

Longest winning paths in Hex

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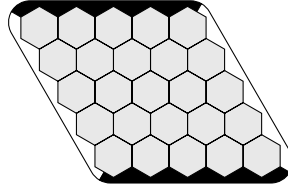
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Abstract

We answer the question: what is the longest winning path on a Hex board of size $n \times n$?

1 Winning paths in Hex

The game of Hex is played between two players on a rhombic board made of hexagonal cells, like this:



The players, Black and White, take turns putting a stone of their color on some cell of the board. Stones are never moved or removed. Black's goal is to connect the top edge to the bottom edge with black stones, and White's goal is to connect the left edge to the right edge with white stones. Hex has many interesting properties, among which is the fact that there is always exactly one winner: the game cannot end in a draw [1].

In this paper, we are interested in the length of possible winning paths, answering a question of Yusufaly [2]. Without loss of generality, we take Black's point of view. By a *winning connection*, we mean a set of black stones connecting the top edge to the bottom edge. By a *winning path*, we mean a winning connection that is minimal, in the sense that none of its proper subsets is a winning connection. The *length* of a winning path is the number of stones in it, and a winning path is *optimal* for a given board size if it is as long as possible. Note that the question we are interested in is the existence of winning paths, not whether such paths could occur in an actual game.

Obviously, on a board of size 5×5 , Black needs at least 5 stones to make a winning path, since Black needs at least one stone in each row, as shown in Figure 1(a). On the other hand, the longest possible winning path has length 11. There are 23 different winning paths of length 11, and a few of them are shown in Figure 1(b). The set of stones shown in Figure 1(c) is a winning connection, but not a winning path, because it is not minimal: Black can remove 8 of the stones and still have a winning connection.

For readers who like a good puzzle, consider the winning path for a 10×10 board that is shown in Figure 2(a). This path has length 46 and is not optimal. Before reading on, try to find a winning path of length 47.

2 Upper bounds on the path length

2.1 Loose bounds

Trivially, the length of a winning path on a board of size $n \times n$ is at most n^2 . Less trivially, the length of a winning path is bounded by $\frac{n^2+1}{2}$. Intuitively, it makes sense that only "about half" of the cells can be

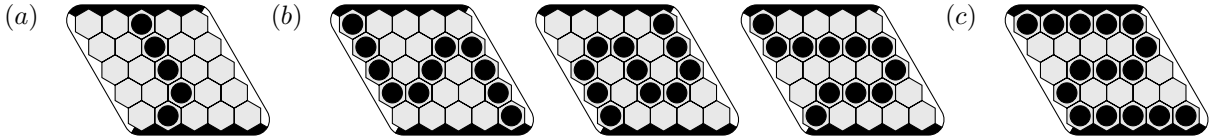


Figure 1: (a) A shortest winning path. (b) Some longest winning paths. (c) A non-minimal connection.

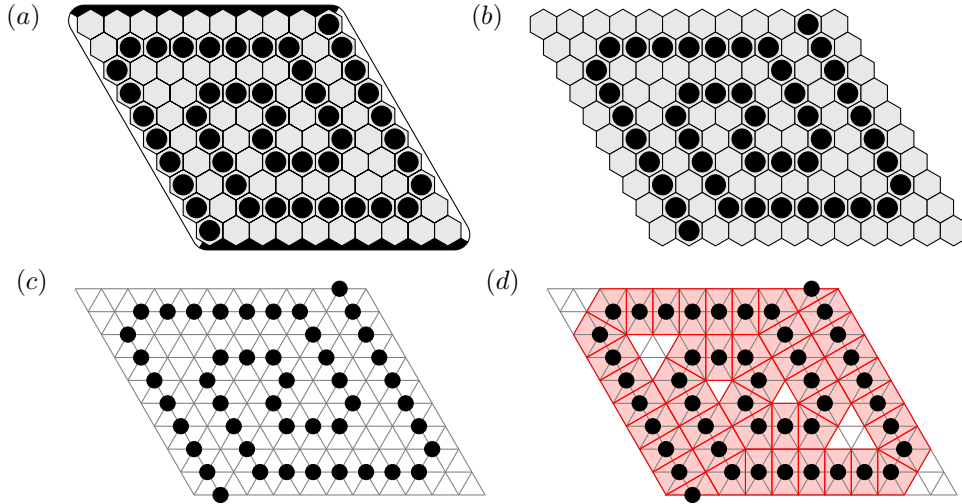
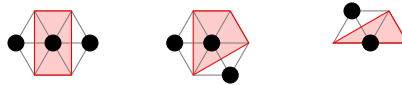


Figure 2: (a) A winning path for 10×10 . Can you find a longer one? (b) The same path, with the white edges replaced by columns of empty cells. (c) The triangular unit grid. (d) Each interior black stone occupies an area of 4 units.

part of a winning path. A more precise way of seeing this is as follows. Consider a winning path, such as the one shown in Figure 2(a). Add two columns of empty cells to represent the left and right board edges, as in Figure 2(b). Then create a triangular grid by connecting the centers of cells, as in Figure 2(c). This triangular grid consists of $2(n+1)(n-1)$ triangles. We call it the *unit grid* for an $n \times n$ -board, and we call the triangles the *unit triangles*. Moreover we define the *unit area* to be the area of a unit triangle. Note that the stones sit on the vertices of the unit grid. Next, we place each black stone inside a polygonal region whose shape is determined by the position of the neighboring stones:



We call the region associated to each stone its *domain*. As shown in Figure 2(d), the domains of all the stones in a winning path are disjoint. Moreover, each domain occupies an area of 4 units, except for the domains of the boundary stones, which only occupy an area of 2 units. So if there are k stones in the path, we have $4(k-1) \leq 2(n+1)(n-1)$, or equivalently $k \leq \frac{n^2+1}{2}$, as claimed.

2.2 Wasted triangles

Later in this section, we will derive better bounds on the path length. To motivate how this will be done, first consider the unit triangles in Figure 2(d) that are not covered by the domain of any stone. We call them *wasted triangles*. There are exactly 18 wasted triangles in Figure 2(d). We also note that the same

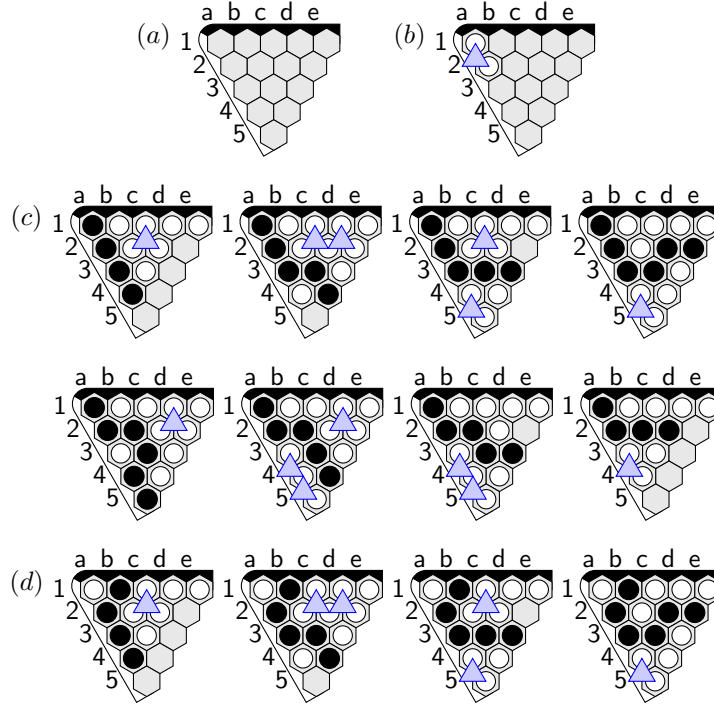


Figure 3: (a) The 15-cell corner region. (b) An upward-pointing wasted triangle when a_1 and b_1 are white.

wasted triangles can equivalently be observed in Figure 2(b), where they correspond to triples of pairwise adjoint empty cells. For a given winning path, let n be the board size, let k be the length of the path, and let t be the number of wasted triangles. These three quantities are related by a simple formula. Namely, by expressing the area of the unit grid in two different ways, we get $4(k-1) + t = 2(n+1)(n-1)$, or equivalently,

$$k = \frac{n^2 + 1}{2} - \frac{t}{4}. \quad (1)$$

It follows that for any fixed board size, maximizing the path length is equivalent to minimizing the number of wasted triangles. Our strategy for finding an upper bound on the path length will be to find a lower bound on the number of wasted triangles.

2.3 Wasted triangles near the corner

It will be helpful to distinguish *upward-pointing* unit triangles (\triangle) from *downward-pointing* ones (∇). The following lemma guarantees that there is at least one upward-pointing wasted triangle near the top left corner of the board.

Lemma 2.1. *Consider a winning path on a board of size at least 5×5 . Then there exists an upward-pointing wasted triangle within the 15-cell corner region shown in Figure 3(a).*

Proof. By case distinction. Consider the board region in Figure 3(a). We use standard Hex coordinates; for example, the cell in the acute corner is a_1 , and its neighbors are b_1 and a_2 . For brevity, we refer to cells that are part of the winning path as “black” and all other cells as “white”. Recall that a wasted triangle can overlap the left edge (cf. Figure 2 (b) and (d)).

- Case 1: a_2 is white. Then a_1 must also be white, since no minimal winning path can pass through it. Then a_1 , a_2 , and the left edge form an upward-pointing wasted triangle, as shown in Figure 3(b).

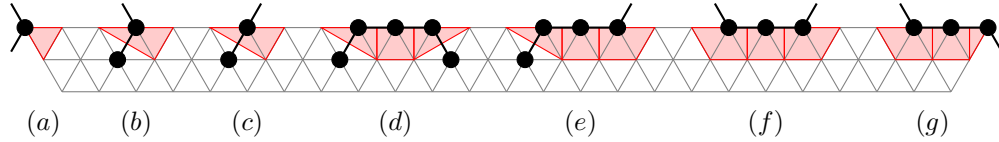
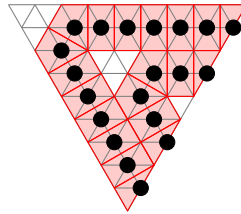


Figure 4: Possible boundary components of winning paths within a triangular region of the unit grid.

- Case 2: a_2 and a_1 are black. We distinguish several subcases, based on how the path continues. These subcases are shown in Figure 3(c). In each case, we know a cell to be white if it next to the top edge, if it already has two adjacent black neighbors, if its unique black neighbor already has two black neighbors, or if there is no room for it to have two black neighbors. In all cases, this is sufficient to identify at least one upward-pointing wasted triangle (and sometimes more than one).
- Case 3: a_2 is black and a_1 is white. In this case, b_1 must be black, and the situation is similar to the previous case, but slightly simpler. It is shown in Figure 3(d). \square

2.4 Boundary components in triangular regions

To motivate the next construction, we return to the unit grid of Figure 2(d), with its winning path and the domains of the path stones. Consider the following triangle-shaped subregion:



This particular region of the unit grid consists of 45 downward-pointing unit triangles (wasted or otherwise) and 36 upward-pointing ones. Therefore, the total number of downward-pointing unit triangles exceeds the number of upward-pointing ones by 9 in this region. Note that the domain of each interior black stone partially or fully overlaps some downward-pointing and some upward-pointing unit triangles. By area, each such domain covers an equal amount of downward- and upward-pointing unit triangles (2 units of each). On the other hand, the domain of each black boundary stone covers 1.5 downward-pointing and 0.5 upward-pointing unit triangles by area. It follows that the total excess of 9 upward-pointing triangles in the region can be accounted for as follows: 4 of them are due to boundary stones, and 5 are due to an excess of downward-pointing wasted triangles over upward-pointing ones. In other words, in the presence of 4 boundary stones, there must be 5 more downward-pointing wasted triangles than upward-pointing ones.

This reasoning generalizes to other triangle-shaped regions. Consider a winning path on some Hex board, and consider some downward-pointing triangle-shaped region of the unit grid. We first examine what happens along the region's boundary. Two stones of the winning path are *consecutive* if they are adjacent. We call a maximal stretch of consecutive stones that lie on the region's boundary a *boundary component*. Figure 4 shows some possible shapes of boundary components for the triangular region. In fact, up to symmetry, direction of neighboring path segments outside the region, and number of repetitions of horizontal path segments, all possible shapes of boundary components are represented in Figure 4. We say that a boundary component is *transversal* if it is of the form shown in Figure 4(b) or (c).

As can be seen by inspecting Figure 4, the union of the domains of the stones in any one boundary component covers, by area, one more upward-pointing than downward-pointing unit triangle. From this we can get a relationship between several quantities. For the given triangular region of the unit grid, let e denote the excess number of downward-pointing over upward-pointing unit triangles in the region. Let

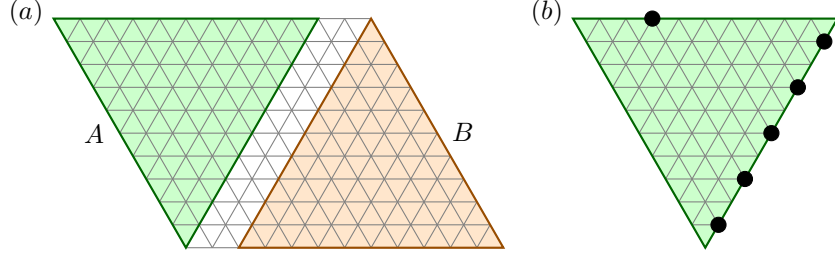


Figure 5: (a) The regions A and B . (b) A winning path has at most $\frac{n+1}{2}$ boundary components on the boundary of region A .

b denote the number of boundary components of the winning path in question. Let t_{down} and t_{up} denote the number of downward- and upwards-pointing wasted triangles in the region, respectively. Since each boundary component reduces the excess by exactly one, we have

$$e = b + t_{\text{down}} - t_{\text{up}}. \quad (2)$$

2.5 Improved bounds on the path length

Using equations (1) and (2) and Lemma 2.1, we can derive improved bounds on the length of winning paths.

Lemma 2.2. *Assume $n \geq 5$, and let k be the length of a winning path on a Hex board of size $n \times n$. Then $k \leq \frac{n^2}{2} - \frac{n}{4} + \frac{1}{4}$.*

Proof. Consider a winning path on a board of size $n \times n$. On the board's unit grid, let A and B be the maximal triangular regions containing the left and right acute corner, respectively, as shown in Figure 5(a). We first consider region A . Within this region, the excess number of downward-pointing over upward-pointing unit triangles is $e = n - 1$. As before, let b be the number of boundary components of the winning path. There can be at most one boundary component along the region's top edge, and at most $\frac{n-1}{2}$ boundary components along its right edge, as shown in Figure 5(b). Therefore, $b \leq 1 + \frac{n-1}{2} = \frac{n+1}{2}$. As before, let t_{down} and t_{up} be the number of downward- and up-ward-pointing wasted triangles in region A . Then using (2), we have $t_{\text{down}} - t_{\text{up}} = e - b \geq (n - 1) - \frac{n+1}{2} = \frac{n}{2} - \frac{3}{2}$, or equivalently, $t_{\text{down}} + t_{\text{up}} \geq \frac{n}{2} - \frac{3}{2} + 2t_{\text{up}}$. Moreover, from Lemma 2.1 and our assumption $n \geq 5$, we know that $t_{\text{up}} \geq 1$, which implies $t_{\text{down}} + t_{\text{up}} \geq \frac{n}{2} + \frac{1}{2}$. In other words, there are at least $\frac{n}{2} + \frac{1}{2}$ wasted triangles in region A . By the symmetric argument, there are also at least $\frac{n}{2} + \frac{1}{2}$ wasted triangles in region B , and therefore, the total number t of wasted triangles in the entire unit grid satisfies $t \geq n + 1$. Let k be the length of the winning path; then by (1), we have $k = \frac{n^2+1}{2} - \frac{t}{4} \leq \frac{n^2+1}{2} - \frac{n+1}{4} = \frac{n^2}{2} - \frac{n}{4} + \frac{1}{4}$, as claimed. \square

The next lemma offers a small improvement over Lemma 2.2 in case $n \equiv 3 \pmod{8}$.

Lemma 2.3. *Assume $n \geq 5$ and $n \equiv 3 \pmod{8}$, and let k be the length of a winning path on a Hex board of size $n \times n$. Then $k \leq \frac{n^2}{2} - \frac{n}{4} - \frac{3}{4}$.*

Proof. The proof is essentially the same as that of Lemma 2.2, with one small twist. As before, we consider regions A and B as in Figure 5(a). As before, we have $e = n - 1$ and $b \leq \frac{n+1}{2}$. We claim that there are at least $\frac{n}{2} + \frac{3}{2}$ wasted triangles in region A . To prove this claim, we consider two cases.

- Case 1: $b < \frac{n+1}{2}$. Then $t_{\text{down}} - t_{\text{up}} = e - b > \frac{n}{2} - \frac{3}{2}$, and with $t_{\text{up}} \geq 1$, this implies $t_{\text{down}} + t_{\text{up}} > \frac{n}{2} + \frac{1}{2}$. Since $\frac{n}{2} + \frac{1}{2}$ is an integer, $t_{\text{down}} + t_{\text{up}}$ must be the at least the next largest integer, so $t_{\text{down}} + t_{\text{up}} \geq \frac{n}{2} + \frac{3}{2}$, as claimed.

- Case 2: $b = \frac{n+1}{2}$. In this case, there must be exactly one boundary component along the top edge of region A , and exactly $\frac{n-1}{2}$ of them along the right edge, as shown in Figure 5(b). Since there is no additional space, each boundary component must consist of a single stone, and therefore must be transversal. The area of region A is $(n-1)^2$ units, which is divisible by 4. The area covered by the domains of stones is also divisible by 4, because each interior stone has a domain of area 4, and each boundary stone has a domain of area 2, and there is an even number of boundary stones. Therefore, the number of wasted triangles within region A must also be divisible by 4. From the proof of Lemma 2.2, we know that there are at least $\frac{n}{2} + \frac{1}{2}$ wasted triangles in region A . Since this number is congruent to 2 modulo 4, there must be at least two more, so at least $\frac{n}{2} + \frac{5}{2}$ wasted triangles in region A . This implies the claim.

We have proved that the number of wasted triangles in region A is at least $\frac{n}{2} + \frac{3}{2}$. Since by symmetry, the same applies to region B , the total number of wasted triangles on the whole grid is at least $n+3$, so by (1), the winning path has length at most $\frac{n^2+1}{2} - \frac{n+3}{4} = \frac{n^2}{2} - \frac{n}{4} - \frac{1}{4}$. Moreover, since $n \equiv 3 \pmod{8}$, the latter quantity is not an integer; the largest integer below it is $\frac{n^2}{2} - \frac{n}{4} - \frac{3}{4}$, proving the lemma. \square

The following theorem summarizes the findings of Lemmas 2.2 and 2.3.

Theorem 2.4. *Assume $n \geq 5$, and let k be the length of a winning path on a Hex board of size $n \times n$. Then we have:*

- (a) *If $n \equiv 0 \pmod{4}$, then $k \leq \frac{n^2}{2} - \frac{n}{4}$.*
- (b) *If $n \equiv 1 \pmod{4}$, then $k \leq \frac{n^2}{2} - \frac{n}{4} - \frac{1}{4}$.*
- (c) *If $n \equiv 2 \pmod{4}$, then $k \leq \frac{n^2}{2} - \frac{n}{4} - \frac{1}{2}$.*
- (d) *If $n \equiv 3 \pmod{8}$, then $k \leq \frac{n^2}{2} - \frac{n}{4} - \frac{3}{4}$.*
- (e) *If $n \equiv 7 \pmod{8}$, then $k \leq \frac{n^2}{2} - \frac{n}{4} + \frac{1}{4}$.*

Proof. Properties (a), (b), (c), and (e) are direct consequences of Lemma 2.2, because in each case, the claimed bound is the largest integer below $\frac{n^2}{2} - \frac{n}{4} + \frac{1}{4}$. Property (d) is Lemma 2.3. \square

3 Sharpness of the bounds

Theorem 3.1. *The bounds in Theorem 2.4 are sharp for all $n \geq 5$ except for $n = 9$.*

Proof. We first consider the case $n \leq 20$. Figure 6 shows a winning path for each $n = 1, \dots, 20$. The length of each path is listed in Figure 7, along with each applicable bound from Theorem 2.4. These paths witness the fact that the bounds are sharp except for $n = 9$. Conversely, the bounds of Theorem 2.4 imply that the paths in Figure 6 are optimal, except perhaps for $n = 1, 2, 3, 4$ and $n = 9$. The optimality of the paths for these remaining cases can be shown by exhaustive search.

Next, we consider the case $n > 20$. Figure 8 shows three constructions for turning a winning path for board size n into a winning path for board size $n+8$. By applying these constructions recursively, starting with the winning paths for $n = 13, \dots, 20$ from Figure 6, we obtain winning paths for all $n > 20$. As a matter of fact, the paths for $n = 13, 14, 15, 16, 18, 19, 20$ in Figure 6 were already obtained by this method.

It remains to show that the resulting paths match the bounds of Theorem 2.4. It is easy to check that given a winning path of length k for board size n , the constructions of Figure 8 yield a winning path for board size $n+8$ of length $k' = k + 8n + 30$. Now suppose that $k = \frac{n^2}{2} - \frac{n}{4} + c$. Then it follows that $k' = k + 8n + 30 = \frac{(n+8)^2}{2} - \frac{n+8}{4} + c$. Therefore, if a path for board size n matches its bound of Theorem 2.4, then so does the constructed path for board size $n+8$. The result follows by induction. \square

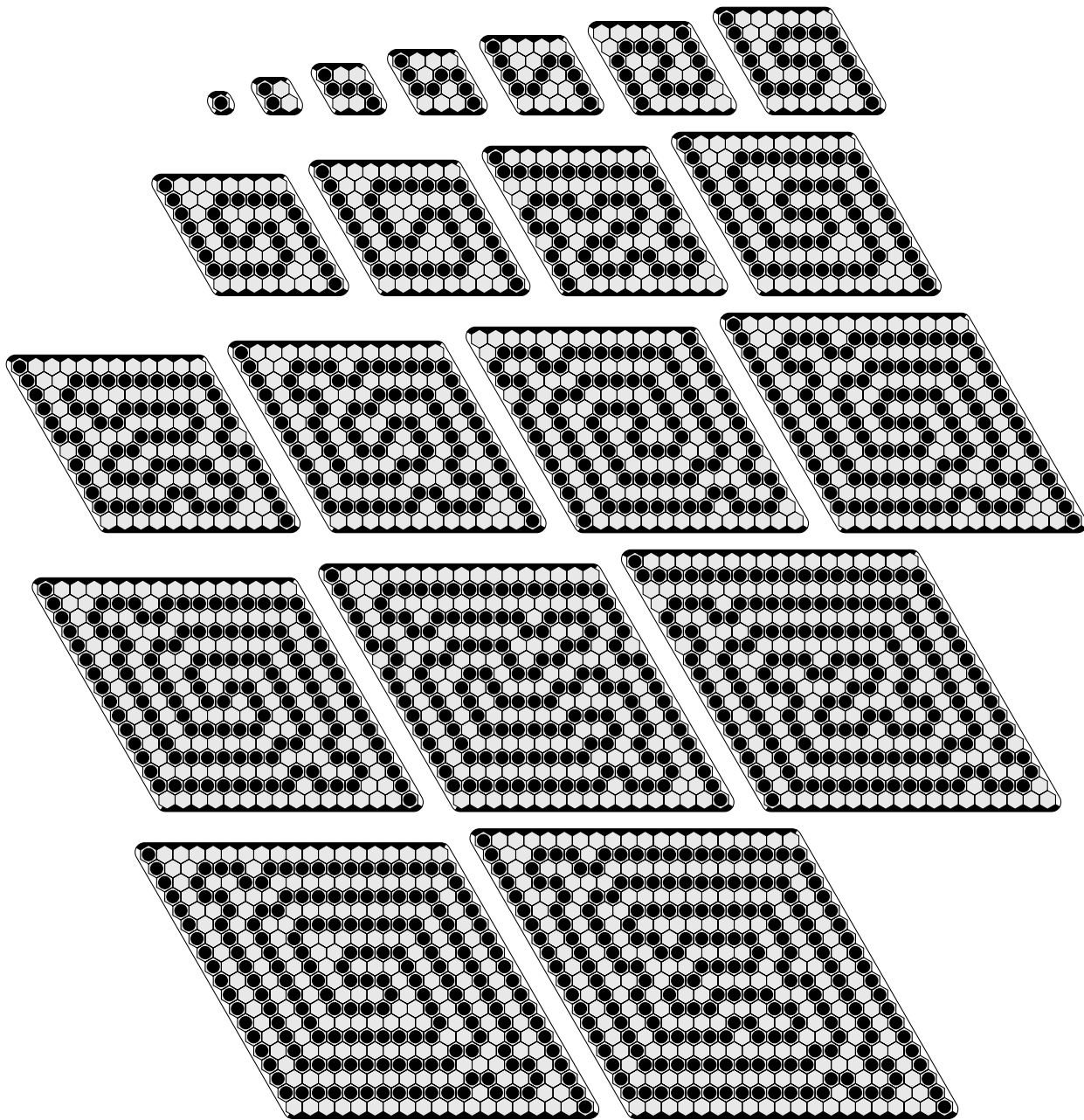


Figure 6: Optimal winning paths for $n = 1, \dots, 20$.

n	length	bound	count
1	1	N/A	1
2	2	N/A	3
3	5	N/A	1
4	8	N/A	4
5	11	11	23
6	16	16	51
7	23	23	20
8	30	30	115
9	37	38	5568
10	47	47	12
11	57	57	3521
12	69	69	40
13	81	81	1058
14	94	94	2104
15	109	109	668
16	124	124	7540
17	140	140	1298
18	157	157	83648
19	175	175	16631833
20	195	195	70630

Figure 7: For each $n = 1, \dots, 20$, the length of the optimal winning path (“length”), the bound on the length predicted by Theorem 2.4 (“bound”), and the number of optimal paths (“count”).

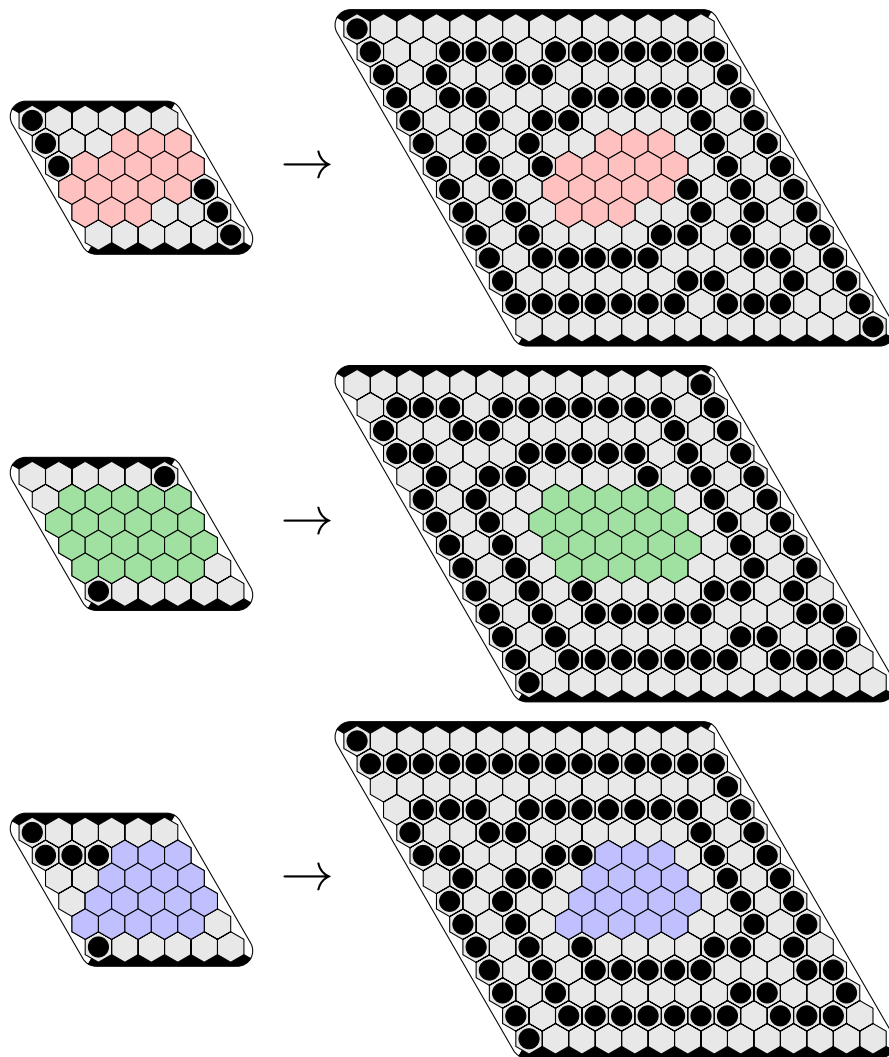
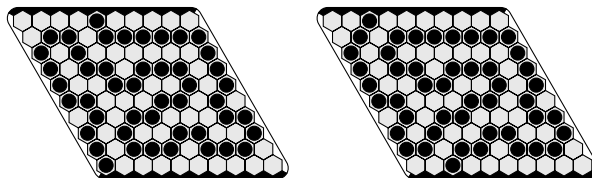


Figure 8: Recursive construction of winning paths for $n > 20$. Given a winning path of length k for board size n of one of the types shown, construct a winning path of length $k + 8n + 30$ for board size $n + 8$.

Remark 3.2. One can also ask *how many* optimal winning paths there are for each board size. The number of optimal winning paths for $n = 1, \dots, 20$ can be calculated by enumeration, and is shown in Figure 7. No formula is currently known for these numbers.

One may note that, compared to other nearby n , the number of solutions is unusually large for $n = 9$, $n = 11$, and $n = 19$. This may be related to the fact that the quantity $\frac{n^2}{2} - \frac{n}{4} - k$ is slightly larger than usual in these cases, where k is the path length. This yields an above-average number of wasted triangles and therefore more potential freedom in choosing winning paths. On the other hand, the number of solutions is unusually small for $n = 10$ and $n = 12$, perhaps reflecting the fact that these solutions are qualitatively different than those of smaller board sizes.

Remark 3.3. All of the winning paths shown in Figures 6 and 8 start and end in a corner of the board. However, not all optimal winning paths have this property. For example, only 2 of the 12 optimal winning paths for 10×10 start and end in a corner. Here are two optimal paths that do not go corner-to-corner:



Remark 3.4. Manually finding an optimal winning path can be a fun and difficult puzzle. Probably the most difficult cases are $n = 10$, $n = 12$, and $n = 17$, because the solutions do not seem to follow any pattern that can be guessed from smaller board sizes.

4 Conclusions and future work

We have determined the length of the longest winning path on Hex boards of size $n \times n$, for all n . More generally, one may ask the same question for non-rhombic boards, i.e., boards of size $m \times n$. While it is possible that our methods can shed some light on this more general question, we have left it for future work.

Acknowledgements

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