Noise and Disturbance for Qubit Measurements

An Information-Theoretic Characterisation

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Introduction

Heisenberg's Uncertainty Principle (informally)

The measurement of one quantum observable introduces an irreversible disturbance into any complementary observable property of the system.

It is crucial to make the distinction between two forms of uncertainty relations:

- Preparation Uncertainty Relations: $\Delta\hat{x}\Delta\hat{p}_x \geq \frac{\hbar}{2}$, etc.
  - Tradeoff between the accuracy of $\hat{x}$ and $\hat{p}_x$ values with which a state can be prepared
- Measurement Uncertainty Relations: $N(\mathcal{M},\hat{x})D(\mathcal{M},\hat{p}_x) \geq \frac{\hbar}{2}$
  - Tradeoff between the accuracy of $\hat{x}$ measurement and associated disturbance of $\hat{p}_x$ for a measurement $\mathcal{M}$
- Different measures of noise and disturbance are possible
  - Entropic measures

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Definitions[1]

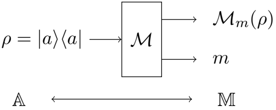
Let A, B be discrete observables and $\mathcal{M} = \{\mathcal{M}_m\}_m$ an instrument.

Noise – $N(\mathcal{M}, A)$

$N(\mathcal{M}, A)$ is the uncertainty in the outcome of the measurement $\mathcal{M}$ for a randomly prepared eigenstate $|a\rangle$: $N(\mathcal{M}, A) = H(\mathbb{A}|\mathbb{M})$.

Disturbance – $D(\mathcal{M}, B)$

$D(\mathcal{M}, B)$ is the uncertainty in a measurement of B following the measurement of $\mathcal{M}$ on a randomly prepared state $|b\rangle$ and the possible application of a correction $\mathcal{E}$: $D(\mathcal{M}, B) = \min_{\mathcal{E}} H(\mathbb{B}|\mathbb{B}')$.



[1] F. Buscemi, M. J. W. Hall, M. Ozawa & M. W. Wilde. PRL 112, 050401, 2014.

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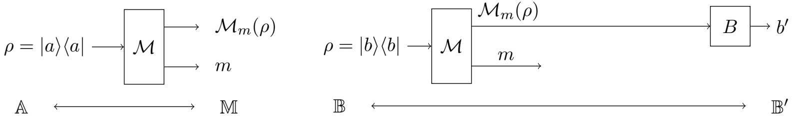

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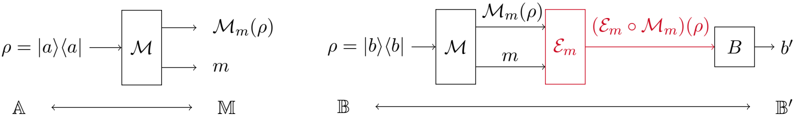

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Noise-Noise & Noise-Disturbance Tradeoffs

There are two separate tradeoffs of interest:

- Joint-measurement noise (noise-noise): how accurately can one measure both A and B?
- Noise-disturbance: how does the accuracy of $\mathcal{M}$ with respect to A affect its disturbance with respect to B?

It is known that for all $A, B, \mathcal{M}$ we have both

$$N(\mathcal{M}, A) + N(\mathcal{M}, B), \quad N(\mathcal{M}, A) + D(\mathcal{M}, B) \geq \log \max_{a,b} |\langle a|b\rangle|^2.$$

- These relations are generally not tight
- We wish to characterise precisely the sets of obtainable noise-noise and noise-disturbance values
  - and thus extract tight relations
- We focus on qubits, the simplest possible systems

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Noise-Noise Uncertainty Relations

For qubits, the set of obtainable $\big(N(\mathcal{M},A),\, N(\mathcal{M},B)\big)$ values is

$$\mathrm{conv}\,\{(H(A|\rho),\, H(B|\rho)) \mid \rho \text{ is any qubit density matrix}\}\,.$$

- This is simply the convex hull of the "entropic preparation uncertainty region" for A and B, for which a tight characterisation was recently proved[2]

For $A = \sigma_z,\ B = \sigma_x$ we have:

$$g\big(H(\sigma_z|\rho)\big)^2 + g\big(H(\sigma_x|\rho)\big)^2 \le 1$$

where g is the inverse of

$$h(x) = -\tfrac{1+x}{2}\log\left(\tfrac{1+x}{2}\right) - \tfrac{1-x}{2}\log\left(\tfrac{1-x}{2}\right).$$

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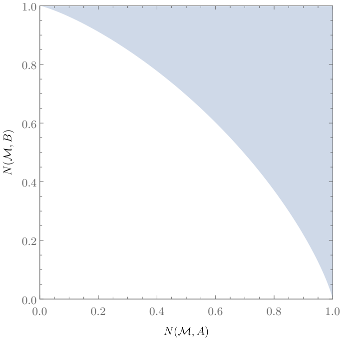
$$N(\mathcal{M},\sigma_z) + N(\mathcal{M},\sigma_x) \ge 1$$



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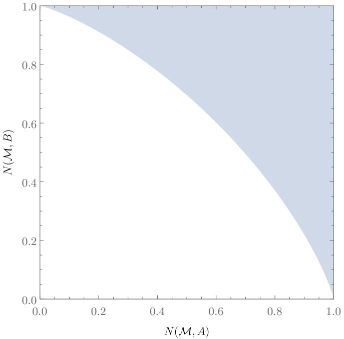

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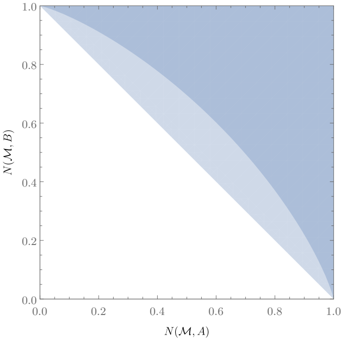

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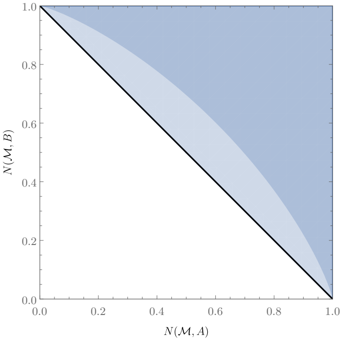

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Let $A = \boldsymbol{a} \cdot \boldsymbol{\sigma}$, $B = \boldsymbol{b} \cdot \boldsymbol{\sigma}$. Generally, we have two cases:

- For $|\boldsymbol{a} \cdot \boldsymbol{b}| \gtrsim 0.391$, the entropic uncertainty region is convex:

$$g\left(N(\mathcal{M}, A)\right)^2 + g\left(N(\mathcal{M}, B)\right)^2 - 2|\boldsymbol{a} \cdot \boldsymbol{b}|\, g\left(N(\mathcal{M}, A)\right)\, g\left(N(\mathcal{M}, B)\right) \leq 1 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$$

  - Two-outcome measurements are optimal

- For $|\boldsymbol{a} \cdot \boldsymbol{b}| \lesssim 0.391$, the entropic uncertainty region is concave
  - No analytic relation, can numerically calculate optimal bound
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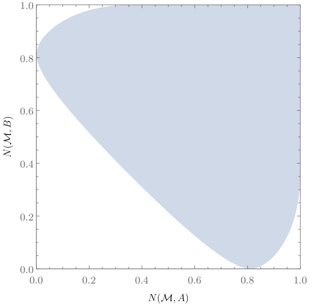

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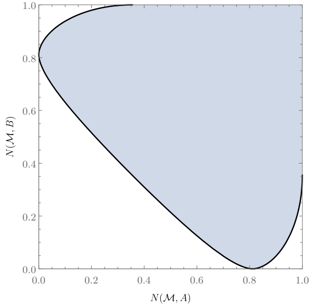

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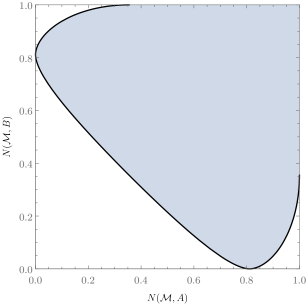

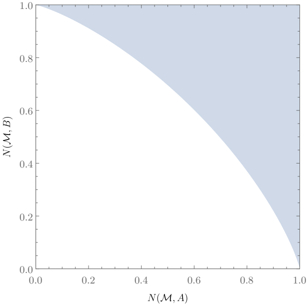

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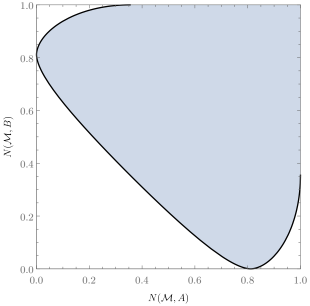

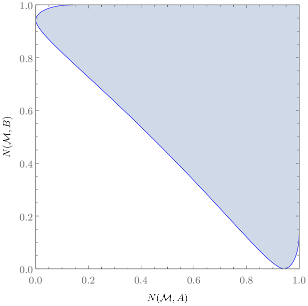

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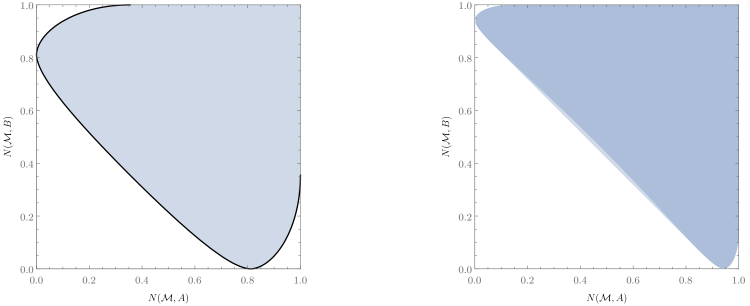

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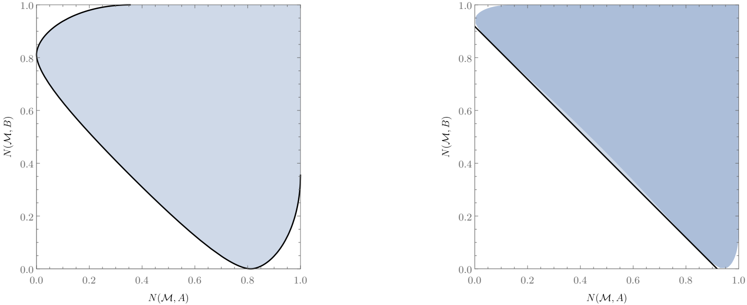

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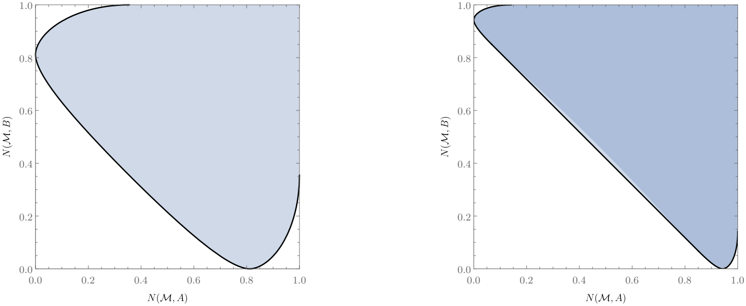


Noise-Disturbance Uncertainty Relations

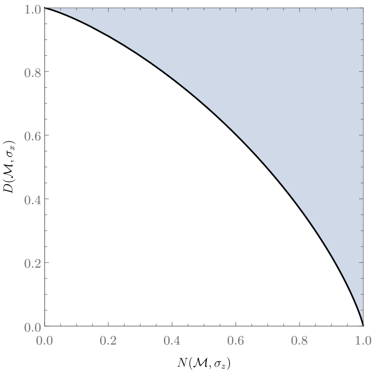
Understanding the noise-disturbance tradeoff requires taking into account the transformation on the state due to $\mathcal{M}$

- The following bound was conjectured recently[3] for $\boldsymbol{a} \cdot \boldsymbol{b} = 0$:

$$g\left(N(\mathcal{M}, \sigma_z)\right)^2 + g\left(D(\mathcal{M}, \sigma_x)\right)^2 \leq 1$$

We prove this bound to be tight under certain conditions:

- If the measurement $\mathcal{M}$ has only 2 outcomes
  - Can be extended to any $\boldsymbol{a}, \boldsymbol{b}$
- If the state is transformed according to the "square-root dynamics" and no correction is applied
  - $\rho \xrightarrow{\mathcal{M}} \sum_m \sqrt{M_m}\rho\sqrt{M_m}$



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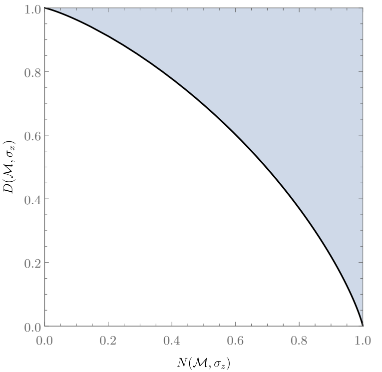

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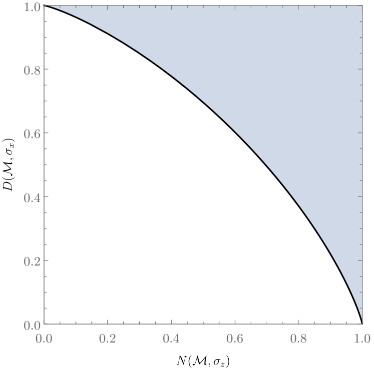

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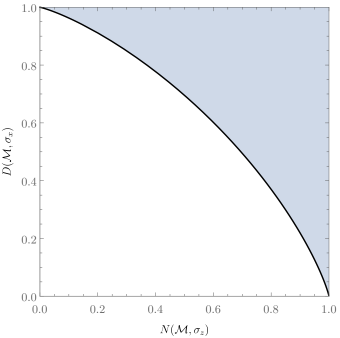
Tightening the Noise-Disturbance Relation

Is this bound tight generally, or can we violate it?

- We find a class of 3-outcome measurements that violate it
- Non-trivial corrections needed
- Numerically, appears to be tight for arbitrary $\mathcal{M}$ (even with 4 or more outcomes)
- Doesn't appear to give a nice relation; parametrically ($0 \leq \theta \leq \frac{\pi}{2}$):

$$(N(\mathcal{M},\sigma_z), D(\mathcal{M},\sigma_x)) = \left(\frac{\cos\theta + h(\sin\theta)}{1+\cos\theta}, \frac{h(\cos\theta)}{1+\cos\theta}\right)$$



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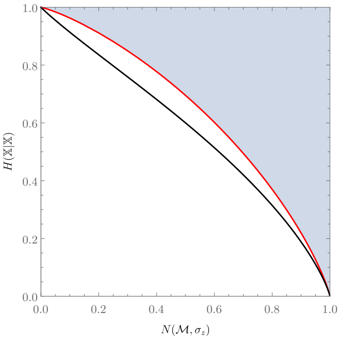

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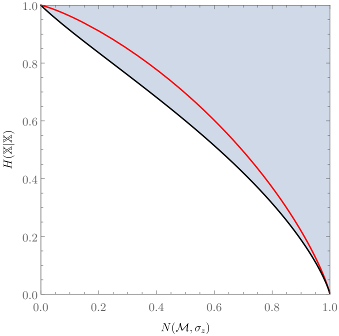

Note that the set of obtainable values is non-convex, and that three-outcome measurements are optimal (cf. noise-noise case)

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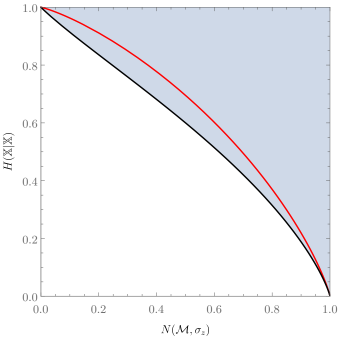

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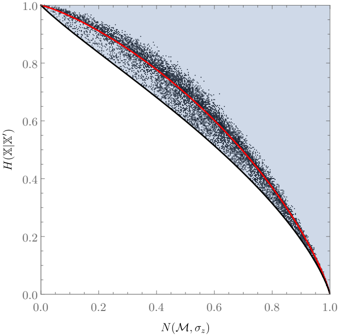

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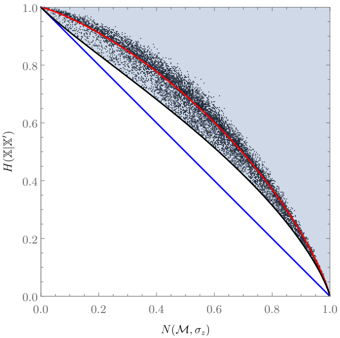

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Conclusions and Future Work

Tight joint-measurement uncertainty relations for qubits

- Four-outcome POVMs needed for "optimal" measurements
- Can readily generalise to 3 or more observables

Conjectured tight characterisation of noise-disturbance region

- Three-outcome measurements with non-trivial corrections or measurement dynamics needed for "optimal" measurements

Several points remain to explore:

- Noise-disturbance for non-orthogonal Pauli measurements
- Measurements that are optimal with respect to both tradeoffs
- Relation to other notions of noise/disturbance

Further information:

- Forthcoming paper
- Abbott *et al.* Mathematics 4, p. 8 (arXiv:1512.02383)

Saturating Noise-Noise Bound

- Any point $\big(N(\mathcal{M},A),N(\mathcal{M},B)\big)$ in $\{(H(A|\rho),\,H(B|\rho))|\rho\}$ can be obtained by a POVM projecting onto ρ, i.e.,

$$\{\tfrac{1}{2}(\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma}),\ \tfrac{1}{2}(\mathbb{1}-\boldsymbol{r}\cdot\boldsymbol{\sigma})\}$$

- Let (u_1,v_1) and (u_2,v_2) be two points obtained by projections onto ρ_1 and ρ_2
- For any $q\in[0,1]$ the POVM

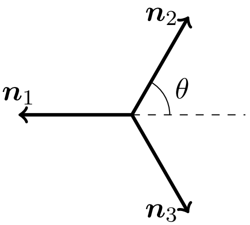
$$\{\tfrac{q}{2}(\mathbb{1}+\boldsymbol{r}_1\cdot\boldsymbol{\sigma}),\ \tfrac{q}{2}(\mathbb{1}-\boldsymbol{r}_1\cdot\boldsymbol{\sigma}),\tfrac{1-q}{2}(\mathbb{1}+\boldsymbol{r}_2\cdot\boldsymbol{\sigma}),\ \tfrac{1-q}{2}(\mathbb{1}-\boldsymbol{r}_2\cdot\boldsymbol{\sigma})\}$$

gives

$$\begin{aligned}\big(N(\mathcal{M},A),N(\mathcal{M},B)\big) &= q(u_1,v_1)+(1-q)(u_2,v_2)\\ &= (qu_1+(1-q)u_2,qv_1+(1-q)v_2)\end{aligned}$$

Improved Noise-Disturbance Bound

- Consider the POVM $M = \{M_1, M_2, M_3\}$ where $M_i = \alpha_i(\mathbb{1} + \boldsymbol{n}_i \cdot \boldsymbol{\sigma})$
  - For it to be valid, need $\sum \alpha_i = 1$ and $\sum \alpha_i \boldsymbol{n}_i = \boldsymbol{0}$
  - Pictorially, the vectors $\alpha_i \boldsymbol{n}_m$ must form a triangle
- For $\theta \in [0, \pi/2]$ take the following arrangement:



- Following measurement outcome m, the system is in state with Bloch-vector $\boldsymbol{n}_m$
- Perform the correction mapping $\boldsymbol{n}_2, \boldsymbol{n}_3 \to \boldsymbol{x}$ and leaving $\boldsymbol{n}_1 = -\boldsymbol{x}$ unchanged