The Functorial Difference Operator

Robert Paré
(Dalhousie University)
pare@mathstat.dal.ca

Topos Institute Colloquium

January 11, 2024

What is an endofunctor of **Set** like?

- Polynomials $F(X) = \sum_{i \in I} X^{A_i}$ (Kock 2009, Spivak *et al.* – see [3, 5])
- Analytic functors $F(X) = \int^{n} X^n \times C_n$ (Joyal 1981 [2])
- Monads (Manes 2002 [4], Szawiel Zawadowski 2015 [7])
- Reduced powers $F(X) = X^{\mathscr{F}}$, $\mathscr{F}$ filter (Blass 1976 [1])

The structure of endofunctors of **Set**

- We study $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ by perturbing X and measuring the change in $F(X)$.

Example

$FX = X^3$

An element of $F(X+1) = (X + \{*\})^3$

$$\begin{array}{rcl}
(x_1, x_2, x_3) & \rightsquigarrow & X^3 \\
\left.\begin{array}{r} (x_1, x_2, *) \\ (x_1, *, x_3) \\ (*, x_2, x_3) \end{array}\right\} & \rightsquigarrow & 3X^2 \\
\left.\begin{array}{r} (x_1, *, *) \\ (*, x_2, *) \\ (*, *, x_3) \end{array}\right\} & \rightsquigarrow & 3X \\
(*, *, *) & \rightsquigarrow & 1
\end{array}$$

- Going from X to $X+1$, F gains $3X^2 + 3X + 1$ elements.

Tautness

Definition (Manes 2002 [4])

A functor is *taut* if it preserves inverse images

$$
\begin{array}{ccc}
A_0 & \rightarrowtail & A \\
{\scriptstyle f_0}\downarrow & \boxed{\mathsf{Pb}} & \downarrow{\scriptstyle f} \\
B_0 & \rightarrowtail & B
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
FA_0 & \longrightarrow & FA \\
{\scriptstyle Ff_0}\downarrow & \boxed{\mathsf{Pb}} & \downarrow{\scriptstyle Ff} \\
FB_0 & \longrightarrow & FB.
\end{array}
$$

A natural transformation $t\colon F \longrightarrow G$ is *taut* if the naturality squares for monomorphisms are pullbacks

$$
A_0 \rightarrowtail A
\quad \Longrightarrow \quad
\begin{array}{ccc}
FA_0 & \longrightarrow & FA \\
{\scriptstyle tA_0}\downarrow & \boxed{\mathsf{Pb}} & \downarrow{\scriptstyle tA} \\
GA_0 & \longrightarrow & GA\ .
\end{array}
$$

The plenitude of tautness

There are plenty of taut functors:

- Polynomial functors
- Analytic functors
- Reduced powers
- Left exact functors
- Functors $\mathbf{Set} \longrightarrow \mathbf{Set}$ that preserve binary coproducts

They are closed under a variety of operations.

We get a sub-2-category $\mathscr{T}\!aut$ of $\mathscr{C}\!at$, whose objects are categories with inverse images, 1-cells are taut functors, 2-cells taut natural transformations.

Limits

Proposition

Assume that $\mathbf{B}$ *has* $\mathbf{I}$*-limits and let* $\Gamma\colon \mathbf{I} \longrightarrow \mathscr{C}at(\mathbf{A},\mathbf{B})$ *be a diagram.*

(1) If ΓI is taut for all I, then $\varprojlim_I \Gamma I$ is taut.

(2) If, furthermore, $\mathbf{I}$ is non-empty and connected and $\Gamma(i)$ is a taut transformation for all $i\colon I \longrightarrow I'$ then $\varprojlim_I \Gamma(I)$ is the limit in $\mathscr{T}aut(\mathbf{A},\mathbf{B})$, i.e., the inclusion $\mathscr{T}aut(\mathbf{A},\mathbf{B}) \rightarrowtail \mathscr{C}at(\mathbf{A},\mathbf{B})$ creates connected limits.

Example

If $F, G\colon \mathbf{A} \longrightarrow \mathbf{B}$ are taut and $\mathbf{B}$ has finite products, then $F \times G$ is taut, but the projections are not.

The constant functor 1 is taut but not terminal in $\mathscr{T}aut(\mathbf{A},\mathbf{B})$.

Confluence

Definition

$\mathbf{I}$ is *confluent* if every span in $\mathbf{I}$ can be completed to a commutative square

$$\forall \quad \begin{array}{ccc} & & I_1 \\ & \overset{\alpha_1}{\nearrow} & \\ I_0 & & \\ & \underset{\alpha_2}{\searrow} & \\ & & I_2 \end{array} \qquad \exists \quad \begin{array}{ccccc} & & I_1 & & \\ & \overset{\alpha_1}{\nearrow} & & \overset{\beta_1}{\searrow} & \\ I_0 & & & & I \\ & \underset{\alpha_2}{\searrow} & & \underset{\beta_2}{\nearrow} & \\ & & I_2 & & \end{array} \qquad \beta_1\alpha_1 = \beta_2\alpha_2.$$

Theorem

$\mathbf{I}$-*colimits commute with inverse images in* $\mathbf{Set}$ *if and only if* $\mathbf{I}$ *is confluent.*

Remark

This means $\varinjlim : \mathbf{Set}^{\mathbf{I}} \longrightarrow \mathbf{Set}$ *is taut.*

Colimits

Theorem

(1) Confluent colimits of taut functors in $\mathscr{C}at(\mathbf{A}, \mathbf{Set})$ *are taut.*
(2) The inclusion $\mathscr{T}aut(\mathbf{A}, \mathbf{Set}) \rightarrowtail \mathscr{C}at(\mathbf{A}, \mathbf{Set})$ *creates confluent colimits.*

Example

Coproducts, filtered colimits, quotients by a group action, are all confluent.

Polynomials

$P(X) = \sum_{i \in I} X^{A_i}$ is taut.

A morphism of polynomials is a natural transformation $t(X) \colon P(X) \longrightarrow Q(X)$.

If $Q(X) = \sum_{j \in J} X^{B_j}$, morphisms $P(X) \longrightarrow Q(X)$ correspond to

$$\alpha \colon I \longrightarrow J, \quad \langle f_i \colon B_{\alpha(i)} \longrightarrow A_i \rangle_i \quad .$$

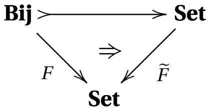
f is *vertical* if α is an identity,
cartesian if all the f_i are isomorphisms.

Proposition

t is taut if and only if all the f_i are epimorphisms.

Analytic functors

- Species $F\colon \mathbf{Bij} \rightarrowtail \mathbf{Set}$
- Analytic functor



(Left Kan extension)

$$
\begin{aligned}
\widetilde{F}(X) &= \int^{n\in\mathbb{N}} X^n \times F(n) \\
&\cong \varinjlim_{a\in F(n)} X^n \\
&\cong \sum_{n\in\mathbb{N}} \left(X^n \times F(n)\right) / S_n
\end{aligned}
$$

Proposition

$\widetilde{F}$ is taut.

Reduced powers

- Filter $\mathscr{F} \subseteq 2^A$ – closed under finite intersections
  – upclosed
- Reduced power $X^{\mathscr{F}}$

$$X^A/\sim \qquad (f \sim g \Leftrightarrow \{a \in A \mid f(a) = g(a)\} \in \mathscr{F})$$

$$\cong \varinjlim_{B \in \mathscr{F}} X^B$$

Proposition
$X^{\mathscr{F}}$ *is taut.*

Note: $X^{\mathscr{F}}$ is not an analytic functor, unless $\mathscr{F}$ is principal $(X^{\langle A_0 \rangle} \cong X^{A_0})$.

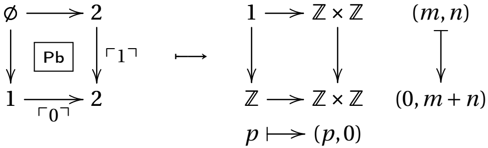
Monads

- The free monoid monad $1 + X + X^2 + \ldots$ is taut.

- The free commutative monoid monad $1 + X + X^2/S_2 + X^3/S_3 + \ldots$ is taut.

- Manes (2002 [4]) Collection monads are finitary taut monads.

- The free abelian group monad is not taut.



- Płonka (1967) [6] - Balanced equations (same variables on both sides).

- Szawiel/Zawadowski (2015) [7] - A finitary monad is taut if and only if it can be defined by balanced equations.

The difference operator

For $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ define $\Delta[F]\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ by $\Delta[F](X) = F(X+1) \setminus F(X)$.

Example

$\Delta[C] = 0 \quad \Delta[X] = 1$

Proposition

If F is taut, then $\Delta[F](X)$ is a taut subfunctor of $F(X+1)$.

Everything hinges on the following fact:

For a diagram of sets and functions

$$\begin{array}{ccc} A_0 & \rightarrowtail & A \\ {\scriptstyle f_0}\downarrow & (*) & \downarrow{\scriptstyle f} \\ B_0 & \rightarrowtail & B \end{array}$$

f restricts to the complements A_0' and B_0' iff $(*)$ is a pullback.

Colimits

Let $\mathbf{Taut} = \mathscr{T}aut(\mathbf{Set}, \mathbf{Set})$.

Proposition

Δ *is a functor, the* difference operator,

$$\Delta : \mathbf{Taut} \longrightarrow \mathbf{Taut}.$$

It preserves confluent colimits

$$\Delta[\varinjlim_{I} \Gamma I] \cong \varinjlim_{I} \Delta[\Gamma I] \ .$$

Corollary

(1) $\Delta[CF] \cong C\Delta[F]$.
(2) $\Delta[F+G] \cong \Delta[F] + \Delta[G]$.

Limits

Proposition

$\Delta[F \times G] \cong (\Delta[F] \times G) + (F \times \Delta[G]) + (\Delta[F] \times \Delta[G])$.

More generally:

Proposition

$$\Delta[\prod_{i \in I} F_i] \cong \sum_{J \subsetneq I} (\prod_{j \in J} F_j) \times (\prod_{k \notin J} \Delta[F_k]).$$

Theorem

Δ *preserves non-empty connected limits:*

$$\Delta[\varprojlim_I \Gamma I] \cong \varprojlim_I \Delta[\Gamma I].$$

Polynomials

- $\Delta[X^A] \cong \sum_{B \subsetneq A} X^B$

Proposition

If $P(X)$ is a polynomial functor, then so is $\Delta[P(X)]$. For $P(X) = \sum_{i \in I} X^{A_i}$,

$$\Delta[P(x)] = \sum_{j \in J} X^{B_j}$$

where $J = \left\{(i, B) \mid i \in I, B \subsetneq A_i\right\}$ and for $j = (i, B), \quad B_j = B$.

- $\Delta[X^n] \cong \sum_{k=0}^{n-1} \binom{n}{k} X^k$

Proposition

If $F(X)$ is a power series functor $\sum_{n=0}^{\infty} C_n X^n$, then $\Delta[F(X)]$ is also a power series $\sum_{n=0}^{\infty} D_n X^n$ where $D_n = \sum_{k=1}^{\infty} \binom{n+k}{k} C_k$.

Analytic functors

Proposition

If $\widetilde{F}(X)$ *is an analytic functor corresponding to the species* $F\colon \mathbf{Bij} \longrightarrow \mathbf{Set}$*, then* $\Delta[\widetilde{F}(X)]$ *is also analytic, corresponding to the species*

$$G(n) = \int^{k \in \mathbb{N}^+} F(n+k).$$

- A G-structure of cardinality n consists of a positive integer k and an equivalence class of F-structures of cardinality $n+k$. Two such structures are equivalent if one is transformed into the other by a bijection fixing the first n elements.

Reduced powers

A filter $\mathscr{F}$ on A induces an equivalence relation on subsets of A

$$\begin{aligned} B \sim C \quad &\Leftrightarrow \quad \{a \in A \mid a \in B \Leftrightarrow a \in C\} \in \mathscr{F} \\ &\Leftrightarrow \quad (B \cap C) \cup (B' \cap C') \in \mathscr{F}. \end{aligned}$$

For every $B \subseteq A$, let $\mathscr{F}_B = \{C \subseteq B \mid C \cup B' \in \mathscr{F}\}$.

Proposition

$\mathscr{F}_B$ is a filter on B and $\quad \Delta[X^{\mathscr{F}}] \cong \sum_{[B] \neq [A]} X^{\mathscr{F}_B}.$

(The sum is over all equivalence classes not equal to $[A]$, one summand for each class.)

Lax chain rule

Theorem

For taut functors F and G there is a taut natural transformation

$$\gamma_{G,F} \colon (\Delta[G]\circ F)\times\Delta[F] \longrightarrow \Delta[G\circ F]$$

which is:
(1) monic,
(2) natural in F and G,
(3) associative

$$\begin{array}{ccc} (\Delta[H]\circ G\circ F)\times(\Delta[G]\circ F)\times\Delta[F] & \xrightarrow{\mathrm{id}\times\gamma_{G,F}} & (\Delta[H]\circ G\circ F)\times\Delta[G\circ F] \\ {\scriptstyle \gamma_{H,G\circ F}\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\gamma_{H,G\circ F}} \\ (\Delta[H\circ G]\circ F)\times\Delta[F] & \xrightarrow[\gamma_{H\circ G,F}]{} & \Delta[H\circ G\circ F] \end{array} \quad ,$$

(4) unitary

$$\begin{array}{ccc} (\Delta[\mathrm{Id}]\circ F)\times\Delta[F] & \xrightarrow{\gamma_{\mathrm{Id},F}} & \Delta[\mathrm{Id}\circ F] \\ \| & & \| \\ 1\times\Delta[F] & \xrightarrow{\cong} & \Delta[F] \end{array} \quad , \qquad \begin{array}{ccc} (\Delta[F]\circ\mathrm{Id})\times\Delta[\mathrm{Id}] & \xrightarrow{\gamma_{F,\mathrm{Id}}} & \Delta[F\circ\mathrm{Id}] \\ \| & & \| \\ \Delta[F]\times 1 & \xrightarrow{\cong} & \Delta[F] \end{array} \quad .$$

Newton series

- For $f : \mathbb{R} \longrightarrow \mathbb{R}$ its *Newton series* is

$$\sum_{n=0}^{\infty} \frac{\Delta^n[f](0)}{n!} x^{\downarrow n} = \sum_{n=0}^{\infty} \Delta^n[f](0) \binom{x}{n}.$$

– $x^{\downarrow n}$ = falling power $x(x-1)\cdots(x-n+1)$

– $\binom{x}{n}$ = "binomial coefficient" $\frac{x(x-1)\cdots(x-n+1)}{n!}$

– $\Delta^n[f]$ is iterated difference

$$\begin{aligned}
\Delta^0[f](x) &= f(x) \\
\Delta^1[f](x) &= f(x+1) - f(x) \\
\Delta^2[f](x) &= f(x+2) - 2f(x+1) + f(x) \\
\Delta^3[f](x) &= f(x+3) - 3f(x+2) + 3f(x+1) - f(x) \\
&\quad etc.
\end{aligned}$$

Iterated difference

Proposition

$\Delta^n[F](X) = \{a \in F(X+n) \mid a \notin F(X+k) \quad \textit{for any proper subset} \quad k \subsetneq n\}$.

S_n acts on $\Delta^n[F](X)$ giving a species $\Delta^*[F](0)$ and a corresponding analytic functor

$$\sum_{n=0}^{\infty} \left(X^n \times \Delta^n[F](0)\right) / S_n.$$

But this won't give $F(X)$, even for polynomials. However $\Delta^n[F](X)$ has more "symmetries".

Proposition

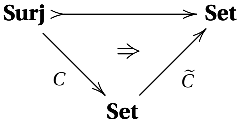
If $e\colon n \twoheadrightarrow m$ *is onto,* $F(X+e)\colon F(X+n) \longrightarrow F(X+m)$ *restricts to*

$$\Delta^e[F](X)\colon \Delta^n[F](X) \longrightarrow \Delta^m[F](X).$$

Soft species

Definition

Let **Surj** be the category of finite cardinals and surjections. A *soft species* is a functor $C\colon \mathbf{Surj} \longrightarrow \mathbf{Set}$. It determines a *soft analytic functor* (semi-analytic in [7]) by left Kan extension along the inclusion of **Surj** into **Set**:



$$\widetilde{C}(X) = \int^{n \in \mathbf{Surj}} C(n) \times X^n.$$

Proposition

Analytic functors are soft analytic. Soft analytic functors are taut.

Soft analytic functors

For $C \colon \mathbf{Surj} \longrightarrow \mathbf{Set}$, an element of $\widetilde{C}(X)$ is an equivalence class

$$[a \in C(n), f \colon n \longrightarrow X].$$

Factor f:

$$\begin{array}{ccc} a & \in & C(n) \\ \big\downarrow & & \big\downarrow{\scriptstyle C(k)} \\ b & \in & C(k) \end{array} \qquad \begin{array}{ccc} n & \xrightarrow{\ f\ } & X \\ {\scriptstyle e}\big\downarrow & & \big\| \\ k & \rightarrowtail & X \end{array} .$$

So every equivalence class has a representation with f monic.

$$\widetilde{C}(X) \cong \sum_{n=0}^{\infty} \Big(C(n) \times \mathrm{Mono}(n, X) \Big) / S_n \cong \sum_{n=0}^{\infty} C(n) \times \binom{X}{n}$$

but only as sets!

Newton series

For $F \colon \mathbf{Set} \longrightarrow \mathbf{Set}$ a taut functor, the sets $\Delta^n[F](0)$ extend to a soft species

$$\Delta^*[F](0) \colon \mathbf{Surj} \longrightarrow \mathbf{Set}.$$

The corresponding soft analytic functor

$$\overline{F}(X) = \widetilde{\Delta^*[F](0)} = \int^{n \in \mathbf{Surj}} \Delta^n[F](0) \times X^n$$

is the *Newton series* of F.

As sets

$$\overline{F}(X) \cong \sum_{n=0}^{\infty} \big(\Delta^n[F](0) \times \mathsf{Mono}\ (n, X)\big) / S_n \cong \sum_{n=0}^{\infty} \Delta^n[F](0) \times \binom{X}{n}.$$

Compare with:

$$\sum_{n=0}^{\infty} \frac{\Delta^n[f](0)}{n!} x^{\downarrow n} = \sum_{n=0}^{\infty} \Delta^n[f](0) \binom{x}{n}.$$

Fundamental theorem of functorial differences

Let $\mathbf{SoftSp}$ be the category $\mathbf{Set}^{\mathbf{Surj}}$ of soft species and natural transformations.

Theorem

(1) $\widetilde{(\ )}$ *gives a functor* $\mathbf{SoftSp} \longrightarrow \mathbf{Taut}$.

(2) $F \longmapsto \langle \Delta^n[F](0) \rangle_n$ *gives a functor* $\Delta^* : \mathbf{Taut} \longrightarrow \mathbf{SoftSp}$.

(3) $\widetilde{(\ )}$ *is left adjoint to* Δ^*.

(4) The unit is an isomorphism $C \xrightarrow{\cong} \Delta^*[\widetilde{C}](0)$.

Corollary

The Newton sum of a soft analytic functor "converges to it".

Conclusion

We've:

- Identified taut functors as the context to develop a functorial calculus of differences.
- Discovered confluent colimits which are central.
- Generalized the sum and product rules to colimits and limits.
- Established a lax chain rule.
- Expressed Newton summation as a left adjoint.

A multivariable version is in preparation.

Thank you!

References

[1] Andreas Blass.
Exact functors and measurable cardinals.
Pacific J. Math., 63(2):335–346, 1976.

[2] André Joyal.
Une théorie combinatoire des séries formelles.
Advances in Mathematics, 42(1):1–82, October 1981.

[3] Joachim Kock.
Notes on polynomial functors.
`http://mat.uab.cat/~kock/cat/polynomial.html`, August 16 2016.

[4] Ernest G. Manes.
Taut monads and $T0$-spaces.
Theoretical Computer Science, 275:79–109, 2002.

[5] Nelson Niu and David I. Spivak.
Polynomial functors: A mathematical theory of interaction.
`https://github.com/ToposInstitute/poly`, October 11 2023.

[6] Jerzy Płonka.
On a method of construction of abstract algebras.
Fundamenta Mathematicae, 61(2):183–189, 1967.

[7] Stanislaw Szawiel and Marek Zawadowski.
Monads of regular theories.
Appl.Categor Struct, 23:215–262, 2015.