a family of left G_i-sets. F is taut.

If we take all the G_i to be trivial we get polynomial functors and if we take S_n, $n \in \mathbb{N}$ we get the "classical" divided power series.

2.3. Analytic functors. Joyal's original definition of *analytic functor* [9] was in the context of his reformulation of combinatorics in categorical terms and was strictly finitary. They have since been generalized in many directions (see [7] and references there).

In [9], a *species (of structure)* was defined as a functor $F\colon \mathbf{Bij} \longrightarrow \mathbf{Bij}$, where $\mathbf{Bij}$ is the category of finite cardinals and bijections, but for general purposes there is no problem in taking $F\colon \mathbf{Bij} \longrightarrow \mathbf{Set}$. So F is a sequence of left S_n-sets, Fn, one for each $n \in \mathbb{N}$.

A species F determines an *analytic functor* $\widetilde{F}$ as the left Kan extension of F along the inclusion of $\mathbf{Bij}$ in $\mathbf{Set}$



$\widetilde{F}$ is given by the formula

$$\widetilde{F}X = \int^{n \in \mathbb{N}} X^n \times Fn$$

which can be calculated as a colimit

$$\widetilde{F}X = \varinjlim_{a \in Fn} X^n$$

taken over the category of elements of F, $\mathbf{El}(F)$. As $\mathbf{Bij}$ is a groupoid, so is $\mathbf{El}(F)$, and therefore $\widetilde{F}$ is taut.

2.3.1. Proposition. *The classical analytic functors are taut.*

Let's reformulate this a bit in order to compare it with power series and divided power series. Just like any colimit, a colimit over a groupoid is the coproduct of the individual colimits over the connected components of the groupoid. If we choose a representative object in each component, then these colimits are colimits taken over groups, the automorphism groups of the chosen objects. So an analytic functor can be written as

$$\sum_{n=0}^{\infty} X^n \otimes_{S_n} C_n$$

for a family of left S_n-sets C_n. The C_n are the Fn from above, the notation chosen meant to suggest "coefficients", and to agree more with the previous sections.

Just to be clear, $X^n \otimes_{S_n} C_n$, which is the coend or colimit from above, is explicitly described as the set of all equivalences classes $[x_1, \ldots, x_n; c] = \langle x_i \rangle \otimes c$ for $x_i \in X$ and $c \in C$. The equivalence relation is

$$(x_1, \ldots, x_n; c) \sim (y_1, \ldots, y_n; d) \text{ iff } \exists \sigma \in S_n(\bigwedge_i y_i = x_{\sigma i}) \wedge (c = \sigma d)$$

or put differently

$$\langle x_i \rangle \otimes \sigma d = \langle x_{\sigma i} \rangle \otimes d \ .$$

Now, $X^n \cong X^n \otimes_{S_n} S_n$ so power series are analytic, and $X^{[n]} \cong X^n \otimes_{S_n} 1$ so divided power series are also analytic. In fact, the extended divided power series are analytic functors. For any subgroup G of S_n, the left cosets of G, S/G form a left S_n-set as usual, and

$$X^n \otimes_{S_n} S_n/G \cong X^n/G \ .$$

Note in passing that

$$X^n/S_n \times X^m/S_m \cong X^{n+m}/(S_n \times S_m) \cong X^{n+m} \otimes_{S_{n+m}} (S_{n+m}/S_n \times S_m)$$

which is the proper way of generalizing the equation

$$x^{[n]}x^{[m]} = \binom{n+m}{n} x^{[n+m]} \ .$$

The cardinality of $S_{n+m}/S_n \times S_m$ is indeed $\binom{n+m}{n}$.

But analytic functors can also be presented as extended divided power series, so determine the same class of taut functors just presented differently. Indeed, if C is a left S_n-set, it is a sum of indecomposables and these are isomorphic to left cosets of some subgroup of S_n. So

$$C = \sum_{i \in I} S_n/G_i$$

and then

$$X^n \otimes_{S_n} C \cong \sum_{i \in I} X^n/G_i \ .$$

So our fanciful $\sum 1/G_i$ from the previous section is just $\sum S_n/G_i$ in the language of analytic functors.

We summarize the above discussion in the following.

2.3.2. Proposition. *Extended divided power series and analytic functors determine the same class of taut functors.*

We can generalize analytic functors, as we did for divided powers, allowing arbitrary powers not just finite ones.

2.3.3. Definition. A *generalized analytic functor* is given by

$$FX = \sum_{i \in I} X^{A_i} \otimes_{G_i} C_i$$

where $\langle G_i \rangle_{i \in I}$ is a family of groups, and $\langle A_i \rangle_{i \in I}$ and $\langle C_i \rangle_{i \in I}$ are families of left G_i-sets.

As in the finitary case we have the following.

2.3.4. Proposition.

(1) Generalized analytic functors are taut.

(2) Polynomial functors are generalized analytic.

(3) Generalized divided power series give the same class of taut functors as generalized analytic functors.

2.4. Reduced powers. So far all of our examples, interesting in their own right, have been special cases of generalized symmetric power series functors, which brings us to reduced powers, which will give a new class.

Reduced powers are a lot like symmetric powers in that they are (with one exception) quotients of powers, i.e. can be defined as equivalence classes of functions from a fixed exponent into a variable set, but in fact give us a completely new class of taut functors as we will see below. The reader is referred to Blass' paper [3], mentioned in the introduction, where he proves, among other things, that every left exact endofunctor is a directed union of reduced powers.

2.4.1. Definition. Let A be a set. A *filter* $\mathcal{F}$ on A is a set of subsets of A, $\mathcal{F} \subseteq 2^A$, such that

(1) $\mathcal{F}$ is closed under finite intersections, i.e. $A \in \mathcal{F}$ (empty intersection) and $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F}$.

(2) $\mathcal{F}$ is up-closed $A_1 \in \mathcal{F}$ & $A_1 \subseteq A_2 \Rightarrow A_2 \in \mathcal{F}$.

$\mathcal{F}$ is *proper* if $\emptyset \notin \mathcal{F}$.

2.4.2. Examples.

(1) The set of cofinite (finite complement) subsets of any infinite set, e.g. $\mathbb{N}$.

(2) The set of subsets $A \subseteq \mathbb{N}$ such that there exists $n_0 \in A$ with $k \geq n_0 \Rightarrow k \in A$.

(3) If $A_0 \subseteq A$ is a non-empty subset, then the set of all A_1 containing A_0 is a *principal filter*.

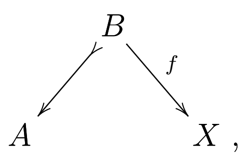
2.4.3. Definition. Let A be a set and $\mathcal{F}$ a filter on A. The *reduced power* $X^{\mathcal{F}}$ is the colimit

$$X^{\mathcal{F}} = \varinjlim_{B \in \mathcal{F}} X^B.$$

As $X^{\mathcal{F}}$ is a filtered colimit of representables we immediately get the following.

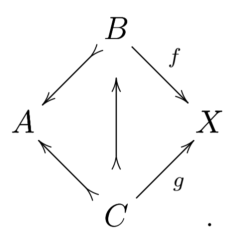
2.4.4. Proposition. *For any filter $\mathcal{F}$, the reduced power $X^{\mathcal{F}}$ gives a left exact endofunctor of* **Set***, in particular it is taut.*

Just from the definition as a colimit we see that $X^{\mathcal{F}}$ consists of equivalence classes of partial functions from A into X,



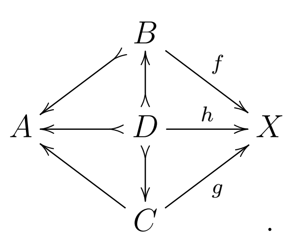
with $B \in \mathcal{F}$. The equivalence relation is generated by restriction

$f \sim g$ if



Because the colimit is filtered, the equivalence relation can be described as

$$f \sim g \Leftrightarrow \quad \text{there exists} \quad A \longleftarrow\!\!\prec D \xrightarrow{h} X \quad \text{such that} \quad D \in \mathcal{F} \quad \text{and}$$



An even more amenable description is that

$$f \sim g \quad \Leftrightarrow \quad \{a \in B \cap C \mid fa = ga\} \in \mathcal{F}.$$

The reduced power $X^{\mathcal{F}}$ is often defined as follows:

$$X^{\mathcal{F}} = \{f\colon A \longrightarrow X\}/\sim = X^A/\sim$$

where $f \sim g$ iff $\{a \in A \mid fa = ga\} \in \mathcal{F}$. This is a lot easier to handle and justifies the term "reduced power". It is equivalent to the colimit definition except when $\mathcal{F}$ is trivial, i.e. contains $\emptyset$, and so all subsets. In that case, the colimit definition gives

$$X^{\mathcal{F}} = 1$$

the constant functor with value 1, whereas the quotient of the representable gives

$$X^A/\sim = \begin{cases} \emptyset & \text{if } X = 0 \\ 1 & \text{otherwise.} \end{cases}$$

This functor is not taut as, e.g.

$$\begin{array}{ccc} 0 & \rightarrowtail & 1 \\ \downarrow & \boxed{\text{Pb}} & \downarrow 1 \\ 1 & \underset{0}{\rightarrowtail} & 2 \end{array} \quad \longmapsto \quad \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array} .$$

We don't want to exclude the improper filter in our definition but we nevertheless work with the reduced power definition, which we find easier, and check the degenerate case separately.

If $\mathcal{F}$ is a principal filter generated by $A_0 \subseteq A$, then it is easily seen that $X^{\mathcal{F}} \cong X^{A_0}$. But we have:

2.4.5. Proposition. *If $\mathcal{F}$ is a non-principal filter, then the reduced power functor $X^{\mathcal{F}}$ is not a generalized symmetric power series.*

Proof. Suppose $X^{\mathcal{F}} \cong \sum_{i \in I} X^{B_i}/G_i$. If we let $X = 1$, we get $1 \cong I$, so there's only one term in the series X^B/G where B is a left C-set, $X^{\mathcal{F}} \cong X^B/G$. Now, $X^{\mathcal{F}}$ preserves finite products, so X^B/G will also. Consider the special case

$$(B \times B)^B/G \cong B^B/G \times B^B/G.$$

An element on the left is an equivalence class of pairs $[\langle\phi, \psi\rangle]$ for $\phi, \psi\colon B \longrightarrow B$, and two pairs are equivalent

$$\langle\phi, \psi\rangle \sim \langle\phi', \psi'\rangle \Leftrightarrow \exists\, \sigma \in G(\sigma\phi = \phi' \wedge \sigma\psi = \psi')\ . \tag{1}$$

(Wlog we have assumed that G is a subgroup of S_B.)

An element on the right is a pair of equivalence classes $\langle[\phi], [\psi]\rangle$ with two being equal

$$\langle[\phi], [\psi]\rangle = \langle[\phi'], [\psi]\rangle \Leftrightarrow \exists\, \sigma, \tau \in G(\sigma\phi = \phi' \wedge \tau\psi = \psi)\ . \tag{2}$$

Of course (1) ⇒ (2). That's the canonical product comparison morphism. If (2) ⇒ (1) then for any $\sigma \in G$, we have $\langle [\mathrm{id}], [\mathrm{id}] \rangle = \langle [\sigma], [\mathrm{id}] \rangle$ so $[\langle \mathrm{id}, \mathrm{id} \rangle] = [\langle \sigma, \mathrm{id} \rangle]$ i.e. there is $\sigma' \in G$ such that

$$\sigma' \cdot \mathrm{id} = \sigma \quad \text{and} \quad \sigma' \cdot \mathrm{id} = \mathrm{id}$$

so $\sigma = \mathrm{id}$, i.e. G is trivial.

So now we're reduced to $X^{\mathcal{F}} = X^B$, a representable. This means that $X^{\mathcal{F}}$ preserves all products, which implies that $\mathcal{F}$ is closed under arbitrary intersections and that means $\mathcal{F}$ is principal, contrary to our assumption. ∎

Non trivial reduced powers are quotients of representables which are projective, so if $\mathcal{F}$ is a filter on A and $\mathcal{G}$ one on B, a natural transformation $t\colon X^{\mathcal{F}} \longrightarrow X^{\mathcal{G}}$ lifts

$$\begin{array}{ccc} X^A & \overset{\bar{t}}{\dashrightarrow} & X^B \\ \downarrow & & \downarrow \\ X^{\mathcal{F}} & \xrightarrow[t]{} & X^{\mathcal{G}} \end{array}$$

which is equivalent to a function $\phi\colon B \longrightarrow A$. Not every ϕ will give a natural transformation which descends to the reduced powers, and two ϕ's that do, may give the same reduced transformation. For an element of $X^{\mathcal{F}}$, i.e. an equivalence class $[f]$ of functions $f\colon A \longrightarrow X$, the induced t is given by

$$t(X)[f] = [f\phi] \ .$$

Some basic set theoretical calculations lead to the following result, from [18] Proposition VI.3 (also contained in Theorem 3 of [3]).

2.4.6. Proposition. *For filters $\mathcal{F}$ and $\mathcal{G}$ on A and B respectively, the natural transformations between the reduced power functors*

$$t\colon X^{\mathcal{F}} \longrightarrow X^{\mathcal{G}}$$

are in bijection with equivalence classes of functions $\phi\colon B \longrightarrow A$ such that for each $A_0 \in \mathcal{F}$ we have $\phi^{-1}A_0 \in \mathcal{G}$. ϕ is equivalent to $\psi\colon B \longrightarrow A$ if

$$\{b \in B \mid \phi(b) = \psi(b)\} \in \mathcal{G} \ .$$

We think of the elements of $\mathcal{F}$ (or $\mathcal{G}$) as large subsets of A (resp. B). Then natural transformations between the reduced powers correspond to equivalence classes of functions which map large subsets of A to large subsets of B by inverse image, much like continuous functions do for opens. We now characterize those classes $[\phi]$ for which the corresponding transformation is taut as those that map large subsets of B to large subsets of A by direct image, like open maps. First a lemma.

2.4.7. LEMMA. *Let $z\colon Y \rightarrowtail X$ be a monomorphism. An element $[f]$ of $X^{\mathcal{F}}$ is in (the image of) $Y^{\mathcal{F}}$ if and only if there is an $A_0 \in \mathcal{F}$ and a factorization*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X \\ \uparrow & & \uparrow{\scriptstyle z} \\ A_0 & \overset{}{\underset{g}{\dashrightarrow}} & Y \end{array} .$$

PROOF. If $Y = \emptyset$ then $Y^{\mathcal{F}} = \emptyset$ and the result is trivial. Otherwise, if there are A_0 and g as above, we can extend g to $\bar{g}\colon A \longrightarrow Y$. Then $z\bar{g}$ and f agree on A_0 so $[f] = [z\bar{g}]$, i.e. $[f]$ in $Y^{\mathcal{F}}$.

In the other direction, if $[f]$ is in $Y^{\mathcal{F}}$, i.e. $[f] = [z\bar{g}]$ for some $\bar{g}\colon A \longrightarrow Y$, then f and $z\bar{g}$ agree on some $A_0 \in \mathcal{F}$ and we take $g = \bar{g}|_{A_0}$. ∎

2.4.8. PROPOSITION. *Let $\mathcal{F}$ and $\mathcal{G}$ be filters on A and B respectively, and $\phi\colon B \longrightarrow A$ a function mapping elements of $\mathcal{G}$ to elements of $\mathcal{F}$ by inverse image. Then the induced natural transformation*

$$t\colon X^{\mathcal{F}} \longrightarrow X^{\mathcal{G}}$$

$$[f] \longmapsto [f\phi]$$

is taut if and only if

$$B_0 \in \mathcal{G} \Rightarrow \phi(B_0) \in \mathcal{F}.$$

PROOF. First assume t is taut and let $B_0 \subseteq B$ Then we get $\phi(B_0) \subseteq A$ and a pullback square, by tautness

$$\begin{array}{ccc} \phi(B_0)^{\mathcal{F}} & \rightarrowtail & A^{\mathcal{F}} \\ {\scriptstyle t(\phi(B_0))}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle tA} \\ \phi(B_0)^{\mathcal{G}} & \rightarrowtail & A^{\mathcal{G}} \end{array} .$$

Consider the class $[1_A] \in A^{\mathcal{F}}$. It gets sent to $[\phi]$ in $A^{\mathcal{G}}$ by $t(A)$ and $[\phi]$ is in $\phi(B_0)^{\mathcal{G}}$ (as ϕ factors through $\phi(B_0)$). So $[1_A]$ is in $\phi(B_0)^{\mathcal{F}}$ and thus there are $A_0 \in \mathcal{F}$ and a restriction

$$\begin{array}{ccc} A & \xrightarrow{\ 1_A\ } & A \\ \uparrow & & \uparrow \\ A_0 & \dashrightarrow & \phi(B_0) \end{array} .$$

So $A_0 \subseteq \phi(B_0)$ and we get $\phi(B_0) \in \mathcal{F}$.

Now assume that ϕ preserves large sets (i.e., elements of the filter) and consider a mono $Y \rightarrowtail X$. We wish to show that

$$\begin{array}{ccc} Y^{\mathcal{F}} & \rightarrowtail & X^{\mathcal{F}} \\ {\scriptstyle tY}\downarrow & & \downarrow{\scriptstyle tX} \\ Y^{\mathcal{G}} & \rightarrowtail & X^{\mathcal{G}} \end{array}$$

is a pullback, so let $[f] \in X^{\mathcal{F}}$ be such that $[f\phi] = tX[f] \in Y^{\mathcal{G}}$. By the lemma, there are $B_0 \in \mathcal{G}$ and a restriction g

$$\begin{array}{ccccc} B & \xrightarrow{\ \phi\ } & A & \xrightarrow{\ f\ } & X \\ \uparrow & & & & \uparrow \\ B_0 & & \overset{g}{\dashrightarrow} & & Y \end{array}$$

and g factors through the image $\phi(B_0)$

$$\begin{array}{ccccc} B & \xrightarrow{\ f\ } & A & \xrightarrow{\ f\ } & X \\ \uparrow & & \uparrow & & \uparrow \\ B_0 & \twoheadrightarrow & \phi(B_0) & \longrightarrow & Y \end{array} .$$

Because $\phi(B_0) \in \mathcal{F}$ we get $[f] \in Y^{\mathcal{F}}$. This gives our pullback and t is taut. ∎

There is a category of filters $\mathbb{F}$ introduced in [18] and further studied in [5], whose objects are pairs $(A, \mathcal{F})$ with A a set and $\mathcal{F}$ a filter on it and whose morphisms are equivalences of functions as in Proposition 2.4.6. It is proved in [18] that the epimorphisms in $\mathbb{F}$ are precisely the $[\phi]$ when ϕ satisfies the conditions of 2.4.8.

2.5. Monads. Another instance where endofunctors appear naturally in category theory is in the theory of monads and, in fact, that's what Manes was studying when he introduced taut functors [12]. His interests lay in applications to computer science (collection monads, e.g.) and categorical topology (T_0 spaces, e.g.), so many of his examples centred around lists, e.g. the free monoid monad, and around filters. Among his examples were indeed the free monoid or semigroup monads and the filter monad $\mathbb{F}$. $\mathbb{F}(X)$ is the set of filters on X well-known to be a monad. He showed that not only is $\mathbb{F}$ taut (functor, unit and multiplication are taut) but a monad $\mathbb{T}$ is taut iff it admits a taut morphism of monads $\mathbb{T} \longrightarrow \mathbb{F}$ [12]. He also showed that any submonad of $\mathbb{F}$ is taut (Theorem 3.12), so that the ultrafilter monad β is taut. See *loc. cit.* for more examples.

As just mentioned the free monoid monad is taut. In fact it's a polynomial monad

$$TX = 1 + X + X^2 + \ldots$$

The free commutative monoid monad is also taut as it is a divided powers monad

$$TX = 1 + X + X^2/S_2 + X^3/S_3 + \dots$$

On the other hand the free group monad or the free Abelian group monad are not taut. Consider, e.g., the free Abelian group monad and the following inverse image diagram in **Set**

$$\begin{array}{ccccc} \emptyset & \rightarrowtail & 2 & 0 & 1 \\ \downarrow & & \downarrow & \downarrow & \downarrow \\ 1 & \rightarrowtail & 2 & 1 & 1 \\ \\ 0 & \longmapsto & 0 & & \end{array} .$$

If we apply T we get

$$\begin{array}{ccccc} 1 & \rightarrowtail & \mathbb{Z} \oplus \mathbb{Z} & & (m, n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \rightarrowtail & \mathbb{Z} \oplus \mathbb{Z} & & (0, m+n) \\ \\ n & \longmapsto & (n, 0) & & \end{array}$$

which is not a pullback. The pullback is $\{(m, n) | m + n = 0\}$. The difference between monoids and groups is that in groups there are equations with different variables on either side, e.g. $xx^{-1} = 1$.

In 1967, Płonka [15] studied universal algebras defined by what he termed regular equations, equations that have the same variables on both sides. Szawiel and Zawadowski [17] have shown that finitary monads are taut (which they call semi-analytic) iff they can be presented by regular equations.

Other non-finitary monads considered by Manes are the covariant power-set monad which is taut and the double dualization monad which is not.

We examine the tautness of the covariant power-set functor P as it comes up in the next section. Consider the inverse image diagram in **Set**

$$\begin{array}{ccc} f^{-1}B & \rightarrowtail & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B & \rightarrowtail & C \end{array} .$$

The pullback of PB along Pf

$$\begin{array}{ccc} (Pf)^{-1}(PB) & \rightarrowtail & PA \\ \downarrow & & \downarrow{\scriptstyle Pf} \\ PB & \rightarrowtail & PC \end{array}$$

consists of the set of all subsets of A, $A_0 \subseteq A$ such that $f(A_0) \subseteq B$ which is equivalent to $A_0 \subseteq f^{-1}B$, i.e. $(Pf)^{-1}(PB) = P(f^{-1}B)$, which means that P is taut.

P doesn't preserve all pullbacks though. Just by cardinality arguments we see that for cardinals $m, n > 2$, the pullback

$$\begin{array}{ccc} mn & \longrightarrow & n \\ \downarrow & & \downarrow \\ m & \longrightarrow & 1 \end{array}$$

is not preserved: the cardinality of $Pm \times_{P1} Pn$ is less than the cardinality of $Pm \times Pn = 2^{m+n}$ which itself is less than 2^{mn}, the cardinality of $P(mn)$.

2.6. Dirichlet series. Classically, Dirichlet series are series of the form

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

which could be written as

$$\sum_{n=1}^{\infty} c_n \left(\frac{1}{n}\right)^s$$

and generalize to **Set**

$$\sum_{i \in I} C_i L_i^X \tag{*}$$

for sets I, C_i, L_i, X. This is the definition given in [13], a coproduct of contravariant representables. It would be nice to get an actual endofunctor of **Set** rather than a functor $\mathbf{Set}^{op} \longrightarrow \mathbf{Set}$. We already have an example of this, the covariant powerset functor $PX = 2^X$. We can bootstrap this to get other examples, $(2^A)^X \cong 2^{A \times X}$. We could also take 3^X whose elements are nested pairs of subsets to which direct image also applies. And so on.

What makes this work is that the base L is a complete lattice, a sup lattice to be precise, and functoriality is given by left Kan extension.

2.6.1. Proposition. *Let L be a sup complete lattice. Then left Kan extension makes L^X into a taut endofunctor of* **Set**.

PROOF. Let $f\colon A \longrightarrow B$ be a function, then

$$L^f\colon L^A \longrightarrow L^B$$

is given by

$$L^f(\phi)(b) = \bigvee_{f(a)=b} \phi(a) \ ,$$

the left Kan extension of ϕ along f

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ & \leq & \\ {\scriptstyle \phi}\searrow & & \swarrow{\scriptstyle L^f_{(f)}} \\ & L & \end{array} \quad .$$

It is well-known, and easily seen, that this makes $L^{(\ )}$ into a functor $\mathbf{Set} \longrightarrow \mathbf{Set}$.

Consider the following inverse image diagram in **Set**

$$\begin{array}{ccc} A_0 & \overset{m}{\rightarrowtail} & A \\ \downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle f} \\ B_0 & \underset{n}{\rightarrowtail} & B \end{array}$$

$A_0 = f^{-1}B_0$. First note that if $\psi_0 \in L^{B_0}$ then its extension to B is given by

$$\begin{array}{rcl} L^n(\psi_0)(b) & = & \bigvee_{\phi(b_0)=b} \psi_0(b_0) \\ & = & \begin{cases} b & \text{if } b \in B_0 \\ \bot & \text{o.w.} \end{cases} \end{array}$$

So we can identify L^{B_0} with its image in L^B, i.e.

$$\{\psi\colon B \longrightarrow L | \psi(b) = \bot \text{ for all } b \notin B_0\}.$$

Now take the pullback

$$\begin{array}{ccc} Pb & \longrightarrow & L^A \\ \downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle L^f} \\ L^{B_0} & \rightarrowtail & L^B \end{array} \ .$$

An element of Pb is $\phi\colon A \longrightarrow L$ such that

$$L^f(\phi) \in L^{B_0}$$

i.e.

$$\bigvee_{f(a)=b} \phi(a) = \bot$$

for all $b \notin B_0$. This means $\phi(a) = \bot$ for all a with $f(a) \notin B_0$, i.e. all $a \notin f^{-1}B_0 = A_0$. Thus Pb is L^{A_0} and L^X is taut. ∎

Other than the few examples of monads mentioned in Section 2.5 all of our examples had rank, i.e. were a small colimit of representables. The functors L^X (and $L^{[X]}$ introduced below) are of unbounded rank (i.e. have no rank), unless $L = 1$.

To see this, define the support of $\phi\colon A \longrightarrow L$ to be $\sigma(\phi) = \{a | \phi(a) \neq \bot\}$. If $f\colon A \longrightarrow B$ and $\psi = L^X(f)(\phi)$, i.e.

$$\psi(b) = \bigvee_{f(a)=b} \phi(a)$$

then $\sigma(\psi)$ is the image of $\sigma(\phi)$ under f as is easily seen. Thus the cardinality of $\sigma(\psi)$ is bounded by that of $\sigma(\phi)$. This means that no set of elements can generate L^X because any element with support bigger than the supports of all the supposed generators could never be attained.

Examples of sup-complete lattices which may be of interest are the open sets of a topological space, the subgroups of a group and of special interest to us here, are the finite ordinals.

The functor L^X is not connected. By Proposition 1.3.12, $\pi_0(L^X) = L^1 = L$, and L^X decomposes into a sum of connected functors, one for each $l \in L$,

$$L^X \cong \sum_{l \in L} F_l \ .$$

The transformation $L^X \longrightarrow L$ is given by sup, so $F_l = \{f\colon X \longrightarrow L \mid \bigvee f(x) = l\}$ which is the same as all functions into $D(l) = \{l' \in L \mid l' \leq l\}$, the down-set of l, whose sup is the top element of $D(l)$, i.e. l itself. This leads to the following:

2.6.2. Definition. Let L be a sup lattice. The *normalized exponential functor* with base L is

$$L^{[X]} = \{f\colon X \longrightarrow L \mid \bigvee f(x) = \top\}$$

where $\top$ is the top element of L.

2.6.3. Proposition.

(1) The normalized exponential is connected and taut.

(2) $L^X \cong \sum_{l \in L} D(l)^{[X]}$.

(3) If L_1, L_2 *are sup lattices, then*

$$L_1^{[X]} \times L_2^{[X]} \cong (L_1 \times L_2)^{[X]} \ .$$

Proof. (1) and (2) are immediate by Corollary 1.3.13. (3) is simply that "sup" and "top" are component-wise in $L_1 \times L_2$. ∎

2.6.4. EXAMPLE. Let $n = \{0, 1, \ldots, n-1\}$ represent the n^{th} cardinal number and $\mathbf{n}$ the corresponding ordinal $0 < 1 < \cdots < n-1$. $\mathbf{n}$ is a complete lattice and its down sets $D(l)$ from above are $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}$ so

$$\mathbf{n}^X \cong \mathbf{1}^{[X]} + \mathbf{2}^{[X]} + \cdots + \mathbf{n}^{[X]} \ .$$

The reason we are interested in connected functors is that it allows for a simple analysis of natural transformations between sums of them. If $\langle F_i \rangle_{i \in I}$ and $\langle G_j \rangle_{j \in J}$ are families of connected endofunctors of **Set**, then a natural transformation

$$t \colon \sum_{i \in I} F_i \longrightarrow \sum_{j \in J} G_j$$

is given by a function $\alpha \colon I \longrightarrow J$ and a family of natural transformations

$$\langle t_i \colon F_i \longrightarrow G_{\alpha(i)} \rangle_{i \in I}.$$

So, if t were an isomorphism then so would α and all the t_i.

If we were to define Dirichlet series as a coproduct

$$\sum_{i \in I} L_i^X$$

as suggested above, there could be different families $\langle L_i \rangle$ giving isomorphic functors, which is not desirable. For example, if $\langle n_i \rangle$ is an unbounded sequence of natural numbers, then the functor

$$\sum_{i \in \mathbb{N}} \mathbf{n}_i^X$$

will have infinitely many $\mathbf{k}^{[X]}$ summads for each k and thus

$$\sum_{i \in \mathbb{N}} \mathbf{n}_i^X \cong \mathbb{N} \times \sum_{k \in \mathbb{N}} \mathbf{k}^{[X]}.$$

So any two unbounded sequences give isomorphic functors.

This motivates the following.

2.6.5. DEFINITION. A *Dirichlet functor* is a coproduct of *normalized exponentials*

$$FX = \sum_{i \in I} L_i^{[X]} \ .$$

So, in particular, by (2) above, L^X defines a Dirichlet functor.

For future reference we record the following:

2.6.6. Proposition. *Dirichlet functors are taut.*

Any natural transformation of Dirichlet functors

$$t\colon \sum_{i\in I} L_i^{[X]} \longrightarrow \sum_{j\in J} M_j^{[X]}$$

must, because the $L_i^{[X]}$ are connected, come from a function $\alpha\colon I \longrightarrow J$ and a family of natural transformations

$$t_i\colon L_i^{[X]} \longrightarrow M_{\alpha(i)}^{[X]} .$$

In particular if

$$\sum_{i\in I} L_i^{[X]} \cong \sum_{j\in J} M_j^{[X]}$$

then there is a bijection $\alpha\colon I \longrightarrow J$ and isomorphisms

$$L_i^{[X]} \cong M_{\alpha(i)}^{[X]} .$$

This raises the question of whether the lattices L_i and $M_{\alpha(i)}$ are themselves isomorphic. This is indeed the case but the proof is not that simple. In fact a similar statement for the non-normalized powers L^X is false.

2.6.7. Proposition. *Any top preserving morphism of sup-lattices $\phi\colon L \longrightarrow M$ induces, by composition, a natural transformation*

$$t_\phi\colon L^{[X]} \longrightarrow M^{[X]} .$$

t_ϕ is taut if and only if ϕ reflects "bottom"

$$\phi(a) = \bot \Rightarrow a = \bot .$$

Proof. For $\lambda\colon X \longrightarrow L$ in $L^{[X]}$, $t_\phi(\lambda)$ is $\phi\lambda\colon X \longrightarrow M$.

$$\bigvee_{x\in X} \phi\lambda(x) = \phi(\bigvee_{x\in X} \lambda(x)) = \phi(\top) = \top ,$$

so $\phi\lambda$ is in $M^{[X]}$.

For naturality, let $f\colon Y \longrightarrow X$ be any function. Then for $\lambda \in L^{[Y]}$ and $x \in X$, going across the top and then down in

$$\begin{array}{ccc} L^{[Y]} & \xrightarrow{L^{[f]}} & L^{[X]} \\ {\scriptstyle t_\phi(Y)}\downarrow & & \downarrow{\scriptstyle t_\phi(X)} \\ M^{[Y]} & \xrightarrow[M^{[f]}]{} & M^{[X]} \end{array}$$

gives

$$t_\phi(X)L^{[f]}(\lambda)(x) = \phi(L^{[f]}(\lambda)(x)) = \phi(\bigvee_{fy=x} \lambda y)$$

and first going down and then over

$$M^{[f]}t_\phi(Y)(\lambda)(x) = M^{[f]}(\phi\lambda)(x) = \bigvee_{fy=x} \phi\lambda(y)$$

which are equal because ϕ preserves sup.

Now assume that ϕ reflects $\bot$ and let f be monic. We can identify $L^{[Y]}$ with its image in $L^{[X]}$, which consists of all $\lambda\colon X \longrightarrow L$ such that $\lambda(x) = \bot$ for $x \notin Y$. And similarly for $M^{[Y]}$.

If $\lambda \in L^{[X]}$ is such that $t_\phi(X)(\lambda)$ is in $M^{[Y]}$, then for $x \notin Y$

$$\phi\lambda(x) = \bot$$

so

$$\lambda(x) = 1$$

and thus λ is in $L^{[Y]}$. The square is a pullback and t_ϕ is taut.

Conversely, if t_ϕ is taut, consider the mono $0\colon 1 \rightarrowtail 2$, giving a pullback

$$\begin{array}{ccc} L^{[1]} & \rightarrowtail & L^{[2]} \\ {\scriptstyle t_\phi(1)}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle t_\phi(2)} \\ M^{[1]} & \rightarrowtail & M^{[2]} \end{array} .$$

The image of $L^{[1]} \rightarrowtail L^{[2]}$ is the singleton $(\top, \bot)$ and similarly for $M^{[1]} \rightarrowtail M^{[2]}$. So if $\phi(a) = \bot$, then $t_\phi(\top, a) = (\top, \bot) \in M^{[1]}$ and then $(\top, a) \in L^{[1]}$, i.e. $a = \bot$. ■

2.6.8. Theorem. *Every natural transformation* $t\colon L^{[X]} \longrightarrow M^{[X]}$ *is of the form* t_ϕ *for a unique top preserving sup-map*

$$\phi\colon L \longrightarrow M\ .$$

Proof. Consider a natural transformation $t\colon L^{[X]} \longrightarrow M^{[X]}$. $t(2)\colon L^{[2]} \longrightarrow M^{[2]}$ takes pairs (a, b) such that $a \vee b = \top$ to pairs $(f(a, b), f'(a, b))$ and naturality of t for the switch map $\sigma\colon 2 \longrightarrow 2$ shows that $f'(a, b) = f(b, a)$. So

$$t(2)(a, b) = (f(a, b), f(b, a))$$

for some function $f\colon L^{[2]} \longrightarrow M$ with $f(a, b) \vee f(b, a) = \top$.

We'll show that $\phi = f(-, \top)\colon L \longrightarrow M$ is a top preserving sup-map and that $t = t_\phi$.

Consider $t(X)\colon L^{[X]} \longrightarrow M^{[X]}$. For $x \in X$, define $\alpha_x\colon X \longrightarrow 2$ by

$$\begin{array}{rcl} \alpha_x(x) & = & 0 \\ \alpha_x(y) & = & 1 \ \text{ for } y \neq x. \end{array}$$

Then $L^{[\alpha_x]}\colon L^{[X]}\longrightarrow L^{[2]}$ takes λ to $(\lambda x, \bigvee_{y\neq x}\lambda y)$ (same for $M^{[\alpha_x]}$). Naturality with respect to α_x

$$\begin{array}{ccc} L^{[X]} & \xrightarrow{t(X)} & M^{[X]} \\ {\scriptstyle L^{[\alpha_x]}}\big\downarrow & & \big\downarrow{\scriptstyle M^{[\lambda_x]}} \\ L^{[2]} & \xrightarrow[t(2)]{} & M^{[2]} \end{array}$$

gives for $\lambda \in L^{[X]}$,

$$(t(X)(\lambda)(x), \bigvee_{y\neq x} t(X)(\lambda)(y)) = (f(\lambda x, \bigvee_{y\neq x}\lambda y), f(\bigvee_{y\neq x}\lambda y, \lambda x)) \tag{1}$$

From which we get

$$t(X)(\lambda)(x) = f(\lambda x, \bigvee_{y\neq x}\lambda y) \tag{2}$$

so that t is completely determined by f, which will be the uniqueness part of our bijection once it's established.

Let $\lambda\colon X\longrightarrow L$ be an arbitrary function and let $l\in L$ be such that $l\vee\bigvee_x \lambda x = \top$ so that $[\lambda, l]\colon X+1\longrightarrow L$ is in $L^{[X+1]}$.

Equality of the second coordinates of (1), when applied to $[\lambda, l]$ with $y = 0$, will give

$$f(\bigvee_x \lambda x, l) = \bigvee_x t(X+1)[\lambda, l](x)$$

and using (2), we get

$$f(\bigvee_x \lambda x, l) = \bigvee_x f(\lambda x, l\vee\bigvee_{y\neq x}\lambda y) \tag{3}$$

If we take $l = \top$, then we get

$$f(\bigvee_x \lambda x, \top) = \bigvee_x f(\lambda x, \top)$$

so $\phi = f(-, \top)$ preserves $\bigvee$. We also have $\phi(\top) = \top$ as

$$\phi(\top) = f(\top, \top) = f(\top, \top)\vee f(\top, \top) = \top.$$

It remains only to show that $t_\phi = t$. We have from (2)

$$t(X)(\lambda)(x) = f(\lambda x, \bigvee_{y\neq x}\lambda y)$$

and

$$t_\phi(X)(\lambda)(x) = \phi\lambda(x) = f(\lambda x, \top).$$

A special case of (3) gives for $a \vee b \vee c = \top$

$$f(a \vee b, c) = f(a, c \vee b) \vee f(b, c \vee a).$$

If $a \vee c = \top$, and $b = a$ we get

$$\begin{aligned} f(a,c) &= f(a \vee a, c) = f(a, c \vee a) \vee f(a, c \vee a) \\ &= f(a, \top) \vee f(a, \top) = f(a, \top) \end{aligned}$$

so that $f(a,c)$ is independent of c and we do get $t = t_\phi$. ∎

2.6.9. Corollary. *If $L^{[X]} \cong M^{[X]}$ then the lattices L and M are isomorphic.*

Proof. The bijection $\phi \longleftrightarrow t_\phi$ is functorial in ϕ. ∎

Perhaps surprisingly, a similar result doesn't hold for the full powers L^X. L^X can be decomposed into a coproduct of reduced powers

$$L^X \cong \sum_{l \in L} D(l)^{[X]}$$

with $D(l) = \{a \in L \mid a \leq l\}$. So a natural transformation

$$t \colon L^X \longrightarrow M^X$$

is given by an arbitrary function $\alpha \colon L \longrightarrow M$ and a family of natural transformations

$$t_l \colon D(l)^{[X]} \longrightarrow D(\alpha(l))^{[X]}$$

which correspond to top preserving sup-maps

$$\phi_l \colon D(l) \longrightarrow D(\alpha(l)).$$

Thus $L^X \cong M^X$ if and only if there is a bijection $\alpha \colon L \longrightarrow M$ such that the down sets $D(l)$ and $D(\alpha l)$ are isomorphic. In particular, $L = D(\top) \cong D(\alpha\top)$ so L is isomorphic to a sublattice of M and vice versa, and if L and M are finite then $L \cong M$, but not in general.

Let L be the closed subset of the unit interval $[0,1]$ given by

$$\{1\} \cup [1/2, 1/3] \cup \{1/4\} \cup [1/5, 1/6] \cup \cdots \cup \{0\}$$

and M given by

$$[1, 1/2] \cup \{1/3\} \cup [1/4, 1/5] \cup \cdots \cup \{0\}.$$

L and M are sup complete because they are closed and bound subsets of $[0,1]$. There are in each case four types of down sets $D(a)$.

(1) If $a \neq 0$ comes from a singleton $\{a\}$ in L or M, then $D(a) \cong L$ and there are countably many in each case.

(2) If a is an interval $[c, d)$ in L or M, then $D(a) \cong M$ and there are the power of the continuum of these for each of L and M.

(3) If a is the right end point of an interval $[c, a]$, then $D(a) \cong L + \{\top\}$ and there are countably many of these for each of L and M.

(4) If $a = 0$ then $D(a) = \{0\}$ in both cases.

We conclude that $L^X \cong M^X$, but L is not isomorphic to M because in L the top element, 1, is isolated but it's not in M.

In a more speculative vein, suppose we want something resembling the familiar Dirichlet series,

$$FX = \sum_{n=1}^{\infty} C_n (1/n)^{[X]}$$

we would need sup lattices n_* to play the role of $1/n$. We would like $n_* \times m_* \cong (nm)_*$ so that our new Dirichlet functors multiply in the familiar way. We also want the nullary version $1_* \cong 1$. This means that n_* is determined (up to isomorphism) by the prime factors of n. In order for these new Dirichlet functors to be closed under the difference operator Δ, to be introduced in the next section, we would like the down sets of n_* to be of the same form. This more or less (though perhaps not quite) forces the following definition.

2.6.10. Definition.

(1) If p is a prime number, then p_* is the totally ordered set

$$p_* = \{q \leq p \mid q \ \text{ is prime}\},$$

including $p_0 = 1$, the 0^{th} prime.

(2) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ is the prime decomposition of n, then $n_* = (p_{1*})^{\alpha_1} \times (p_{2*})^{\alpha_2} \times \cdots \times (p_{k*})^{\alpha_k}$.

So $1_* = \{1\}$, $2_* = \{1, 2\}$, $3_* = \{1, 2, 3\}$, $5_* = \{1, 2, 3, 5\}$, $7_* = \{1, 2, 3, 5, 7\}, \ldots$ that is, we allocate ordinals in order to each prime, so we have

$$1_* \cong \mathbf{1}, \quad 2_* \cong \mathbf{2}, \quad 3_* \cong \mathbf{3}, \quad 5_* \cong \mathbf{4}, \quad 7_* \cong \mathbf{5}, \ldots$$

We then extend this to all n by cartesian product

$$4_* = 2_* \times 2_*, \quad 6_* = 2_* \times 3_*, \quad \ldots \quad 12_* = 2_* \times 2_* \times 3_* \quad \ldots$$

To be precise, we take the prime factors in increasing order, as illustrated above.

The lattices n_* are all different, as one would hope.

2.6.11. PROPOSITION. *Let* $\mathbf{n}_1, \ldots, \mathbf{n}_k$ *and* $\mathbf{m}_1, \ldots, \mathbf{m}_l$ *be finite ordinals* > 1. *If the lattices* $\prod \mathbf{n}_i$ *and* $\prod \mathbf{m}_j$ *are isomorphic then* $k = l$ *and there is a permutation of the subscripts* σ *such that* $\mathbf{n}_i = \mathbf{m}_{\sigma i}$ *for all* i.

PROOF. Let $\phi\colon \prod \mathbf{n}_i \longrightarrow \prod \mathbf{m}_i$ be a lattice isomorphism. The atoms in $\prod \mathbf{n}_i$ are $e_i = (0, \ldots 0, 1, 0 \ldots 0)$ with a 1 in the i^{th} coordinate. Similarly for the atoms e'_j in $\prod \mathbf{m}_j$. ϕ preserves (and reflects) atoms so we have a bijection σ on the subscripts such that $\phi(\mathbf{e}_i) = \mathbf{e}'_{\sigma i}$. In particular, $k = l$.

For any i, the set

$$A_i = \{a \in \prod \mathbf{n}_i \mid e_j \not\leq a \text{ for all } j \neq i\}$$

is the set $\{(0, 0 \ldots r \ldots 0) \mid r \in \mathbf{n}_i\}$ which is isomorphic to $\mathbf{n}_i$.

If B_j is the similarly defined set for $\prod \mathbf{m}_j$, then ϕ restricts to an isomorphism

$$A_i \xrightarrow{\cong} B_{\sigma(i)}$$

and so $\mathbf{n}_i = \mathbf{m}_{\sigma i}$ for all i. ■

With these n_* we can define what we call, for lack of a better name, sequential Dirichlet functors.

2.6.12. DEFINITION. A *sequential Dirichlet functor* is one of the form

$$FX = \sum_{n=1}^{\infty} C_n n_*^{[X]}$$

for any arbitrary sequence of sets $C_n, n \in \mathbb{N}^+$.

2.6.13. PROPOSITION. *Let* $GX = \sum_{n=1}^{\infty} D_n n_*^{[X]}$ *be another sequential Dirichlet functor. Then* $F \times G$ *is also a sequential Dirichlet functor. In fact we have*

$$(F \times G)(X) \cong \sum_{n=1}^{\infty} \sum_{rs=n} (C_r \times D_s) n_*^{[X]} \ .$$

PROOF. This follows simply using distributivity, the isomorphisms $r_*^{[X]} \times s_*^{[X]} \cong (rs)_*^{[X]}$, and then collecting like terms. ■

We end this section with even more speculation.

2.6.14. DEFINITION. The sequential Dirichlet series

$$Z(X) = \sum_{n=1}^{\infty} n_*^{[X]}$$

is called the *zeta functor*.

The Euler product formula is

$$\sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{p\ prime} \sum_{k\in\mathbb{N}} \frac{1}{p^{ks}}\ .$$

We get a similar formula for the Zeta functor though we have to replace the infinite product by the colimit of its finite factors.

Let P be a finite set of primes and P^* the set of all n whose prime factors lie in P.

2.6.15. Proposition.

(1) For P a finite set of primes, we have an isomorphism

$$\sum_{n\in P^*} n_*^{[X]} \cong \prod_{p\in P} \sum_{k\in\mathbb{N}} p_*^{k[X]}$$

(2)

$$Z(X) \cong \varinjlim \prod_{p\in P} \sum_{k\in\mathbb{N}} p_*^{k[X]}$$

where the colimit is taken over all finite sets of primes P.

Proof.

(1) An element on the left is an $n = \prod_{p\in P} p^{k_p}$ and a function

$$\phi\colon X \longrightarrow \prod_{p\in P} p_*^{k_p}$$

whose sup is the top element.

An element on the right is P-tuple of functions

$$\phi_p\colon X \longrightarrow p_*^{k_p}$$

whose sup is the top element, which corresponds bijectively to the ϕ above.

(2) Take $\varinjlim$ of the isos in (1), and note that $Z(X)$ is the colimit of the left sides.

∎

2.6.16. Remark. If we list the primes in increasing order $p_1, p_2, \ldots$, as usual, we can restrict the colimit to the final subset of initial segments and get

$$Z(X) \cong \varinjlim_{n} \prod_{p \,\leq\, p_n} p_*^{k[X]}$$

which some may prefer.

3. The difference operator

Given a functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$, we wish to study how it grows as the input set grows. Given a set A, we want to perturb it a bit $A \rightsquigarrow A'$ and measure the change $FA \rightsquigarrow FA'$. The smallest perturbing of A is simply adding a new element, and the perturbation is the coproduct injection

$$j\colon A \longrightarrow A + 1 \ .$$

We want to see what new elements F has acquired in passing from A to $A+1$, i.e. the elements in the set difference

$$F(A+1) \setminus \mathrm{Im} F(j) \ .$$

If $A \neq 0$ then j is a split mono so that $F(j)$ is also a mono and we can write, by abuse of notation,

$$F(A+1) \setminus FA \ .$$

One shouldn't expect this to be functorial in A but, perhaps somewhat surprisingly, it is for taut functors.

3.1. Definition and functorial properties. Let $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be a functor. For any set A define

$$\Delta[F](A) = F(A+1) \setminus \mathrm{Im}(j)$$

for $j\colon A \longrightarrow A+1$ the coproduct injection.

The $\setminus$ is set difference, not something usually considered by category theorists as it is not functorial. But it is more functorial than one might think. The following lemma will be pivotal in our discussion.

3.1.1. Lemma. *Let $f\colon A \longrightarrow B$ be a function, $A_0 \subseteq A$, $B_0 \subseteq B$ subsets, and assume f restricts to $f_0\colon A_0 \longrightarrow B_0$*

$$\begin{array}{ccc} A_0 & \rightarrowtail & A \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ B_0 & \rightarrowtail & B \end{array} \qquad (*)$$

Then f restricts to $f \setminus f_0 \colon A \setminus A_0 \longrightarrow B \setminus B_0$

$$\begin{array}{ccc} A \setminus A_0 & \rightarrowtail & A \\ {\scriptstyle f\setminus f_0}\downarrow & & \downarrow{\scriptstyle f} \\ B \setminus B_0 & \rightarrowtail & B \end{array} \qquad (**)$$

iff $(*)$ *is a pullback diagram, i.e.* $A_0 = f^{-1}B_0$. *When this is the case,* $(**)$ *will also be a pullback diagram.*

Proof. It is a triviality, though perhaps worth mentioning, that a function $f \colon A \longrightarrow B$ restricts to $A_0 \longrightarrow B_0$ iff

$$a \in A_0 \Rightarrow fa \in B_0 \ ,$$

and the resulting square is a pullback iff

$$a \in A_0 \Leftrightarrow fa \in B_0 \ .$$

This is equivalent to

$$a \notin A_0 \Leftrightarrow fa \notin B_0$$

whence the lemma. ■

We will find it useful to have a name for the functor $X \longmapsto X + 1$. Let's call it S for *successor*

$$SX = X + 1 \ .$$

S could also stand for *shift* as it will be used for precomposing, as in the proposition below.

3.1.2. Proposition. *If $F \colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is taut, then $\Delta[F]$ is a taut subfunctor of FS.*

Proof. Let $f \colon A \longrightarrow B$ be any function. Then, by tautness of F

$$\begin{array}{ccc} FA & \overset{Fj_A}{\rightarrowtail} & F(A+1) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle F(f+1)} \\ FB & \underset{Fj_B}{\rightarrowtail} & F(B+1) \end{array}$$

is a pullback, so by Lemma 3.1.1, $F(f+1)$ restricts to

$$\begin{array}{ccc} F(A+1) \setminus \mathrm{Im}(Fj_A) & \rightarrowtail & F(A+1) \\ \downarrow & & \downarrow{\scriptstyle F(f+1)} \\ F(B+1) \setminus \mathrm{Im}(Fj_B) & \rightarrowtail & F(B+1) \end{array}$$

which makes $\Delta[F]$ into a subfunctor of FS. Furthermore, this square is also a pullback so the inclusion

$$\Delta[F] \rightarrowtail FS$$

is taut, and by Proposition 1.1.2, part (6), $\Delta[F]$ is also taut. ∎

3.1.3. Corollary. *For F taut, the transformation induced by Fj and the inclusion*

$$F + \Delta[F] \longrightarrow FS$$

is an isomorphism.

3.1.4. Definition. $\Delta[F]$ is called the *difference functor* of F.

Notation: Taut functors preserve monos so $F(j)$ is monic, and we will identify FX with its image in $F(X+1)$, so $\Delta[F](X) = F(X+1) \setminus FX$.

3.1.5. Proposition. *If $t\colon F \longrightarrow G$ is a taut transformation, then $tS\colon FS \longrightarrow GS$ restricts to a taut transformation $\Delta t\colon \Delta[F] \longrightarrow \Delta[G]$*

$$\begin{array}{ccc} \Delta[F] & \rightarrowtail & FS \\ {\scriptstyle \Delta[t]}\downarrow & & \downarrow{\scriptstyle tS} \\ \Delta[G] & \rightarrowtail & GS \end{array} .$$

Proof. t is taut so

$$\begin{array}{ccc} FX & \overset{Fj}{\rightarrowtail} & F(X+1) \\ {\scriptstyle tX}\downarrow & & \downarrow{\scriptstyle t(X+1)} \\ GX & \underset{Gj}{\rightarrowtail} & G(X+1) \end{array}$$

is a pullback, so by Lemma 3.1.1, $t(X+1)$ restricts to

$$\begin{array}{ccc} F(X+1) \setminus FX & \rightarrowtail & F(X+1) \\ \downarrow & & \downarrow{\scriptstyle t(X+1)} \\ G(X+1) \setminus GX & \rightarrowtail & G(X+1) \end{array}$$

giving $\Delta[t](X)$ and a pullback square. $\Delta[t]$ is automatically natural.

For any mono $A \rightarrowtail B$ we have

$$\begin{array}{ccccc} \Delta[F]A & \rightarrowtail & \Delta[F]B & \rightarrowtail & F(B+1) \\ {\scriptstyle \Delta[t]A}\downarrow & & \downarrow{\scriptstyle \Delta[t]B} & & \downarrow{\scriptstyle t(B+1)} \\ \Delta[G]A & \rightarrowtail & \Delta[G]B & \rightarrowtail & G(B+1) \end{array} = \begin{array}{ccccc} \Delta[F]A & \rightarrowtail & F(A+1) & \rightarrowtail & F(B+1) \\ {\scriptstyle \Delta[t]A}\downarrow & & \downarrow{\scriptstyle t(A+1)} & & \downarrow{\scriptstyle t(B+1)} \\ \Delta[G]A & \rightarrowtail & G(A+1) & \rightarrowtail & G(B+1) \end{array} .$$

The second, third and fourth squares are pullbacks so the first square is also a pullback. Thus $\Delta[t]$ is taut. ∎

This defines the *difference functor*

$$\Delta\colon \mathbf{Taut} \longrightarrow \mathbf{Taut}$$

on the category of taut endofunctors of **Set** and taut natural transformations.

3.2. Commutation properties. We give functorial analogues of the usual properties of finite differences.

3.2.1. Proposition. *Let $C\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be the constant functor with value C, then $\Delta[C] = 0$.*

3.2.2. Proposition. *For $FX = X$, the identity functor $\mathbf{Set} \longrightarrow \mathbf{Set}$, the difference is $\Delta[X] = 1$.*

3.2.3. Remark. There is a (unique) natural transformation $X \longrightarrow 1$ but if we take differences we get 1 and 0 and there is no transformation $1 \longrightarrow 0$. This shows that the tautness condition in Proposition 3.1.5 is necessary.

For any functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ we write CF for the product of the constant functor C with F, which is isomorphic to the coproduct of C copies of F.

3.2.4. Proposition. $\Delta[CF] \cong C\Delta[F]$.

3.2.5. Proposition. $\Delta[F+G] \cong \Delta[F] + \Delta[G]$.

The two previous propositions, easy to prove directly, are special cases of a much more general result, namely that Δ commutes confluent colimits of taut diagrams.

3.2.6. Theorem. *Let $\mathbf{I}$ be a small confluent category and $\Gamma\colon \mathbf{I} \longrightarrow \mathbf{Taut}$, then we have an isomorphism*

$$\varinjlim_{I} \Delta[\Gamma I] \cong \Delta\big[\varinjlim_{I} \Gamma(I)\big] \ .$$

Proof. If $\alpha\colon I \longrightarrow J$ in $\mathbf{I}$, then by assumption $\Gamma(\alpha)$ is a taut transformation $\Gamma(I) \longrightarrow \Gamma(J)$ and so $\Gamma(\alpha)S$ restricts to a taut natural transformation $\Delta[\Gamma(\alpha)]\colon \Delta[\Gamma I] \longrightarrow \Delta[\Gamma J]$

$$\begin{array}{ccc}
\Delta[\Gamma I] & \rightarrowtail & \Gamma(I)S \\
{\scriptstyle \Delta\Gamma(\alpha)}\downarrow & & \downarrow{\scriptstyle \Gamma(\alpha)S} \\
\Delta[\Gamma J] & \rightarrowtail & \Gamma(J)S \ .
\end{array}$$

This makes $\Delta\Gamma$ into another diagram $\mathbf{I} \longrightarrow \mathbf{Taut}$ and we have a natural isomorphism

$$\Delta[\Gamma I] + \Gamma(I) \xrightarrow{\ \cong\ } \Gamma(I) \circ S \ .$$

Colimits commute with coproducts, and precomposing with S, so

$$\varinjlim_{I} \Delta[\Gamma I] + \varinjlim_{I} \Gamma(I) \xrightarrow{\cong} \big(\varinjlim_{I} \Gamma\big) \circ S$$

which by Proposition 1.3.14 means that

$$\varinjlim_{I} \Delta[\Gamma I] \cong \Delta\big[\varinjlim_{I} \Gamma(I)\big] \ .$$

■

Proposition 3.2.4 is a special case with $\mathbf{I}$ a discrete category with C elements and $\Gamma\colon \mathbf{I} \longrightarrow \mathbf{Taut}$ the constant diagram with value F, and Proposition 3.2.5 with $\mathbf{I} = 2$. Of course, more generally

$$\Delta\Big[\sum_i F_i\Big] \cong \sum_i \Delta[F]_i \ .$$

There is a product rule for finite differences.

3.2.7. Theorem. *If* $F, G\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ *are taut, then*

$$\Delta[F \times G] \cong (\Delta[F] \times G) + (F \times \Delta[G]) + (\Delta[F] \times \Delta[G]) \ .$$

Proof. If we take the product of the two isomorphisms

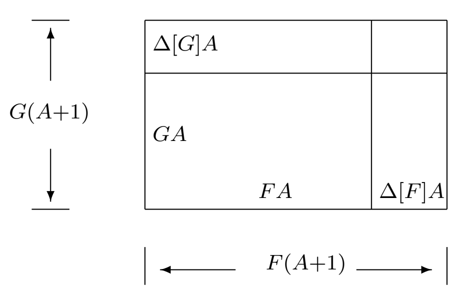
$$\Delta[F] + F \xrightarrow{\cong} FS \quad \text{and} \quad \Delta[G] + G \xrightarrow{\cong} GS$$

and use distributivity of product over sum we get

$$(\Delta[F] \times \Delta[G]) + (\Delta[F] \times G) + (F \times \Delta[G]) + (F \times G) \xrightarrow{\cong} FS \times GS = (F \times G)S$$

and the result follows by Lemma 1.3.14. ■

This is purely a set theoretical result clearly illustrated by



.

The reader can easily write down, or just imagine, the seven terms for $\Delta[F \times G \times H]$ gotten by applying the theorem several times. In fact, easy set theoretical techniques will reveal the formula for infinite products.

3.2.8. THEOREM. *Given a set I and a family $\langle F_i \rangle$ of taut functors* **Set** $\longrightarrow$ **Set**, *we have*

$$\Delta\Big[\prod_{i\in I} F_i\Big] \cong \sum_{J\subsetneq I}\Big(\prod_{j\in J} F_j\Big) \times \Big(\prod_{k\notin J} \Delta[F_k]\Big),$$

(the sum is taken over proper subsets J of I).

To complete the commutativity/distributivity properties of Δ with limits, we have the following.

3.2.9. THEOREM. *Let* **I** *be non-empty and connected, and* $\Gamma\colon \mathbf{I} \longrightarrow \mathbf{Taut}$ *a taut diagram. Then*

$$\Delta\big[\varprojlim_I \Gamma(I)\big] \cong \varprojlim_I \Delta[\Gamma I]\ .$$

PROOF. Because Γ takes its values in **Taut**, we get a diagram $\Delta\Gamma\colon \mathbf{I} \longrightarrow \mathbf{Taut}$ such that

$$\Delta[\Gamma I] + \Gamma(I) \xrightarrow{\ \cong\ } \Gamma(I) \circ S\ ,$$

just like in the proof of Theorem 3.2.6. In **Set**, non-empty connected limits commute with coproducts, so

$$\varprojlim_I \Delta[\Gamma I] + \varprojlim_I \Gamma(I) \xrightarrow{\ \cong\ } \varprojlim \Gamma(I) \circ S$$

and the result follows by Proposition 1.3.14. ∎

Although arbitrary limits of taut functors are taut, for the theorem it is necessary that the transition morphisms $\Gamma(\alpha)\colon \Gamma(I) \longrightarrow \Gamma(J)$ be taut. One sees the problem immediately when attempting to apply Δ to the pullback

$$\begin{array}{ccc} F \times G & \longrightarrow & G \\ \downarrow & & \downarrow \\ F & \longrightarrow & 1\ . \end{array}$$

3.3. THE LAX CHAIN RULE. Generally speaking there is no good chain rule for finite differences, notwithstanding the work of Alvarez-Picallo and Pacaud-Lemay [1], which deals with a different situation. Ideally, we would have

$$\Delta[G \circ F] \cong (\Delta[G] \circ F) \times \Delta[F]$$

but this fails even for such simple functors as $F(X) = G(X) = X^2$. Indeed, an easy calculation shows that in this case

$$\Delta[G \circ F](X) \cong 4X^3 + 6X^2 + 4X + 1$$

whereas

$$(\Delta[G] \circ F(X)) \times \Delta[F](X) \cong 4X^3 + 2X^2 + 2X + 1\ .$$

However, for functors, there is a lot of extra room to maneuver and we get a comparison morphism, which will be an isomorphism only in the simplest of cases as the above example shows, but with good properties nonetheless.

We will sometimes write $\circ$ for composition of functors, where we think it makes things clearer.

3.3.1. Theorem. *For taut functors* $F, G\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ *we have a natural comparison*

$$\gamma\colon (\Delta[G] \circ F) \times \Delta[F] \longrightarrow \Delta[G \circ F]$$

the chain rule transformation. γ *is taut and monic.*

Proof. Let A be a set and take an element $x \in \Delta[F](A)$. This gives a function

$$\phi_x = [Fj_A, x]\colon FA + 1 \longrightarrow F(A+1) \ ,$$

which is Fj_A on the first summand and x on the second. As x is not in the image of Fj_A, ϕ_x is monic and

$$\begin{array}{ccc} FA & \overset{j_{FA}}{\rightarrowtail} & FA+1 \\ \Vert & & \downarrow{\scriptstyle \phi_x} \\ FA & \underset{Fj_A}{\rightarrowtail} & F(A+1) \end{array}$$

is a pullback. G is taut so

$$\begin{array}{ccc} GFA & \overset{G(j_{FA})}{\rightarrowtail} & G(FA+1) \\ \Vert & & \downarrow{\scriptstyle G(\phi_x)} \\ GFA & \underset{GF(j_A)}{\rightarrowtail} & GF(A+1) \end{array}$$

is also a pullback, and by Lemma 3.1.1, $G(\phi_x)$ restricts to γ_x giving another pullback

$$\begin{array}{ccc} \Delta[G](FA) & \rightarrowtail & G(FA+1) \\ {\scriptstyle \gamma_x}\downarrow & & \downarrow{\scriptstyle G(\phi_x)} \\ \Delta[G \circ F](A) & \rightarrowtail & GF(A+1) \ . \end{array}$$

If we put all these γ_x together by taking the coproduct of the top arrows we get our $\gamma = [\gamma_x]_x$ and

$$\begin{array}{ccc} \Delta[G](FA) \times \Delta[F]A & \rightarrowtail & G(FA+1) \times \Delta[F]A \\ {\scriptstyle [\gamma_x]_x}\downarrow & & \downarrow{\scriptstyle [G(\phi_x)]_x} \\ \Delta[G \circ F](A) & \rightarrowtail & GF(A+1) \ , \end{array}$$

also a pullback.

We have one such γ for each A, so we should write $\gamma(A) = [\gamma_x(A)]_x$. To check naturality of γ it is sufficient to check the commutativity of the naturality square on each injection, that is that the square labelled (?) below commutes for each $x \in \Delta[F](A)$:

$$\begin{array}{ccccc} \Delta[G](FA) & \xrightarrow{\Delta[G](Ff)} & \Delta[G](FB) & \rightarrowtail & G(FB+1) \\ \downarrow{\scriptstyle \gamma_x(A)} & (?) & \downarrow{\scriptstyle \gamma_y(B)} & (1) & \downarrow{\scriptstyle G(\phi_y)} \\ \Delta[G\circ F](A) & \xrightarrow[\Delta[G\circ F](f)]{} & \Delta[G\circ F](B) & \rightarrowtail & GF(B+1) \end{array} ,$$

this for an arbitrary function $f\colon A \longrightarrow B$ and $y = F(f+1)(x)$. Compare this with the following diagram

$$\begin{array}{ccccc} \Delta[G](FA) & \rightarrowtail & G(FA+1) & \xrightarrow{G(Ff+1)} & G(FB+1) \\ \downarrow{\scriptstyle \gamma_x(A)} & (2) & \downarrow{\scriptstyle G(\phi_x)} & (3) & \downarrow{\scriptstyle G(\phi_y)} \\ \Delta[G\circ F](A) & \rightarrowtail & GF(A+1) & \xrightarrow[GF(f+1)]{} & GF(B+1) \end{array} .$$

The composites of the two top arrows of each diagram are equal by definition of the functoriality of $\Delta[G]$, and the same holds for the bottom arrows but for $\Delta[G\circ F]$. (1) and (3) commute by definition of γ_x and γ_y respectively. As the bottom arrow of (1) is monic, (?) will commute if (3) does.

(3) is G of the diagram

$$\begin{array}{ccc} FA+1 & \xrightarrow{Ff+1} & FB+1 \\ \downarrow{\scriptstyle \phi_x} & (4) & \downarrow{\scriptstyle \phi_y} \\ F(A+1) & \xrightarrow[F(f+1)]{} & F(B+1) \end{array}$$

which on injections is

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow{\scriptstyle Fj_A} & (5) & \downarrow{\scriptstyle Fj_B} \\ F(A+1) & \xrightarrow[F(B+1)]{} & F(B+1) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 1 & = & 1 \\ \downarrow{\scriptstyle x} & (6) & \downarrow{\scriptstyle y} \\ F(A+1) & \xrightarrow[F(f+1)]{} & F(B+1) \end{array}$$

each of which commutes, the second by definition of y. So (?) commutes establishing naturality of γ.

If f is monic so is $F(f+1)$ and then (6) is a pullback. (5) is always a pullback by tautness so in this case (4) is a pullback, so (3) is too, and (1) and (2) are pullbacks by definition of γ_x and γ_y, so (?) will be a pullback, showing that γ is taut.

To show that γ is monic, first recall that for each x, γ_x itself is monic (being the restriction of $G(\phi_x)$). So it is only necessary to show that the γ_x are pairwise disjoint. Let $x \neq x'$, then

$$\begin{array}{ccc} FA & \xrightarrow{j_{FA}} & FA+1 \\ {\scriptstyle j_{FA}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_x} \\ FA+1 & \xrightarrow[\phi_{x'}]{} & F(A+1) \end{array}$$

is a pullback and G of it is too. So we have the following pullbacks

$$\begin{array}{ccccc} 0 & \longrightarrow & GFA & \xrightarrow{G(j_{FA})} & G(FA+1) \\ \big\downarrow & & {\scriptstyle G(j_{FA})}\big\downarrow & & \big\downarrow{\scriptstyle G(\phi_x)} \\ \Delta[G](FA) & \longrightarrow & G(FA+1) & \xrightarrow[G(\phi_{x'})]{} & GF(A+1) \end{array}.$$

The bottom arrow can be written as $\gamma_{x'}$ followed by the inclusion, so if we pull back in stages we get

$$\begin{array}{ccccc} 0 & \longrightarrow & \Delta[G](FA) & \longrightarrow & G(FA+1) \\ \big\downarrow & & \big\downarrow{\scriptstyle \gamma_x} & & \big\downarrow{\scriptstyle G(\phi_x)} \\ \Delta[G](FA) & \xrightarrow[\gamma_{x'}]{} & \Delta[G \circ F](A) & \longrightarrow & GF(A+1) \end{array}$$

so γ_x and $\gamma_{x'}$ are disjoint, giving the desired result, that γ is monic. ∎

For $x \in \Delta[F]A$ and $y \in \Delta[G](FA)$, we have

$$\gamma(y, x) = G(\phi_x)(y)$$

which in fact is defined for all $x \in F(A+1)$ and $y \in G(FA+1)$ giving an element of $GF(A+1)$. This "full γ" is neither taut nor monic. The above proof shows, in part, that if $x \notin FA$ and $y \notin GFA$, then $\gamma(y, x) \notin GFA$, and with these restrictions we do get tautness and monicity.

The chain rule transformation is natural in F and G.

3.3.2. Theorem. *Let F, F', G, G' be taut functors and $t\colon F \longrightarrow F'$ $u\colon G \longrightarrow G'$ be taut transformations, then the following diagram commutes*

$$\begin{CD}
(\Delta[G]\circ F)\times\Delta[F] @>{\gamma_{G,F}}>> \Delta[G\circ F]\\
@V{(\Delta u)\circ F\times\Delta t}VV @VV{\Delta(u\circ t)}V\\
(\Delta[G']\circ F')\times\Delta[F'] @>>{\gamma_{G',F'}}> \Delta(G'\circ F')\ .
\end{CD}$$

Proof. In fact, for any A

$$\begin{CD}
G(FA+1)\times F(A+1) @>{\gamma_{G,F}}>> GF(A+1)\\
@V{u\circ(tA+1)\times t(A+1)}VV @VV{(u\circ t)(A+1)}V\\
G'(F'A+1)\times F'(A+1) @>>{\gamma_{G',F'}}> G'F'(A+1)
\end{CD}$$

commutes, *a fortiori* its restriction to the diagram of the statement. The $u\circ t$ on the right is the horizontal composition of natural transformations and expands to the composite on the right below. The $u\circ(tA+1)$ is also a horizontal composition and can be written as on the left here:

$$\begin{CD}
G(FA+1)\times F(A+1) @>{\gamma_{G,F}}>> GF(A+1)\\
@V{u(FA+1)\times F(A+1)}VV @VV{uF(A+1)}V\\
G'(FA+1)\times F(A+1) @>>{\gamma_{G',F}}> G'F(A+1)\\
@V{G'(tA+1)\times t(A+1)}VV @VV{G't(A+1)}V\\
G'(F'A+1)\times F'(A+1) @>>{\gamma_{G',F'}}> G'F'(A+1)\ .
\end{CD}$$

For $x\in F(A+1)$, the restriction of the top square to the x^{th} injection is

$$\begin{CD}
G(FA+1) @>{G(\phi_x)}>> GF(A+1)\\
@V{u(FA+1)}VV @VV{uF(A+1)}V\\
G'(FA+1) @>>{G(\phi_x)}> G'F(A+1)
\end{CD}$$

which commutes by naturality of u.

The second diagram, restricted to the x^{th} injection is

$$\begin{array}{ccc} G'(FA+1) & \xrightarrow{G'(\phi_x)} & G'F(A+1) \\ {\scriptstyle G'(tA+1)}\downarrow & & \downarrow{\scriptstyle G't(A+1)} \\ G'(F'A+1) & \xrightarrow[G'(\phi_{x'})]{} & G'F'(A+1) \end{array}$$

where $x' = t(A+1)(x)$. This diagram is G' of

$$\begin{array}{ccc} FA+1 & \xrightarrow{[F_{j_A},x]} & F(A+1) \\ {\scriptstyle tA+1}\downarrow & & \downarrow{\scriptstyle t(A+1)} \\ F'A+1 & \xrightarrow[{[F'_{j_A},x']}]{} & F'(A+1) \end{array}$$

which commutes, on the first summand by naturality of t and on the second by definition of x'. ∎

We also have the following associativity and unit laws for γ.

3.3.3. THEOREM. *For taut functors F, G, H we have the following commutativities:*
(1)

$$\begin{array}{ccc} (\Delta[H]\circ G\circ F)\times(\Delta[G]\circ F)\times\Delta[F] & \xrightarrow{\mathrm{id}\times\gamma_{G,F}} & (\Delta[H]\circ G\circ F)\times\Delta[G\circ F] \\ {\scriptstyle \gamma_{H,G}\circ F\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \gamma_{H,G\circ F}} \\ (\Delta[H\circ G]\circ F)\times\Delta[F] & \xrightarrow[\gamma_{H\circ G,F}]{} & \Delta[H\circ G\circ F] \end{array} ,$$

(2)

$$\begin{array}{ccc} (\Delta[\mathrm{Id}]\circ F)\times\Delta[F] & \xrightarrow{\gamma_{\mathrm{Id},F}} & \Delta[\mathrm{Id}\circ F] \\ \| & & \| \\ 1\times\Delta[F] & \xrightarrow{\cong} & \Delta[F] \end{array} ,$$

(3)

$$\begin{array}{ccc} (\Delta[F]\circ\mathrm{Id})\times\Delta[\mathrm{Id}] & \xrightarrow{\gamma_{F,\mathrm{Id}}} & \Delta[F\circ\mathrm{Id}] \\ \| & & \| \\ \Delta[F]\times 1 & \xrightarrow{\cong} & \Delta[F] \end{array} .$$

Proof. (1) Let A be a set. We'll show that

$$\begin{array}{ccc} H(GFA+1)\times G(FA+1)\times F(A+1) & \xrightarrow{\mathrm{id}\times\gamma_{G,F}(A)} & H(GFA+1)\times GF(A+1) \\ {\scriptstyle \gamma_{HG}(FA)\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \gamma_{H,GF}(A)} \\ HG(FA+1)\times F(A+1) & \xrightarrow[\gamma_{HG,F}(A)]{} & HGF(A+1) \end{array}$$

commutes. Evaluate this diagram at an element (x,y,z) of the domain

$$\begin{array}{ccc} (z,y,x) & \longmapsto & (x,G(\phi_x)(y))=(z,w) \\ \downarrow & & \downarrow \\ (H(\phi_y)(z),x) & \longmapsto & HG(\phi_x)H(\phi_y)(z) \overset{?}{=} H(\phi_w)(z)\ . \end{array}$$

So we have to show that $HG(\phi_x)H(\phi_y)=H(\phi_w)$ for $w=G(\phi_x)(y)$, and it is sufficient to show that $G(\phi_x)\phi_y=\phi_w$, i.e. that

$$\begin{array}{ccc} GFA+1 & \xrightarrow{\phi_y} & G(FA+1) \\ & {\scriptstyle \phi_w}\searrow & \downarrow{\scriptstyle G(\phi_x)} \\ & & GF(A+1) \end{array}$$

commutes. Restricting to the summands we have

$$\begin{array}{ccc} GFA & \xrightarrow{Gj_{FA}} & G(FA+1) \\ & {\scriptstyle GF(j_A)}\searrow & \downarrow{\scriptstyle G[Fj_A,x]} \\ & & GF(A+1) \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{y} & G(FA+1) \\ & {\scriptstyle w}\searrow & \downarrow{\scriptstyle G(\phi_x)} \\ & & GF(A+1) \end{array}$$

each of which commutes, the first by functoriality of G, the second by definition of w. This proves (1).

In (2) (and (3)) we denote the identity functor on **Set**, which we have been calling X, by Id. Also 1 denotes the constant functor with value 1, i.e. the terminal endofunctor. Then $\Delta[\mathrm{Id}](X)=(X+1)\setminus X=1$, i.e. $\Delta[\mathrm{Id}]=1$. To calculate $\gamma_{\mathrm{Id},F}$, take $x\in F(A+1)$ and consider γ on the x^{th} summand:

$$\mathrm{Id}(\phi_x)\colon \mathrm{Id}(FA+1)\longrightarrow \mathrm{Id}F(A+1)$$

i.e. just $\phi_x\colon FA+1\longrightarrow F(A+1)$. Then $y\in\Delta[\mathrm{Id}](FA)$ must be $*$ and $\phi_x(*)=x$. So

$$\gamma_{\mathrm{id},F}(*,x)=x$$

which is what (2) is asserting.

For (3) consider

$$\gamma\colon F(\mathrm{Id}A+1)\times \mathrm{Id}(A+1) \longrightarrow F\mathrm{Id}(A+1)$$

i.e.

$$\gamma\colon F(A+1)\times (A+1) \longrightarrow F(A+1)\ .$$

For $x \in A+1$, $\phi_x\colon A+1 \longrightarrow A+1$ is the identity on A but $\phi_x(*) = x$. If we take $x \in \Delta[\mathrm{Id}](A)$, x must be $*$ so $\phi_* = 1_{A+1}\colon A+1 \longrightarrow A+1$ and $\gamma_* = 1_{F(A+1)}$. This, when restricted to $\Delta[F](A)\times 1$, is what (3) says. ∎

Getting a comparison

$$\gamma\colon \Delta[G]\circ F \times \Delta[F] \rightarrowtail \Delta[G\circ F]$$

in that direction is a bit surprising. Normally we would expect a morphism *into* a product and, as $\Delta[G\circ F]$ is a kind of cokernel, a morphism out of it. One might think that there is a comparison in the reverse direction for which γ is a splitting. Looking more carefully we see that it seems unlikely because we would need a natural transformation

$$\Delta[G\circ F] \longrightarrow \Delta[F]\ .$$

So from an element of $GF(X+1)$ we would need to construct an element of $F(X+1)$ in a natural way. Nothing comes to mind but it's a bit difficult to pin down precisely.

Consider the following example which illustrates well the nature of γ. Let $F(X)$ be arbitrary (taut) and $G(X) = X^3$. Then $\Delta[G](X) = 3X^2+3X+1$ so that

$$\Delta[G](FX)\times\Delta[F](X) = 3\Big(F(X)^2\times\Delta[F](X)\Big) + 3\Big(F(X)\times\Delta[F](X)\Big) + \Delta[F](X)\ .$$

On the other hand $(G\circ F)(X) = F(X)\times F(X)\times F(X)$ so we can use the product rule (Theorem 3.2.7) (three times) to get

$$\Delta[G\circ F](X) = 3\Big(F(X)^2\times\Delta[F]X\Big) + 3\Big(F(X)\times(\Delta[F](X))^2\Big) + \Big(\Delta[F](X)\Big)^3\ .$$

Then γ is the identity on the first three of the seven summands, and given by diagonals on the remaining ones. As γ is component-wise on the summands, any splitting would have to be too. For the first three summands there is only one choice, but for summands of the form $F(X)\times\Delta[F](X) \longrightarrow F(X)\times\Delta[F](X)^2$ we have the two projections, and for the last summand there are three. This gives us $2\cdot 2\cdot 2\cdot 3 = 24$ "canonical" splittings (for γ at F and G). But there may be many more depending on F. The simplest possible $\Delta[F](X)$ is $X+1$ for $F(X) = X^{[2]} = X^2/S_2$. Then the component of γ on the last summand

$$\Delta[F](X) \longrightarrow (\Delta[F](X))^3 \quad = \quad (X+1 \longrightarrow X^3+3X^2+3X+1)$$

$$\begin{array}{ccc} x & \longmapsto & (x,x,x) \\ * & \longmapsto & * \ . \end{array}$$

Now there are only three splittings of $X \longrightarrow X^3$, the projections (Yoneda) and only one for $1 \longrightarrow 1$, but it's arbitrary for the six terms in the middle, giving 648 splittings, by our count (could be wrong, but there are lots of them). But none is natural in G. Any permutation of 3, $\sigma \in S_n$ gives an automorphism of G which percolates down to the same permutation on the X^3 in $(\Delta[F](X))^3$, so no one projection would be invariant. The upshot is that there is no global splitting of γ natural in F and G.

It will be useful in Subsection 4.5 if we express the difference operator with its lax chain rule in tangent category terms (see [6] for definitions). Our tangent space for **Set** is $P_1\colon \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$. Given a taut functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$, we define $D(F)\colon \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set} \times \mathbf{Set}$ by

$$D(F)(A,B) = (FA, \Delta[F](A) \times B)\ .$$

Note that $D(F)$ is "linear" in the second variable, i.e. $D(F)(A,-)$ preserves colimits, and as such is completely determined by $D(F)(A,1)$.

The chain rule says that D is a monoidal functor on taut endofunctors.

3.3.4. THEOREM. *Let* $\mathbf{End}_{Taut}(\mathbf{Set})$ *and* $\mathbf{End}_{Taut}(\mathbf{Set} \times \mathbf{Set})$ *be the strict monoidal categories of taut endofunctors on* **Set** *and* $\mathbf{Set} \times \mathbf{Set}$ *respectively, with taut natural transformations, and tensor given by composition. Then D defines a monoidal functor*

$$\mathbf{End}_{Taut}(\mathbf{Set}) \longrightarrow \mathbf{End}_{Taut}(\mathbf{Set} \times \mathbf{Set})\ .$$

In the present context we might even go so far as to write a general object of $\mathbf{Set} \times \mathbf{Set}$ as $(A, \Delta A)$ where ΔA is just another object, independent of A but thought of as an increment in A, just like the dx in $f(x)dx$. Then

$$D(F)(A, \Delta A) = (FA, \Delta[F](A) \times \Delta A)\ .$$

4. Differences for the special classes

We give explicit formulas for the difference operator on the various classes of taut functors studied in Section 2.

4.1. POLYNOMIALS. Let $F(X) = X^A$ for some set A. An element $\phi \in \Delta[F](X) = (X+1)^A \setminus X^A$ is a function $\phi\colon A \longrightarrow X+1$ with $* \in \mathrm{Im}(\phi)$ ($*$ is the unique element of 1). If $S = \{a \in A | \phi(a) \neq *\}$ is the support of ϕ, then ϕ is uniquely determined by its restriction to S, and the condition $* \in \mathrm{Im}(\phi)$ is that S is a proper subset of A. This produces a bijection

$$\Delta[X^A] \cong \sum_{S \subsetneq A} X^S$$

where the coproduct is taken over all proper subsets of A. The bijection is natural as $(f+1)^A\colon (X+1)^A \longrightarrow (Y+1)^A$ preserves support for any $f\colon X \longrightarrow Y$.

A polynomial functor is a small coproduct of powers

$$P(X) = \sum_{i \in I} X^{A_i}$$

for a family $\langle A_i \rangle_{i \in I}$ of sets and as Δ commutes with coproducts we have

$$\Delta[P](X) = \sum_{S \subsetneq A_i} X^S$$

where the coproduct is taken over all $i \in I$ and proper subsets $S \subsetneq A_i$.

If A is a finite cardinal n, we can group like powers of X together to get

$$\Delta[X^n] = \sum_{k=0}^{n-1} \binom{n}{k} X^k$$

where $\binom{n}{k}$ is, as usual, the binomial coefficient.

This is all we need for the following result.

4.1.1. Proposition.

(1) If $P(X)$ is a polynomial functor then so is $\Delta[P](X)$

(2) If $P(X)$ is a power series functor, so is $\Delta[P](X)$

(3) If $P(X)$ is a finitary polynomial functor, so is $\Delta[P](X)$.

4.2. Divided powers. Let n be a positive integer. Then $X^{[n]} = X^n/S_n$ consists of equivalence classes of n-tuples $[x_1 \dots x_n]$ with $[x_1 \dots x_n] = [y_1 \dots y_n]$ iff there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$ for all i. So $\Delta[X^{[n]}]$ consists of such equivalence classes with $x_i \in X$ or $x_i = *$ for all i and at least one $x_i = *$. Within each of these equivalence classes there are those n-tuples with all the $*$'s at the end. If there are k x's for X (and $n-k$ stars) then the equivalence relation reduces to the existence of $\sigma \in S_k$ with $y_i = x_{\sigma i}$. This way we see that

$$\Delta[X^{[n]}] \cong X^{[0]} + X^{[1]} + \cdots + X^{[n-1]}.$$

If FX is a divided power series

$$FX = C_0 + C_1 X^{[1]} + C_2 X^{[2]} + \cdots = \sum_{n=0}^{\infty} C_n X^{[n]}$$

then

$$\begin{aligned}\Delta[F](X) &= \sum_{i=1}^{\infty} C_i + \left(\sum_{i=2}^{\infty} C_i\right) X^{[1]} + \left(\sum_{i=3}^{\infty} C_i\right) X^{[2]} + \dots \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=n+1}^{\infty} C_i\right) X^{[n]}.\end{aligned}$$

4.2.1. Proposition. *If $F(X)$ is a divided power series then so is $\Delta[F](X)$.*

$\Delta[X^{[A]}]$ for infinite sets A is more complicated. For example, for $A = \mathbb{N}$, we have for each n, equivalence classes of the form $(x_0, x_1, \ldots, x_{n-1}, *, *, *, \ldots)$ giving an $X^{[n]}$ summand, but there are also equivalence classes of the form

$$(* * * \cdots * \, x_n x_{n+1} x_{n+2} \ldots), n > 0$$

giving countably many $X^{[\mathbb{N}]}$ summands. And there's even one more type of equivalence class with infinitely many $*$'s and infinitely many $x_i \in X$, which we can represent as

$$(x_0, *, x_2, *, x_4, *, \ldots)$$

This gives us an extra copy of $X^{[\mathbb{N}]}$. Thus

$$\Delta[X^{[\mathbb{N}]}] \cong \mathbb{N} \times X^{[\mathbb{N}]} + \sum_{n=0}^{\infty} X^{[n]} \ .$$

A more general approach will lead to a better understanding. Let G be a group and A a left G-set. Then

$$\Delta[X^A/G]$$

can be described as follows. G acts on the set $P'A$ of proper subsets of A by application of the action element-wise

$$B \subsetneq A \quad \overset{g}{\longmapsto} \quad gB = \{gb \,|\, b \in B\}.$$

Let $S \subseteq P'A$ be a choice of representative for each orbit of this action. Then we have:

4.2.2. Proposition.

$$\Delta[X^A/G] \cong \sum_{B \in S} A^B/\mathrm{Stab}(B)$$

where $\mathrm{Stab}(B) = \{g \in G \,|\, gB = B\}$, *the stabilizer of* B.

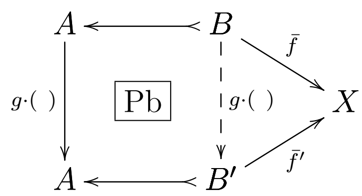
Proof. An element of $\Delta[X^A/G]$ is an equivalence class $[f]$ of functions $f\colon A \longrightarrow X + 1$ with $* \in \mathrm{Im}(f)$. The equivalence relation is given by $f \sim f'$ iff there is $g \in G$ with

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X+1 \\ {\scriptstyle g\cdot(\ )}\downarrow & \nearrow_{f'} & \\ A & & \end{array} \ .$$

A function into $X + 1$ is equivalent to a partial function

$$A \leftarrowtail B \xrightarrow{\ \bar{f}\ } X$$

and $* \in \mathrm{Im}(f)$ is equivalent to B being a proper subset of A. The equivalence relation translates to the existence of $g \in G$ such that



(*)

and the square being a pullback is equivalent to $g(B) = B'$.

Any $(B', \bar{f}')$ is equivalent to a $(B, \bar{f})$ with $B \in S$ (there's a $g \in G$ which maps B' to B, and defines $\bar{f}$ by $(*)$ above), and for these (B, f) the equivalence relation reduces to $(*)$ with $g \in \mathrm{Stab}(B)$. ∎

4.2.3. Corollary. *The class of (generalized) divided power series is closed under the difference operator* Δ.

4.3. Analytic functors.

4.3.1. Proposition. *Let C be a left S_n-set ($n \in \mathbb{N}$). Then there exist S_k-sets C_k, $k = 0, 1, \ldots, n-1$ such that*

$$\Delta[X^n \otimes_{S_n} C] \cong \sum_{k=0}^{n-1} X^k \otimes_{S_k} C_k \ .$$

Proof. An element of $\Delta[X^n \otimes_{S_n} C] = \big((X+1)^n \otimes_{S_n} C\big) \setminus (X^n \otimes_{S_n} C)$ is an equivalence class $[x_1, \ldots, x_n; c]$ where $c \in C$ and $x_i \in X$ or $x_i = *$ for all i, with at least one $*$. $[x_1, \ldots, x_n; c] = [y_1, \ldots, y_n; d]$ iff there exists $\sigma \in S_n$ such that $y_i = x_{\sigma i}$ for all i and $c = \sigma d$.

The number of $*$'s is invariant under the action of S_n so is an invariant of the equivalence class, but it is also invariant under the functorial action (i.e. for functions $f\colon X \longrightarrow Y$), so that $\Delta[X^n \otimes_{S_n} C]$ decomposes into a coproduct

$$\sum_{k=0}^{n-1} \Phi_k$$

of endofunctors Φ_k, where k is the number of x's that are *not* stars.

Let C_k be the set of equivalence classes $[c]$ for $c \in C$ with $c \sim d$ iff there is $\sigma \in S_n$ such that $\sigma(i) = i$ for all $i \le k$ and $c = \sigma d$. S_k acts on C_k by $\tau[c] = [\bar{\tau} c]$ where $\bar{\tau}$ is τ extended to a permutation on n by the identity, i.e.

$$\bar{\tau}(i) = \begin{cases} \tau(i) & 1 \le i \le k \\ i & k < i \le n \ . \end{cases}$$

The action is well defined because such an extension will commute with the σ's above as they act on disjoint sets. So

$$\tau[c] = [\bar{\tau}c] = [\bar{\tau}\sigma d] = [\sigma\bar{\tau}d] = [\bar{\tau}d] \ .$$

The claim is that $\Phi_k \cong X^k \otimes_{S_k} C_k$. Each equivalence class in Φ_k has a representative of the form $[x_1, \ldots x_k, *, \ldots, *; c]$ (several in fact).

We define

$$\phi \colon \Phi_k \longrightarrow X^k \otimes_{S_k} C_k$$

$$\phi[x_1 \ldots x_k, *, \ldots, *; c] = [x_1, \ldots x_k, [c]] \ .$$

ϕ is well defined. Indeed, let $[x_1, \ldots, x_k, *, \ldots, *; c] = [y_1 \ldots y_k, *, \ldots, *; d]$ so there exists $\sigma \in S_n$ such that $c = \sigma d$, $y_i = x_{\sigma i}$, and the $*$'s get permuted among themselves. Let $\tau \in S_k$ be σ restricted to $\{1, \ldots, k\}$, and $\bar{\tau}$ the extension of τ to $\{1, \ldots, n\}$ by the identity on $i > k$. Then

$$\tau[d] = [\bar{\tau}d] = [\bar{\tau}\sigma^{-1}c] = [c]$$

because $\bar{\tau}$ and σ agree on $i \le k$, i.e. $\bar{\tau}\sigma^{-1}(i) = i$. As we have $y_i = x_{\sigma(i)} = x_{\tau(i)}$ for $i \le k$, we have

$$[x_1, \ldots, x_k; [c]] = [y_1, \ldots, y_k; [d]]$$

and ϕ is well defined.

We show that ϕ is one to one. Suppose $[x_1, \ldots, x_k; [c]] = [y_1, \ldots, y_k; [d]]$, i.e. there exists $\tau \in S_k$ such that $y_i = x_{\tau i}$ and $\tau[d] = [c]$. Thus $[\bar{\tau}d] = [c]$ ($\bar{\tau}$ as above) which means there is a $\sigma \in S_n$ such that $\sigma(i) = i$ for $i \le k$ and $c = \sigma\bar{\tau}d$. For $i > k$, $\sigma(i) > k$ so

$$\begin{array}{ll} \sigma\bar{\tau}(i) = \tau(i) & i \le k \\ \sigma\bar{\tau}(i) > k & i > k \end{array}$$

giving our permutation in S_n making

$$[x_1 \ldots x_n, *, \ldots, *; c] = [y_1, \ldots y_k, *, \ldots, *; d] \ .$$

It is clear that ϕ is onto and just as clearly natural, and so is the required isomorphism. ∎

4.3.2. Corollary. *If $C \colon \mathbf{Bij} \longrightarrow \mathbf{Set}$ is a species and*

$$FX = \sum_{n=0}^{\infty} X^n \otimes_{S_n} C(n)$$

the corresponding analytic functor, then

$$\Delta[F](X) \cong \sum_{n=0}^{\infty} X^n \otimes_{S_n} \left(\sum_{l=n+1}^{\infty} C(l)_n \right) \ .$$

Proof. By the proposition

$$\Delta[F](X) \cong \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} X^k \otimes_{S_k} C(n)_k \right)$$

and grouping like powers of X together we get the desired result. ∎

The C_k above may seem mysterious, at the very least a bit opaque. Passing to permutations on infinite sets will force us into a more conceptual presentation. Rather than choose a proper subset of each cardinality as we did in the case of $[n]$ we consider them all at once.

Let A be an arbitrary set and construct a groupoid $\mathbb{P}(A)$ ($\mathbb{P}$ for proper powerset) whose objects are proper subsets $A_0 \subsetneq A$ and whose morphisms are bijections σ of A preserving the subset, i.e.

$$\frac{A_0 \longrightarrow B_0}{\sigma \colon A \longrightarrow A \quad \text{s.t.} \quad \sigma A_0 = B_0} \; .$$

For each set X we get a functor

$$X^{(\ )} \colon \mathbb{P}(A)^{op} \longrightarrow \mathbf{Set}$$

with X^{A_0} the set of functions $A_0 \longrightarrow X$, and for $\sigma \colon A_0 \longrightarrow B_0$

$$X^{\sigma} \colon X^{B_0} \longrightarrow X^{A_0}$$

$$(g \colon B_0 \longrightarrow X) \longmapsto (A_0 \xrightarrow{\sigma_0} B_0 \xrightarrow{s} X)$$

$$\begin{array}{ccccc} A & \longleftarrow & A_0 & & \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_0} & \searrow^{X^{\sigma}(g)} & \\ & & & & X \\ & & & \nearrow_{g} & \\ A & \longleftarrow & B_0 & & . \end{array}$$

Let C be a left S_A-set. The action of S_A on C restricts to a functor

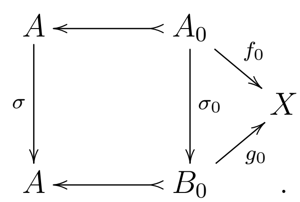
$$\bar{C} \colon \mathbb{P}(A) \longrightarrow \mathbf{Set}$$

$$\begin{array}{ccc} A_0 & \longmapsto & C \\ \downarrow{\scriptstyle \sigma} & \mapsto & \downarrow{\scriptstyle \sigma \circ (\ )} \\ B_0 & \longmapsto & C \; . \end{array}$$

4.3.3. Proposition.

$$\Delta[X^A \otimes_{S_A} C] \cong \int^{A_0 \in \mathbb{P}(A)} X^{A_0} \times \bar{C}.$$

Proof. An element of $\Delta[X^A \otimes_{S_A} C]$ is an equivalence class of pairs $[f\colon A \longrightarrow X+1, c]$ for f a function with $*$ in its image and $c \in C$, with $[f,c] = [g,d]$ iff there is $\sigma \in S_A$ with $f = g\sigma$ and $d = \sigma c$. $f\colon A \longrightarrow X+1$ is a partial map $A \leftarrowtail A_0 \xrightarrow{f_0} X$ with A_0 a proper subset, and $f = g\sigma$ means



This is exactly a description of a general element of $\int^{A_0} X^{A_0} \times C$. ∎

We can simplify things by choosing a skeleton of $\mathbb{P}(A)$. $A_0 \cong B_0$ iff there exists $\sigma\colon A \longrightarrow A$ in S_A such that $\sigma A_0 = B_0$ which implies that the cardinality of A_0 is equal to that of B_0, $\#A_0 = \#B_0$, but the complements of A_0 and B_0 also have the same cardinality, $\#A_0' = \#B_0'$. And this is sufficient to have $A_0 \cong B_0$. Thus for every pair of non-zero cardinals $\kappa = (\kappa_1, \kappa_2)$ such that $\kappa_1 + \kappa_2 = \#A$, we choose a subset $A_\kappa \subsetneq A$ with $\#A_\kappa = \kappa_i$ and $\#A_\kappa' = \kappa_2$, and take the full subcategory $\mathbb{P}'(A)$ of $\mathbb{P}(A)$ determined by these. $\mathbb{P}'(A)$ is now the coproduct of the groups $S_{A_\kappa} \times S_{A_\kappa'}$, which are the groups of automorphisms of the A_k. The coend is then the coproduct of the coends over each of these groups, which is what we've been writing as the tensor product. Thus we have the following:

4.3.4. Corollary.
$$\Delta(X^A \otimes_{S_A} C) = \sum_\kappa X^{A_\kappa} \otimes_{S_{A_\kappa} \times S_{A_\kappa'}} C$$
where the coproduct is taken over $\kappa = (\kappa_1, \kappa_2)$ *with* $\kappa_1, \kappa_2 > 0$ *and* $\kappa_1 + \kappa_2 = \#A$.

This is still not in the form we would like. What we would like is to express it in terms of $\otimes$ over symmetric groups. However

$$X^{A_\kappa} \otimes_{S_{A_\kappa} \times S_{A_\kappa'}} C \cong X^{A_\kappa} \otimes_{S_{A_\kappa}} \left(S_{A_\kappa} \otimes_{S_{A_\kappa} \times S_{A_\kappa'}} C \right)$$

where the S_{A_κ} in the middle is given the left S_{A_κ} action by left multiplication and the right $S_{A_\kappa} \times S_{A_\kappa'}$ action by right multiplication after projecting onto S_{A_κ}.

We can further analyze $S_{A_\kappa} \otimes_{S_{A_\kappa} \times S_{A_\kappa'}} C$. An element is an equivalence class of pairs (σ, c) with $\sigma \in S_{A_\kappa}$ and $c \in C$, with $(\sigma, c) \sim (\tau, d)$ iff there exist $\rho \in S_{A_\kappa}$ and $\rho' \in S_{A_\kappa'}$ such that
$$\tau = \sigma\rho \quad \text{and} \quad c = (\rho + \rho')d\ .$$
Each equivalence class has representatives of the form (id, c) and for these the equivalence relation reduces to $(\mathrm{id}, c) \sim (\mathrm{id}, d)$ iff there exists $\rho' \in S_{A_\kappa'}$ such that $c = (\mathrm{id} + \rho')d$. So

$$S_{A_\kappa} \otimes_{S_{A_\kappa} \times S_{A_\kappa'}} C \cong C_\kappa$$

with $C_\kappa = C/\sim$ where

$$c \sim d \Leftrightarrow \exists \rho' \in S_{A_{\kappa'}}(c = (\mathrm{id} + \rho')d) \ .$$

The upshot of this long discussion is that:

4.3.5. Corollary.

$$\Delta[X^A \otimes_{S_A} C)] = \sum X^{A_\kappa} \otimes_{A_\kappa} C_\kappa \ ,$$

and Δ of a generalized analytic functor is again one.

4.4. Reduced powers. Let $\mathcal{F}$ be a filter on A. An element of $(X+1)^{\mathcal{F}}$ can be identified with an equivalence class of partial functions $A \leftarrowtail A_0 \xrightarrow{f} X$ with $(A \leftarrowtail A_0 \xrightarrow{f} X) \sim (A \leftarrowtail B_0 \xrightarrow{g} X)$ if and only if

$$\{a \in A_0 \cap B_0 \mid f(a) = g(x)\} \cup (A_0' \cap B_0') \in \mathcal{F}$$

where A_0' and B_0' are the complements of A_0 and B_0 respectively.

If we take $X = 1$, the f is redundant so that an element of $(1+1)^{\mathcal{F}}$ may be identified with an equivalence class of subsets $A_0 \subseteq A$ with $A_0 \sim B_0$ if and only if

$$(A_0 \cap B_0) \cup (A_0' \cap B_0') \in \mathcal{F}.$$

The canonical map $(X+1)^{\mathcal{F}} \longrightarrow (1+1)^{\mathcal{F}}$ partitions $(X+1)^{\mathcal{F}}$ into a disjoint union indexed by the equivalence classes of subsets $[A_0]$,

$$(X+1)^{\mathcal{F}} \cong \coprod_{[A_0]} (X+1)^{\mathcal{F}}_{[A_0]}$$

where an element of $(X+1)^{\mathcal{F}}_{[A_0]}$ corresponds to a partial function $A \leftarrowtail B_0 \xrightarrow{g} X$ if and only if $B_0 \sim A_0$.

For any subset $A_0 \subseteq A$ let

$$\mathcal{F}_{A_0} = \{A_1 \subseteq A_0 \mid A_1 \cup A_0' \in \mathcal{F}\} \ .$$

4.4.1. Proposition. *$\mathcal{F}_{A_0}$ is a filter on A_0 and $(X+1)^{\mathcal{F}}_{[A_0]} \cong X^{\mathcal{F}_{A_0}}$.*

Proof. $\mathcal{F}_{A_0}$ is easily seen to be a filter. An element of $X^{\mathcal{F}_{A_0}}$ is an equivalence class of functions $[A_0 \xrightarrow{t} X]$.

$$[A_0 \xrightarrow{f} X] = [A_0 \xrightarrow{g} X] \Longleftrightarrow \{a \in A_0 \mid fa = ga\} \in \mathcal{F}_{A_0}$$

$$\Longleftrightarrow \{a \in A_0 \mid fa = ga\} \cup A_0' \in \mathcal{F}$$

$$\Longleftrightarrow [A \leftarrowtail A_0 \xrightarrow{f} X] = [A \leftarrowtail A_0 \xrightarrow{g} X] \ \text{ in } \ (X+1)^{\mathcal{F}}.$$

So taking $[A_0 \xrightarrow{f} X]$ in $X^{\mathcal{F}}_{A_0}$ to $[A \longleftarrow A_0 \xrightarrow{f} X]$ in $(X+1)^{\mathcal{F}}$ is well-defined and one-to-one. To see that it's onto, first assume that $X \neq \emptyset$, and take $[A \longleftarrow B_0 \xrightarrow{g} X]$ in $(X+1)^{\mathcal{F}}_{[A_0]}$ and define $f\colon A_0 \longrightarrow X$ by

$$f(a) = \begin{cases} g(a) & \text{if } a \in A_0 \cap B_0 \\ \text{arbitrary, otherwise} \end{cases}$$

Then

$$\begin{aligned} \{a \in A_0 \cap B_0 \mid f(a) = g(a)\} &\cup (A_0' \cap B_0') \\ = (A_0 \cap B_0) &\cup (A_0' \cap B_0') \in \mathcal{F} \end{aligned}$$

because $A_0 \sim B_0$. If $X = \emptyset$ then $(X+1)^{\mathcal{F}}_{[A_0]}$ is also empty unless $[A_0] = [\emptyset]$, and $X^{\mathcal{F}_{A_0}}$ which is contained in $(X+1)^{\mathcal{F}}_{[A_0]}$ is also empty. For $[A_0] = [\emptyset]$, $(X+1)^{\mathcal{F}}_{[A_0]} \cong 1$ and so is $X^{\mathcal{F}_{[A_0]}}$ as $\emptyset \in \mathcal{F}_{[A_0]}$. ■

4.4.2. Corollary. $\Delta[X^{\mathcal{F}}] = \sum_{[A_0] \neq [A]} X^{\mathcal{F}_{A_0}}$
where the coproduct is taken over a set of representatives of the $\mathcal{F}$-equivalence classes of subsets $A_0 \subseteq A$ with $[A_0] \neq [A]$.

Proof. $(X+1)^{\mathcal{F}} \cong \sum X^{\mathcal{F}}_{A_0}$ over all classes and $\mathcal{F}_A = \mathcal{F}$. ■

4.4.3. Corollary. *If $\mathcal{U}$ is an ultrafilter, then*

$$\Delta[X^{\mathcal{U}}] \cong 1\ .$$

Proof. As $\mathcal{U}$ is an ultrafilter, for any subset $A_0 \subseteq A$, either $A_0 \in \mathcal{U}$ or $A_0' \in \mathcal{U}$. If $A_0 \in \mathcal{U}$ then $[A_0] = [A]$. If $A_0' \in \mathcal{U}$, then $[A_0] = [\emptyset]$ for

$$(A_0 \cap \emptyset) \cup (A_0' \cap \emptyset') = A_0' \in \mathcal{U}.$$

Thus there are only two classes $[A]$ and $[\emptyset]$ so $\Delta[X^{\mathcal{U}}] = X^{\mathcal{F}_\emptyset} \cong 1$ as $\mathcal{F}_\emptyset = 2^A$. ■

This gives an example of two functors, not differing by a constant, with the same finite difference, namely X and $X^{\mathcal{U}}$.

We could define *reduced power series* to be coproducts

$$\sum_{i \in I} C_i \times X^{\mathcal{F}_i}$$

and get that Δ of such is again one. Apart from $\Delta[X^{\mathcal{F}}]$ we don't know of any naturally arising examples.

Note that all filters $\mathcal{F}$ on finite sets are principal, i.e. the set of all subsets of A containing some fixed subset A_0, namely the intersection of all elements of $\mathcal{F}$. Then $X^{\mathcal{F}} \cong X^{A_0}$, so reduced powers are an essentially infinite phenomenon, infinite powers, so there would be no corresponding thing in algebra or analysis.

4.5. MONADS. All of the previous examples were variations on the power series theme. The filter monad $\mathbb{F}$ is of a different nature. It was central to Manes' paper introducing taut functors [12]. Not only is the functor $\mathbb{F}$ taut but the unit η and multiplication μ are taut transformations, i.e. $\mathbb{F}$ is a taut monad.

Recall that $\mathbb{F}(X) = \{\mathcal{F} \mid \mathcal{F} \text{ is a filter on } X\}$. If $f\colon X \longrightarrow Y$ then we have $\mathbb{F}(f)(\mathcal{F}) = \{Y_0 \subseteq Y \mid f^{-1}Y_0 \in \mathcal{F}\}$. The unit $\eta\colon X \longrightarrow \mathbb{F}(X)$ takes an element x to the principal ultrafilter generated by x, i.e. $\{X_0 \subseteq X \mid x \in X_0\}$. The multiplication is a bit harder, and won't concern us here.

4.5.1. PROPOSITION.

$$\Delta[\mathbb{F}] = \mathbb{F} \ .$$

PROOF. Let $\mathcal{F}$ be a filter on $X + 1$ and

$$\begin{aligned} \mathcal{F}_0 &= \{X_0 \subseteq X \mid X_0 \in \mathcal{F}\}, \\ \mathcal{F}_1 &= \{X_1 \subseteq X \mid X_1 \cup \{*\} \in \mathcal{F}\}. \end{aligned}$$

$\mathcal{F}_0$ may be empty, but if not it's a filter on X and $\mathcal{F}_1$ is always a filter on X. Clearly $\mathcal{F}_0 \subseteq \mathcal{F}_1$. If $\mathcal{F}_0 \neq \emptyset$, then $X \in \mathcal{F}_0$ so $X \in \mathcal{F}$. Then for any $X_1 \in \mathcal{F}_1$, $X_1 = X \cap (X_1 \cup \{*\})$ is in $\mathcal{F}$ so $X_1 \in \mathcal{F}_0$ and $\mathcal{F}_0 = \mathcal{F}_1$.

Conversely, given any filter $\mathcal{F}$ on X we get two filters on $X + 1$

$$\bar{\mathcal{F}} = \{X_0 \cup \{*\} \mid X_0 \in \mathcal{F}\}$$

and

$$\bar{\bar{\mathcal{F}}} = \mathcal{F} \cup \{X_0 \cup \{*\} \mid X_0 \in \mathcal{F}\} \ .$$

Thus the filters on $X + 1$ fall into two disjoint classes, the $\bar{\mathcal{F}}$'s and the $\bar{\bar{\mathcal{F}}}$'s, and the $\bar{\bar{\mathcal{F}}}$'s are precisely the images of $\bar{\mathcal{F}}$'s in $\mathbb{F}(X)$ under the inclusion $\mathbb{F}(j)\colon F(X) \longrightarrow F(X + 1)$. Consequently, $\Delta[\mathbb{F}](X)$ consists of all the $\bar{\mathcal{F}}$'s and is isomorphic to $\mathbb{F}(X)$ itself. ■

If $\mathbb{F}'$ is the submonad of $\mathbb{F}$ of proper filters then for any filter $\mathcal{F}$ on X, $\bar{\mathcal{F}}$ will always be proper whereas $\bar{\bar{\mathcal{F}}}$ will only be proper when $\mathcal{F}$ is. This gives the following result.

4.5.2. PROPOSITION.

$$\Delta[\mathbb{F}'] \cong \mathbb{F} \ .$$

The ultrafilter monad, usually called β, is also a submonad of $\mathbb{F}$. An ultrafilter $\mathcal{U}$ on $X + 1$ will be of the form $\bar{\bar{\mathcal{F}}}$ only if $\mathcal{F}$ is the powerset PX and then $\mathcal{U}$ will be $\langle * \rangle$, the principal ultrafilter determined by $*$. On the other hand $\bar{\bar{\mathcal{F}}}$ will be an ultrafilter iff $\mathcal{F}$ is one on X. It follows that $\beta(X + 1) \cong \beta X + 1$ and we get:

4.5.3. Proposition. $\Delta[\beta] = 1$.

In fact, as is well known and easy to prove, β preserves finite coproducts and $\beta 1 = 1$, from which the proposition follows, without the analysis of $\mathbb{F}$.

The covariant powerset monad $\mathbb{P}$ can also be considered a submonad of $\mathbb{F}$, by the "principal filter inclusion",

$$A \subseteq X \longmapsto \langle A \rangle = \{X_0 \subseteq X \mid A \subseteq X_0\} \ .$$

For $f\colon X \longrightarrow Y$,

$$\begin{aligned} \mathbb{F}(f)(\langle A \rangle) &= \{Y_0 \subseteq Y \mid f^{-1}Y_0 \in \langle A \rangle\} \\ &= \{Y_0 \subseteq Y \mid A \subseteq f^{-1}Y_0\} \\ &= \{Y_0 \subseteq Y \mid fA \subseteq Y_0\} \\ &= \langle fA \rangle \ . \end{aligned}$$

So we could analyze $\Delta[\mathbb{P}]$ in terms of $\mathcal{F}$, but it is easier done directly, though it's really the same thing.

A subset of $X+1$ either contains $*$ or not. In the first case it is of the form $X_1 \cup \{*\}$ for $X_1 \subseteq X$ and in the second it's just $X_0 \subseteq X$, the image of X_0 under $\mathbb{P}(j)$. This gives:

4.5.4. Proposition. $\Delta[\mathbb{P}] \cong \mathbb{P}$.

Although $\Delta[\mathbb{F}] = \Delta[\mathbb{F}'] = \mathbb{F}$, $\Delta[\mathbb{P}] = \mathbb{P}$ and $\Delta[\beta] = 1$ are all monads it's the exception rather than the rule that Δ of a taut monad is again a monad. Note that, although $\Delta[\beta] = 1$ is a monad, it is not a taut one as the unit $X \longrightarrow 1$ is not taut. Even for such a basic monad as $TX = M \times X$, for M a monoid, its difference is the constant functor M which may look like a monad but it's not (unless $M = 1$).

The reason is that for (T, η, μ) a taut monad, we have a pullback

$$\begin{array}{ccc} X+1 & \xrightarrow{\eta(X+1)} & (X+1) \\ \uparrow & \boxed{\text{Pb}} & \uparrow \\ X & \xrightarrow[\eta X]{} & T(X) \end{array} ,$$

so, in passing to the difference $\Delta[T](X) = T(X+1) \setminus T(X)$, we've removed the unit.

However, as we saw in Theorem 3.3.4, looking at the difference operator as producing an endomorphism of the "tangent space", we get a monoidal functor

$$D\colon \mathbf{End}_{Taut}(\mathbf{Set}) \longrightarrow \mathbf{End}_{Taut}(\mathbf{Set} \times \mathbf{Set})$$

which does preserve monoids, i.e. monads. So, for a taut monad (T, η, μ) on $\mathbf{Set}$ we get a taut monad $D(T, \eta, \mu)$ on $\mathbf{Set} \times \mathbf{Set}$

$$\begin{array}{lcl} \mathbf{Set} \times \mathbf{Set} & \longrightarrow & \mathbf{Set} \times \mathbf{Set} \\ (A, B) & \longmapsto & (TA, \Delta[T](A) \times B) \ . \end{array}$$

The unit is

$$(A, B) \xrightarrow{(\eta A, hA \times B)} (TA, \Delta[T](A) \times B)$$

where hA is defined by

$$\begin{array}{ccc} 1 & \xrightarrow{hA} & \Delta[T](A) \\ {\scriptstyle j_2}\downarrow & & \downarrow \\ A+1 & \xrightarrow[\eta(A+1)]{} & T(A+1) \end{array}$$

which exists because η is taut. It is just $\Delta[\eta](A)$.

The multiplication comes from the chain rule transformation. It is

$$(T^2A, \Delta[T](TA) \times \Delta[T](A) \times B) \xrightarrow{(\mu A, mA \times B)} (TA, \Delta[T](A) \times B)$$

where mA is the composite

$$\Delta[T](TA) \times \Delta[T](A) \xrightarrow{\gamma} \Delta[T^2](A) \xrightarrow{\Delta[\mu](A)} \Delta[T](A)\ .$$

The B looks just tacked on with nothing to do with T or A, but they are intertwined via the associativity law for the monad $D(T, \eta, \mu)$. We won't look at this directly, but it is apparent when we consider Eilenberg-Moore algebras for it. An algebra is

$$(\alpha, \beta)\colon (TA, \Delta[T](A) \times B) \longrightarrow (A, B)$$

such that (A, α) is a T-algebra and β makes the following diagrams commute

$$\begin{array}{ccc} 1 \times B & \xrightarrow{h \times B} & \Delta[T](A) \times B \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \beta} \\ & & B \end{array}$$

and

$$\begin{array}{ccc} \Delta[T](TA) \times \Delta[T](A) \times B & \xrightarrow{\Delta[T](\alpha) \times B} & \Delta[T](A) \times B \\ {\scriptstyle \gamma \times B}\downarrow & & \\ \Delta[T^2](A) \times B & & \downarrow{\scriptstyle \beta} \\ {\scriptstyle \Delta[\mu] \times B}\downarrow & & \\ \Delta[T](A) \times B & \xrightarrow[\beta]{} & B\ . \end{array}$$

We see the α appearing in the top row in the equations for β.

4.6. Dirichlet functors. The analysis of $\mathbb{P}$ easily generalizes to the covariant exponentials L^X.

4.6.1. Proposition. *Let L be a sup-lattice, then*

$$\Delta[L^X] \cong L_* \times L^X$$

where $L_* = \{l \in L \mid l \neq \bot\}$.

Proof. An element of $L^{(X+1)}$ is a function $X+1 \xrightarrow{[\phi,l]} L$ where $\phi\colon X \longrightarrow L$ and $l \in L$. If $f\colon X \longrightarrow Y$ and $L^f[\phi,l] = [\psi,m]\colon Y+1 \longrightarrow L$, then $\psi(y) = \bigvee_{fx=y} \phi(x)$ and m is the sup over all pre-images of $*$, i.e. $\{*\}$, so $m = l$, i.e. $L^{(X+1)} \cong L^X \times L$, as functors.

$L^j(\phi) = [\phi,\bot]\colon X+1 \longrightarrow L$ where $\phi\colon X \longrightarrow L$ and $\bot$ is the bottom element of L, i.e. the sup of the empty set, $j^{-1}\{*\}$. This gives the result. ∎

Recall that for $l \in L$, a sup lattice, $D(l)$ denotes the down-set of l, $\{l' \in L \mid l' \leq l\}$, which is a sub sup lattice of L.

4.6.2. Proposition. *Let $C_l = \{l' \in L \mid l \vee l' = \top$ and $l' \neq \bot\}$. Then*

$$\Delta[L^{[X]}] \cong \sum_{l \in L} C_l \cdot D(l)^{[X]} \;.$$

Proof. The proof is similar to that of the previous proposition. An element of $\Delta[L^{[X]}]$ is a function $[\phi,l']\colon X+1 \longrightarrow L$ for $l' \neq \bot$ and $\bigvee_{x\in X} \phi(x) \vee l' = \top$. If we let $l = \bigvee_{x \in X} \phi(x)$ then $\phi \in D(l)^{[X]}$. And there's one such ϕ for each $l' \neq \bot$ and $l \vee l' = \top$. ∎

4.6.3. Corollary. *Δ of a Dirichlet functor is a Dirichlet functor.*

4.6.4. Corollary. *(1) If $\top$ is join-irreducible then*

$$\Delta[L^{[X]}] = L_* \cdot L^{[X]} + \sum_{l \neq \top} D(l)^{[X]} \;.$$

(2) $\Delta[\mathbf{n}^{[X]}] = (n-1)\cdot \mathbf{n}^{[X]} + (\mathbf{n-1})^{[X]} + (\mathbf{n-2})^{[X]} + \cdots + \mathbf{2}^{[X]} + \mathbf{1}^{[X]}$.

(3) $\Delta[p_*^{[X]}] = \pi(p)\cdot p_*^{[X]} + \sum q_*^{[X]} + 1$, *the coproduct taken over all primes $< p$.*

Proof. (1) $\top$ join-irreducible means $l \vee l' = \top \Rightarrow l = \top$ or $l' = \top$, so

$$C_\top = L_* = \{l' \mid l' \neq \bot\} \;\text{ and if }\; l \neq \top$$

then $C_l = \{\top\} \cong 1$.
(2) The top element of $\mathbf{n}$ is join-irreducible, so this is a special case of (1).
(3) This is a special case of (2), given the definition of p_*. ∎

4.6.5. PROPOSITION. Δ *of a sequential Dirichlet functor is again one.*

PROOF. By Corollary 4.6.4 (3) we know that for a prime p, $\Delta p_*^{[X]}$ is a sequential Dirichlet functor. For a composite n, n_* is a cartesian product of p_*'s so by Proposition 2.6.13, $\Delta n_*^{[X]}$ is also a sequential Dirichlet functor. Finally, Δ of a sequential Dirichlet functor is a coproduct of $\Delta n_*^{[X]}$'s so is again one. ■

5. A Newton summation formula

For a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, the difference operator Δ

$$\Delta[f](x) = f(x+1) - f(x)$$

can be iterated giving discrete versions of higher derivatives. Thus

$$\Delta^2[f](x) = f(x+2) - 2f(x+1) + f(x)$$

$$\Delta^3[f](x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x)$$

and so on.

The Newton series is a discrete analog of Taylor series, trying to recover f or some reasonable approximation of f, from specific values $\Delta^k[f](a)$. It takes the form

$$g(x) = \sum_{n=0}^{\infty} \frac{\Delta^n[f](a)}{n!}(x-a)^{\downarrow n} \tag{1}$$

$$= \sum_{n=0}^{\infty} \binom{x-a}{n} \Delta^n[f](a) \tag{2}$$

where $(x-a)^{\downarrow n}$ is the *falling power*

$$(x-a)(x-a-1)(x-a-2)\ldots(x-a-n+1)$$

and $\binom{x-a}{n}$ is the "binomial coefficient"

$$\frac{(x-a)(x-a-1)\ldots(x-a-n+1)}{n!}\ .$$

The formulas (1) and (2) look sufficiently combinatorial to suggest that there may be a similar formula for taut endofunctors of **Set**, and indeed there is. That is what we develop in this section.

This section has significant overlap with [17] but it is hard to isolate what precisely. Their objective was completely different from ours, concentrating on monads, theories, and operads. But crucial points here appear at various points in their work. They clearly recognized the importance of the category **Surj** of finite cardinals and surjections (there called $\mathbb{S}$) and its relation to soft analytic functors (called semi-analytic). Their

Theorem 2.2 is basically Part I of our main theorem below. In particular the $A[n]$ of their Lemma 2.5 is $\Delta^n[F](0)$ by our Corollary 5.1.3.

Neither of the formulas (1) or (2) generalize directly to functors, even for $a = 0$. We need more structure on the higher differences $\Delta^n[F]$, and to study this we need a better understanding of them.

The suggestive notation $\Delta[F](X) = F(X+1) \setminus F(X)$ is too suggestive here. Blindly applying the difference twice would give

$$\begin{aligned} \Delta^2[F](X) &= (F(X+2) \setminus F(X+1)) \setminus (F(X+1) \setminus F(X)) \\ &= F(X+2) \setminus F(X+1) \end{aligned}$$

which is definitely wrong, as can easily be seen by taking $F(X) = X^2$. We must be mindful of which injection we are complementing.

To track the injections, let's take two different one-point sets $1_a = \{a\}$ as $1_b = \{b\}$ with corresponding difference operators

$$\begin{aligned} \Delta_a[F](X) &= F(X+1_a) \setminus F(X) \\ \Delta_b[F](X) &= F(X+1_b) \setminus F(X)\ . \end{aligned}$$

Thus we have

$$\begin{aligned} \Delta_a[\Delta_b[F]](X) &= \Delta_b[F](X+1_a) \setminus \Delta_b[F](X) \\ &= \Big(F(X+1_a+1_b) \setminus F(X+1_a)\Big) \setminus \Big(F(X+1_b) \setminus F(X)\Big) \\ &= F(X+1_a+1_b) \setminus \Big(F(X+1_a) \cup F(X+1_b)\Big) \end{aligned}$$

as $F(X) \subseteq F(X+1_a)$.

This leads to the following. Let $S_A\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be the translation functor $S_A(X) = X + A$, which is obviously taut. For any taut functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ and any set X, let $D_A[F](X)$ be the subset of $F(X+A)$ consisting of those $a \in F(X+A)$ not in the image of $F(X+A_0) \rightarrowtail F(X+A)$ for any proper subset $A_0 \subsetneq A$. As there will be much talk of proper subsets in what follows, we will use an arrow with a double tail $A_0 \twoheadrightarrow A$ to indicate a monomorphism which is not an isomorphism.

5.1.1. Proposition. *$D_A[F]$ is a taut subfunctor of FS_A.*

Proof. First we show it's a subfunctor. Let $f\colon X \longrightarrow Y$ and $a \in FS_A(X) = F(X+A)$. Then $F(f+A)(a) \in F(Y+A)$. Let $A_0 \twoheadrightarrow A$ be a proper subset of A. Then as

$$\begin{array}{ccc} X + A_0 & \rightarrowtail & X + A \\ {\scriptstyle f+A_0}\downarrow & & \downarrow{\scriptstyle f+A} \\ Y + A_0 & \rightarrowtail & Y + A \end{array}$$

is a pullback and as F is taut

$$\begin{array}{ccc} F(X+A_0) & \rightarrowtail & F(X+A) \\ \downarrow{\scriptstyle F(f+A_0)} & & \downarrow{\scriptstyle F(f+A)} \\ F(Y+A_0) & \rightarrowtail & F(Y+A) \end{array}$$

is also a pullback. Thus if $F(f+A)(a)$ were in $F(Y+A_0)$, a would be in $F(X+A_0)$. So if $a \in D_A[F](X)$, $F(f+A)(a)$ is in $D_A[F](Y)$, i.e. $D_A[F]$ is a subfunctor of FS_A.

$D_A[F](X)$ is defined to be the complement of

$$\bigcup F(X+A_0) \subseteq F(X+A) \ ,$$

the union taken over all proper subsets $A \rightarrowtail A$. Thus $D_A[F]$ is a complemented subfunctor of FS_A and by Proposition 1.3.11 is taut. ∎

5.1.2. Proposition. $D_A[D_B[F]] \cong D_{A+B}[F]$.

Proof. An element $a \in D_A[D_B[F]](X)$ is an element $a \in D_B[F](X+A)$ which is not in any $D_B[F](X+A_0)$ for a proper subset $A_0 \subsetneq A$. The first condition means that $a \in F(X+A+B)$ but not in any $F(X+A+B_0)$ for a proper subset $B_0 \subseteq B$. The second condition $((a \notin D_B[F](X+A_0))$ means that a is not in $F(X+A_0+B)$ or there exists a proper subset $B_0 \subsetneq B$ with $a \in F(A+A_0+B_0)$. But this last condition is impossible, for if a were in $F(X+A_0+B_0)$ it would be in $F(X+A+B_0)$ which it is not. The conclusion is that $a \in D_A[D_B[F](X)$ if and only if $a \in F(X+A+B)$ but $a \notin F(X+A+B_0)$ and $a \notin F(X+A_0+B)$. This is equivalent to $a \in F(X+A+B)$ and $a \notin F(X+C)$ for any proper subset $C \subsetneq A+B$ because such a C would be contained in either a A_0+B or a $A+B_0$. The result follows. ∎

5.1.3. Corollary. *For any finite cardinal n*

$$\Delta^n[F] = D_n[F]$$

$$\Delta^n[F](X) = \{a \in F(X+n) | a \notin F(X+n_0) \ \ \textit{for any} \ \ n_0 \rightarrowtail n\}.$$

Proof. $\Delta^0[F] = F = D_0[F]$ and $\Delta^1[F] = \Delta F = D_1[F]$ by definition. The result now follows from the previous proposition by induction. ∎

We are hoping to recover F from the sequence $\Delta^n[F](0)$ via some version of the Newton series, at least for polynomial functors. From the above description of $\Delta^n[F](0)$, it is clear that S_k acts on $\Delta^n[F](0)$ so we get a symmetric sequence (species) and a corresponding analytic functor

$$\sum_{n=0}^{\infty} X^n \otimes_{S_n} \Delta^n[F](0).$$

This looks promising but it doesn't give F even for polynomials. For example $F(X) = X^n$ is connected but the above analytic functor is not. It's defined as a sum of (non trivial) functors. We need something a bit tighter. In fact, not only do bijections act on the $\Delta^n[F](a)$ but epimorphisms do too, in the appropriate sense of course.

Let **Surj** be the category of finite cardinals with surjections as morphisms.

Although it will not be used below, it may be of interest to note that **Surj** is the free symmetric monoidal category generated by a commutative monoid.

5.1.4. Proposition. *For any taut functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ and set A, the family $\langle \Delta^n[F](A)\rangle_n$ is the object part of a functor*

$$\mathbf{Surj} \longrightarrow \mathbf{Set}.$$

Proof. Let $f\colon m \twoheadrightarrow n$ be a surjection and $a \in \Delta^m[F](A)$, so $a \in F(A+m)$ but $a \notin F(A+m_0)$ for any $m_0 \rightarrowtail m$. Then $F(A+f)(a) \in F(A+n)$ and we have to show that it's not in any $F(A+n_0)$ for $n_0 \rightarrowtail n$. Suppose it were. Take the pullback

$$\begin{array}{ccc} m_0 & \rightarrowtail & n \\ \downarrow & \boxed{\text{Pb}} & \downarrow f \\ n_0 & \rightarrowtail & m \end{array} .$$

As usual $m_0 \rightarrowtail m$ is monic, but it is also proper because f is surjective. F being taut produces a pullback

$$\begin{array}{ccc} F(A+m_0) & \rightarrowtail & F(A+m) \\ \downarrow & \boxed{\text{Pb}} & \downarrow F(A+f) \\ F(A+n_0) & \rightarrowtail & F(A+n) \end{array}$$

so if $F(A+f)(a)$ were in $F(A+n_0)$, a would be in $F(A+m_0)$ contrary to the definition of a. ■

Furthermore, the construction $F \rightsquigarrow \langle \Delta^n[F](A)\rangle_n$ is itself functorial. If **Taut** is the category of taut endofunctors of **Set** with taut natural transformations as morphisms, then we have the following.

5.1.5. Proposition. *For any set A, taking iterated differences produces a functor*

$$\Delta^*_A\colon \mathbf{Taut} \longrightarrow \mathbf{Set}^{\mathbf{Surj}} .$$

Proof. For $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ a taut functor, let $\Delta^*_A(F)$ be given by

$$\Delta^*_A(F)(n) = \Delta^n[F](A)$$

the functor of the previous proposition. Let $t\colon F \longrightarrow G$ be a taut transformation. By applying Proposition 3.1.5 iteratively we see that $tS^n\colon FS^n \longrightarrow GS^n$ restricts to a taut transformation

$$\begin{array}{ccc} \Delta^n[F] & \rightarrowtail & FS^n \\ {\scriptstyle \Delta^n[t]}\downarrow & & \downarrow{\scriptstyle tS^n} \\ \Delta^n[G] & \rightarrowtail & GS^n \end{array} .$$

This gives the formula for $\Delta^*_A(t)$, namely $\Delta^*_A(t)(A) = t(A+n)$. So naturality of $\Delta^*_A(F)(m)$ in n is simply

$$\begin{array}{ccc} F(A+m) & \xrightarrow{F(A+f)} & F(A+n) \\ {\scriptstyle t(A+m)}\downarrow & & \downarrow{\scriptstyle t(A+n)} \\ G(A+m) & \xrightarrow[G(A+f)]{} & G(A+n) \end{array}$$

for any surjection $f\colon m \twoheadrightarrow n$, which is just naturality of t itself.

Functoriality of $\Delta^*[t]$ in t follows from the fact that it is a restriction of tS^n. ∎

5.1.6. Definition. Let us call a functor $G\colon \mathbf{Surj} \longrightarrow \mathbf{Set}$ a *soft species* ("soft" because the structures can be compressed). A soft species produces a *soft analytic functor*, the *Newton sum* of G.

$$\widetilde{G}(X) = \int^{n \in \mathbf{Surj}} X^n \times G(n),$$

i.e. , left Kan extension of G along the inclusion J of $\mathbf{Surj}$ into $\mathbf{Set}$

$$\begin{array}{ccc} \mathbf{Surj} & \xrightarrow{J} & \mathbf{Set} \\ & \Rightarrow & \\ {\scriptstyle G}\searrow & & \swarrow{\scriptstyle \mathrm{Lan}_J G = \widetilde{G}} \\ & \mathbf{Set} & \end{array} .$$

This is similar to the definition of analytic functor which is the Kan extension along the inclusion of **Bij** into **Set**. For a species $F\colon \mathbf{Bij} \longrightarrow \mathbf{Set}$ we can take the Kan extension of F in steps

$$\begin{array}{ccccc} \mathbf{Bij} & \xrightarrow{I} & \mathbf{Surj} & \xrightarrow{J} & \mathbf{Set} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \mathrm{Lan}_I F} & \swarrow{\scriptstyle \mathrm{Lan}_{JI} F} & \\ & & \mathbf{Set} & & \end{array}$$

so that every analytic functor is soft analytic: every species can be softened.

It may be of interest to note that the soft species associated to F, $\mathrm{Lan}_I F$ is given by

$$(\mathrm{Lan}_I F)(n) = \sum_{m \twoheadrightarrow n} F(m)/_{\sim}$$

where the sum is taken over all surjections $f\colon m \twoheadrightarrow n$, and the equivalence relation is the quotient of $F(m)$ by the subgroup of S_m consisting of all elements preserving f

$$\begin{array}{ccc} m & \xrightarrow{\sigma} & m \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f} & \\ & n & . \end{array}$$

We can analyze the definition of $\widetilde{G}$ further to connect it to analytic functors and underline the similarity with formulas (1) and (2) for Newton series. From the definition, $\widetilde{G}(X)$ consists of equivalence classes $[a \in G(n), f\colon n \longrightarrow X]$ of triples (n, a, f) where the equivalence relation is generated by the relation $(n, a, f) \sim (m, b, g)$ if there exists a surjection $\sigma\colon n \twoheadrightarrow m$ such that $f = g\sigma$ and $b = G(\sigma)(a)$

$$\begin{array}{ccc} a & \in & G(n) \\ \downarrow & & \downarrow {\scriptstyle G(\sigma)} \\ b & \in & G(m) \end{array} \qquad \begin{array}{ccc} n & \xrightarrow{f} & X \\ {\scriptstyle \sigma}\downarrow & & \| \\ m & \xrightarrow[g]{} & X \end{array} .$$

So the equivalence relation is expressed in terms of a zigzag path of surjections and elements of G satisfying conditions as above. We will see presently that we can assume that the $f\colon n \longrightarrow X$ is monic, σ a bijection and the zigzag paths have length one, i.e. no zigzag at all. Everything we need can be expressed in these terms.

5.1.7. PROPOSITION.

(1) Every equivalence class in $\widetilde{G}(X)$ contains a representation in which f is monic.

(2) Two elements (n, a, f) and (m, b, g) with f and g monic are equivalent if and only if $m = n$ and there is a bijection $\sigma\colon n \longrightarrow m$ with $f = g\sigma$ and $b = G(\sigma)(a)$.

(3) For a morphism $\phi\colon X \longrightarrow Y$, $\widetilde{G}(\phi)[n, a, f] = [m, b, g]$ where g is the image of ϕf and b comes from a as in

$$\begin{array}{ccc} a & \in & G(n) \\ \downarrow & & \downarrow {\scriptstyle G(e)} \\ b & \in & G(m) \end{array} \qquad \begin{array}{ccc} n & \overset{f}{\rightarrowtail} & X \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \phi} \\ m & \underset{g}{\rightarrowtail} & Y \end{array} .$$

(4) If $\phi\colon X \rightarrowtail Y$ is monic, then so is $\widetilde{G}(\phi)$ and an element $[b, m, g]$ in $\widetilde{G}(Y)$ is in the image of $\widetilde{G}(\phi)$ if and only if g factors through ϕ

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow{\scriptstyle \phi} \\ m & \xrightarrow[g]{} & Y \end{array}\ .$$

Proof. (1) For any equivalence class $[n, a, f]$ take the image factorization of f, and since the quotient of a finite cardinal is again one, we get

$$\begin{array}{ccc} a & \in & G(n) \\ \downarrow & & \downarrow{\scriptstyle G(e)} \\ \bar{a} = G(e)(a) & \in & G(\bar{n}) \end{array} \qquad \begin{array}{ccc} n & \xrightarrow{f} & X \\ {\scriptstyle e}\downarrow & & \| \\ \bar{n} & \xrightarrow[\bar{f}]{} & X \end{array}$$

so that $[n, a, f] = [\bar{n}, \bar{a}, \bar{f}]$.

(2) Suppose elements (n, a, f) and (m, b, g) of the same equivalence class are related by a single epimorphism $\sigma\colon n \twoheadrightarrow m$

$$\begin{array}{ccc} a & \in & G(n) \\ \downarrow & & \downarrow{\scriptstyle G(\sigma)} \\ b & \in & G(m) \end{array} \qquad \begin{array}{ccc} n & \xrightarrow{f} & X \\ {\scriptstyle \sigma}\downarrow & & \| \\ m & \xrightarrow[g]{} & X \end{array}\ .$$

Factor f and g, giving

$$\begin{array}{ccccc} n & \xrightarrow{e} & \bar{n} & \xrightarrow{\bar{f}} & X \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \bar{\sigma}} & & \| \\ m & \xrightarrow[e']{} & \bar{m} & \xrightarrow[\bar{g}]{} & X \end{array}\ .$$

We see that $\bar{\sigma}$ is both one-one and onto, so a bijection. Let $\bar{a} = G(e)(a)$ and $\bar{b} = G(e')(b)$.

Then $G(\bar{\sigma})(\bar{a}) = \bar{b}$. So any zigzag path relating (n, a, f) to (m, b, g)

$$\begin{array}{ccccccc}
a & \in & G(n) & \qquad & n & \overset{f}{\rightarrowtail} & X \\
\big\downarrow & & \big\downarrow & & \big\downarrow \sigma_1 & & \big\| \\
a_1 & \in & G(n_1) & & n_1 & \overset{f_1}{\longrightarrow} & X \\
\big\uparrow & & \big\uparrow & & \big\uparrow \sigma_2 & & \big\| \\
a_2 & \in & G(n_2) & & n_2 & \overset{f_2}{\longrightarrow} & X \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\| \\
\vdots & & \vdots & & \vdots & & \vdots \\
\big\uparrow & & \big\uparrow & & \big\uparrow & & \big\| \\
b & \in & G(m) & & m & \underset{g}{\rightarrowtail} & X
\end{array}$$

can be replaced with one in which all the f_i are monic and all the σ_i bijections. Replacing the backward σ's by their inverses gives a path of length 1.

(3) For a morphism $\phi\colon X \longrightarrow Y$, $\widetilde{G}(\phi)$ is given by composition, $\widetilde{G}(\phi)[n, a, f] = [n, a, \phi f]$ and if we want to express this with a monic g we factor ϕf,

$$\begin{array}{ccccccc}
a & \in & (n) & \qquad & n & \overset{f}{\rightarrowtail} & X \\
\big\downarrow & & \big\downarrow G(e) & & \big\downarrow e & & \big\downarrow \phi \\
b & \in & G(m) & & m & \underset{g}{\rightarrowtail} & Y
\end{array}$$

and $[n, a, \phi f] = [m, b, g]$.

(4) A monomorphism in **Set**, $\phi\colon X \rightarrowtail Y$ is split, unless $X = \emptyset \neq Y$, and will be automatically preserved. In case $X = \emptyset$, an element of $\widetilde{G}(\emptyset)$ is of the form $[a \in G(\emptyset), \emptyset \rightarrowtail \emptyset]$ and $\widetilde{G}(\phi)$ will take it to $[a \in G(\emptyset), \emptyset \rightarrowtail Y]$. This is also monic because the equivalence relation is trivial.

For the second part of the statement, the "if" is trivial. So suppose we have an element $[m, b, g]$, with g monic, in the image of $\widetilde{G}(\phi)$ so that there is $[n, a, f]$, with f monic, in $\widetilde{G}(\phi)$ with $[m, b, g] = [n, a, \phi f]$. Then by (2) there is a bijection σ such that

$$\begin{array}{ccccccc}
a & \in & G(n) & \qquad & n \overset{f}{\rightarrowtail} X & \overset{\phi}{\rightarrowtail} & Y \\
\big\uparrow & & \big\uparrow & & \big\uparrow \sigma & & \big\| \\
b & \in & G(m) & & m & \underset{g}{\rightarrowtail} & Y
\end{array}$$

and there is our factorization. ∎

So we see that $\widetilde{G}$ on objects is given by

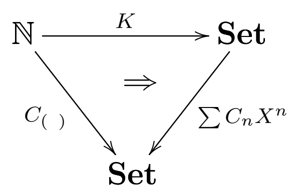
$$\widetilde{G}(X) = \sum_{n=0}^{\infty} G(n) \times \mathrm{Mono}(n, X)/S_n$$

which, if we take $G = \Delta^*[F]$, looks a lot like formula (2) for the Newton sum. It also looks like the formula for an analytic functor. But $\widetilde{G}$ is not the coproduct of functors that this might suggest. First of all $\mathrm{Mono}(n, X)$ is not functorial in X, and secondly $\widetilde{G}$ is a coend over **Surj**, so surjections have to be accounted for. The point is that the extra structure that G has compensates for the shortfalls of $\mathrm{Mono}(n, X)$.

Power series functors

$$\sum_{n=0}^{\infty} C_n X^n$$

are analytic so soft analytic too but we can calculate directly what the corresponding soft species is. We can view such a polynomial functor as a left Kan extension



where $\mathbb{N}$ is the discrete category whose objects are natural numbers, and K takes the number n and sends it to the set of cardinality n

$$K(n) = \{1, 2, \ldots, n\}.$$

The corresponding soft species G is given by

$$G(n) = \sum_{m \twoheadrightarrow n} C_m \ ,$$

one summand for each surjection $m \twoheadrightarrow n$.

5.1.8. Proposition.

(1) *For a soft species G, the corresponding soft analytic functor $\widetilde{G}$ is taut.*

(2) *For a morphism of soft species $t\colon G \longrightarrow H$, i.e. a natural transformation, the corresponding $\tilde{t}\colon \widetilde{G} \longrightarrow \widetilde{H}$ is taut.*

(3) *$\tilde{(\ )}$ defines a functor* **SoftSp** $\longrightarrow$ **Taut**, *from the category of soft species* $\mathbf{Set}^{\mathbf{Surj}}$ *to the category of taut endofunctors of* **Set** *and taut natural transformations.*

Proof. (1) Let

$$\begin{array}{ccc} X_0 & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle \phi_0}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle \phi} \\ Y_0 & \underset{j}{\rightarrowtail} & Y \end{array}$$

be an inverse image diagram. We want to show that

$$\begin{array}{ccc} \widetilde{G}(X_0) & \rightarrowtail & \widetilde{G}(X) \\ {\scriptstyle \widetilde{G}(\phi_0)}\downarrow & & \downarrow{\scriptstyle \widetilde{G}(\phi)} \\ \widetilde{G}(Y_0) & \rightarrowtail & \widetilde{G}(Y) \end{array}$$

is a pullback. Take $[a \in G(a), n \overset{f}{\rightarrowtail} X]$ in $\widetilde{G}(X)$.

$$\begin{aligned} \widetilde{G}(\phi)[a, n, f] &= [a \in G(n), n \rightarrowtail X \xrightarrow{f} Y] \\ &= [G(e)(a) \in G(m), m \overset{g}{\rightarrowtail} Y] \end{aligned}$$

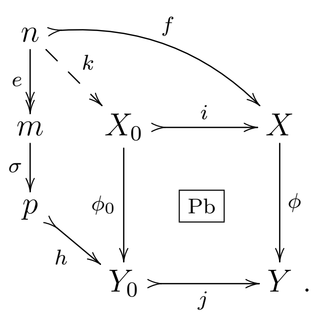
where $n \overset{e}{\twoheadrightarrow} m \overset{g}{\rightarrowtail} Y$ is the image factorization of ϕf. Suppose there is $[b \in G(p), p \overset{h}{\rightarrowtail} Y_0]$ in $\widetilde{G}(Y_0)$ such that

$$[b \in G(p), p \overset{h}{\rightarrowtail} Y_0 \rightarrowtail Y] = [G(e)(a) \in G(m), m \overset{g}{\rightarrowtail} Y].$$

Then by 5.1.7 (2), there exists a bijection $\sigma\colon m \longrightarrow p$ such that

$$\begin{array}{ccc} G(e)(a) & \in & G(m) \\ \downarrow & & \downarrow{\scriptstyle G(\sigma)} \\ b & \in & G(p) \end{array} \qquad \begin{array}{ccccc} m & & \overset{g}{\rightarrowtail} & & Y \\ {\scriptstyle \sigma}\downarrow & & & & \| \\ p & \underset{h}{\rightarrowtail} & Y_0 & \underset{j}{\rightarrowtail} & Y \end{array}$$

i.e. $b = G(\sigma)G(e)(a)$. Consider the diagram



$jh\sigma e = ge = \phi f$ so there exists k as above, and $[n, a, f]$ is in $\widetilde{G}(X_0)$. So $\widetilde{G}$ is taut.

(2) Let $t\colon G \longrightarrow H$ be a natural transformation. Then $\tilde{t}\colon \widetilde{G} \longrightarrow \widetilde{H}$ is defined as follows

$$\tilde{t}(X)\colon \widetilde{G}(X) \longrightarrow \widetilde{H}(X)$$

$$[a \in G(n), n \rightarrowtail X] \longmapsto [t(n)(a) \in H(n), n \rightarrowtail X].$$

We have several things to prove:

(i) Well-defined: If $[a \in G(n), n \rightarrowtail X] = [a' \in G(n'), n' \rightarrowtail X]$ then there is a bijection $\sigma\colon n \longrightarrow n'$ such that

$$\begin{array}{ccccccc}
a & \in & G(n) & , & n & \rightarrowtail & X \\
\big\downarrow & & \big\downarrow{\scriptstyle G(\sigma)} & & \big\downarrow{\scriptstyle \sigma} & & \big\| \\
a' & \in & G(n') & , & n' & \rightarrowtail & X
\end{array}$$

i.e. $a' = G(\sigma)a$. So $H(\sigma)t(n)(a) = t(n')G(\sigma)a = t(n')a'$.

(ii) Natural:

$$\begin{array}{ccccc}
\widetilde{G}(X) & \xrightarrow{\tilde{t}(x)} & \widetilde{H}(X) & & X \\
{\scriptstyle \widetilde{G}(\phi)}\big\downarrow & ? & \big\downarrow{\scriptstyle \widetilde{H}(\phi)} & & \big\downarrow{\scriptstyle \phi} \\
\widetilde{G}(Y) & \xrightarrow[\tilde{t}(Y)]{} & \widetilde{H}(Y) & & Y
\end{array}$$

Choose an element $[a \in G(n), \alpha\colon n \rightarrowtail X]$ of $\widetilde{G}(X)$ and chase it around the diagram

$$\begin{array}{ccc}
[a, \alpha] & \longmapsto & [t(n)(a), \alpha] \\
\big\downarrow & & \big\downarrow \\
[a, \phi\alpha] & \longmapsto & [t(n)(a), \phi\alpha]\ .
\end{array}$$

$\tilde{t}$ is indeed natural.

(iii) Taut: Let

$$\begin{array}{ccc}
X_0 & \rightarrowtail & X \\
{\scriptstyle \phi_0}\big\downarrow & \boxed{\text{Pb}} & \big\downarrow{\scriptstyle \phi} \\
Y_0 & \rightarrowtail & Y
\end{array}$$

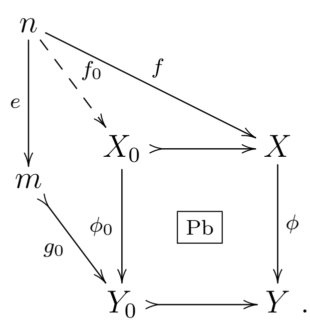
be an inverse image diagram and consider

$$\begin{array}{ccc} \widetilde{G}X_0 & \rightarrowtail & \widetilde{G}X \\ {\scriptstyle \widetilde{G}(\phi_0)}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{G}\phi} \\ \widetilde{G}Y_0 & \rightarrowtail & \widetilde{G}Y\ . \end{array} \qquad (*)$$

If $\widetilde{G}(\phi)[a \in G(n), n \rightarrowtail X] \in \widetilde{G}Y_0$, then it is $[G(e)(a) \in G(m), m \xrightarrow{g} Y]$ where $n \xrightarrow{e} m \xrightarrow{g} Y$ is the image factorization of ϕf. By 5.1.7 (4), $\widetilde{G}(\phi)[a, n]$ is in $\widetilde{G}Y_0$ iff $g\colon m \rightarrowtail Y$ factors through Y_0

$$\begin{array}{ccc} m & \xrightarrow{\quad g \quad} & Y \\ & {\scriptstyle g_0}\searrow \quad \nearrow & \\ & Y_0\ . & \end{array}$$

In the diagram



the outside commutes, so there exists $f_0\colon n \longrightarrow X_0$, as above, i.e. f factors through X_0, and so $[a \in G(n), m \rightarrowtail X] \in \widetilde{G}(X_0)$. Therefore $(*)$ is a pullback, and $\tilde{t}$ is taut.

(3) $\widetilde{(\ )}$ is automatically functorial $\mathbf{Set}^{\mathbf{Surj}} \longrightarrow \mathbf{Set}^{\mathbf{Set}}$ because it is Kan extension. The only question is whether it takes its values in **Taut**, and that's what was proved in (1) and (2). ∎

We are now ready for our main theorem which might be dubbed "The Fundamental Theorem of Functorial Difference Calculus". Part I says "Summing a soft species and then taking differences gives the original soft species" and Part II says "Taking differences of a taut functor and then summing produces a best approximation by a soft analytic functor".

5.1.9. Theorem. [Newton summation]
I. For a soft species $G\colon \mathbf{Surj} \longrightarrow \mathbf{Set}$ *we have a natural isomorphism*

$$G(n) \xrightarrow{\cong} \Delta^*[\widetilde{G}](0).$$

II. $\widetilde{(\ )}\colon \mathbf{SoftSp} \longrightarrow \mathbf{Taut}$ *is left adjoint to* $\Delta^*\colon \mathbf{Taut} \longrightarrow \mathbf{SoftSp}$.

Proof. I. Using Proposition 5.1.7 (1) and Corollary 5.1.3, we see that an element of $\Delta^n[\widetilde{G}](0)$ is an equivalence class

$$[a \in G(k), k \rightarrowtail n]_k$$

which is not equal to any

$$[b \in G(k_0), k_0 \rightarrowtail n_0 \rightarrowtail n]_{k_0}$$

for a proper mono $n_0 \rightarrowtail n$. If $k \rightarrowtail n$ were a proper mono, we would have

$$\begin{array}{ccccccc} a & \in & G(k) & & k & \rightarrowtail & n \\ \downarrow & & \downarrow & & \| & & \| \\ a & \in & G(k) & & k = k & \rightarrowtail & n \end{array},$$

so $[a \in G(n), k \rightarrowtail n]$ would not be in $\Delta^n[\widetilde{G}](0)$. It follows that the elements of $\Delta^n[\widetilde{G}](0)$ are of the form

$$[a \in G(n), \sigma\colon n \longrightarrow n]$$

for σ a bijection. Each such class has a canonical representative with $\sigma = \mathrm{id}$, and for these the equivalence relation is equality, i.e. $\Delta^n[\widetilde{G}](0)$ is in bijection with $G(n)$ itself.

The natural transformation which gives this bijection is

$$\begin{array}{rcl} \eta(n)\colon G(n) & \longrightarrow & \Delta^n[\widetilde{G}](0) \\ a & \longmapsto & [a \in G(n), \mathrm{id}\colon n \longrightarrow n] \,. \end{array}$$

II. Let $G\colon \mathbf{Surj} \longrightarrow \mathbf{Set}$ be a soft species and $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be a taut functor. As $\widetilde{G}$ is the left Kan extension of G along the inclusion $J\colon \mathbf{Surj} \longrightarrow \mathbf{Set}$, we already have a natural bijection of *natural transformations*

$$\frac{\widetilde{G} \xrightarrow{\ t\ } F}{G \xrightarrow[\ u\ ]{} FJ}\,.$$

Note that $\Delta^*[F]$ is a subfunctor of FJ. We will show that in the above bijection t is taut if and only if u factors through $\Delta^*(F) \rightarrowtail FJ$.

First assume t is taut. By Kan extension theory, the u corresponding to t is given by

$$u(n)\colon G(n) \longrightarrow F(n)$$

$$u(n)(a) = t(n)[a \in G(n), \mathrm{id}\colon n \rightarrowtail n]\,.$$

We want to show that $u(n)(a)$ is in fact an element of $\Delta^n[F](0)$, i.e. $u(n)(a)$ is not in any $F(n_0) \rightarrowtail F(n)$ for a proper mono $f\colon n_0 \rightarrowtail n$. Suppose it were, so that we would have $x \in F(n_0)$ such that $F(f)(x) = u(n)(a)$. As t is taut, we have the pullback

$$\begin{array}{ccc} \widetilde{G}(n_0) & \overset{\widetilde{G}(f)}{\rightarrowtail} & \widetilde{G}(n) \\ {\scriptstyle t(n_0)}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle t(n)} \\ F(n_0) & \underset{F(f)}{\rightarrowtail} & F(n) \end{array}$$

and so there is an element $[b \in G(m), m \rightarrowtail n_0]$ in $\widetilde{G}(n_0)$ such that $t(n_0)$ of which is x and $\widetilde{G}(f)[b \in G(m), m \rightarrowtail n_0] = [a \in G(n), \mathrm{id}\colon n \rightarrowtail n]$. This last equation means that there is a bijection $\sigma\colon n \longrightarrow m$ with

$$\begin{array}{ccc} a & \in & G(n) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle G(\sigma)} \\ b & \in & G(m) \end{array} \qquad \begin{array}{ccccc} n & & \xrightarrow{\quad \mathrm{id} \quad} & & n \\ {\scriptstyle\sigma}\Big\downarrow & & & & \Big\| \\ m & \rightarrowtail & n_0 & \rightarrowtail & n \end{array}$$

which implies that $n_0 \rightarrowtail n$ is an iso, i.e. is not proper. This contradicts our choice of n_0, so $u(n)(a)$ is indeed an element of $\Delta^n[F](0)$, and u factors through $\Delta^*[F] \rightarrowtail FJ$.

Now, let $u\colon G \longrightarrow FJ$ factor through $\Delta^*[F]$. The corresponding $t\colon \widetilde{G} \longrightarrow F$ is given by

$$t(X)\colon \widetilde{G}(X) \longrightarrow F(X)$$

$$[a \in G(n), n \overset{f}{\rightarrowtail} X] \longmapsto F(f)u(n)(a)$$

and we want to show that it's taut. To this end, let $\phi\colon Y \rightarrowtail X$ be a monomorphism. We will show that

$$\begin{array}{ccc} \widetilde{G}(Y) & \overset{G(\phi)}{\rightarrowtail} & \widetilde{G}(X) \\ {\scriptstyle t(Y)}\downarrow & & \downarrow{\scriptstyle t(X)} \\ FY & \underset{F\phi}{\rightarrowtail} & FX \end{array}$$

is a pullback. Take $[b \in G(n), g\colon n \rightarrowtail X]$ in $\widetilde{G}(X)$ such that there is $y \in FY$ with

$$\begin{aligned} F(\phi)(y) &= t(X)[n, b, g] \\ &= F(g)u(n)(a)\ . \end{aligned} \qquad (*)$$

We want to show that g factors through ϕ. Take the pullback

$$\begin{array}{ccc} m & \overset{\psi}{\rightarrowtail} & n \\ {\scriptstyle f}\downarrow & \boxed{\text{Pb}} & \downarrow{\scriptstyle g} \\ Y & \underset{\phi}{\rightarrowtail} & X \end{array}$$

and, as F is taut, we get another pullback

$$\begin{array}{ccc} F(m) & \xrightarrow{F(\psi)} & F(n) \\ {\scriptstyle F(f)}\big\downarrow & \boxed{\text{Pb}} & \big\downarrow{\scriptstyle F(g)} \\ F(Y) & \xrightarrow[F(\phi)]{} & F(X) \ . \end{array}$$

Because of $(*)$, there is $x \in F(m)$ such that $F(f)(x) = y$ and $F(\psi)(x) = u(n)(a)$. Since $u(n)(a)$ is in $\Delta^n[F](0)$, ψ cannot be proper so that g will factor through ϕ and $[b \in G(n), s\colon n \rightarrowtail X]$ is in $\widetilde{G}(Y)$. ■

5.1.10. Corollary. *If* $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ *is soft analytic, e.g. power series or analytic, then its Newton series converges to it*

$$F(X) \cong \int^{n \,\in\, \mathrm{Surj}} X^n \times \Delta^n[F](0) \ .$$

Note that what we are calling power series are functors of the form

$$F(X) = \sum_{n=0}^{\infty} C_n X^n$$

and only involve finite powers of X. For polynomial functors as in Definition 2.1.3 involving arbitrary powers of X, the corollary doesn't hold. We merely get a comparison. For example, for an infinite set A, the Newton series of $F(X) = X^A$ converges to the functor

$$G(X) = \{f\colon A \longrightarrow X \mid \text{the image of } f \text{ is finite}\}.$$

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