An operational resource theory of purity

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Introduction

Resource-theoretic approach to the foundations of thermodynamics.

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Resource theories

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Free operations give a preorder on resources: *A more valuable than B* if $A \xrightarrow{\text{free}} B$.

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Free states: only the maximally mixed state $\chi = \frac{1}{d} I$.
Free operations of the form (basic noisy operations)

$$\mathcal{N}\left(\rho_{\mathrm{A}}\right) = \mathrm{tr}_{\mathrm{E}}\left[U_{\mathrm{AE}}\left(\rho_{\mathrm{A}} \otimes \chi_{\mathrm{E}}\right) U_{\mathrm{AE}}^{\dagger}\right]$$

and its topological closure [Shor, Gour et al.].

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Is it specific to QM?

Contents

Section 1

Operational purity

OPTs

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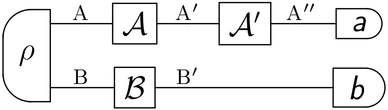

Reversible transformations

A (deterministic) transformation $\mathcal{U}:\mathrm{A}\to\mathrm{B}$ is reversible if there exists $\mathcal{U}^{-1}:\mathrm{A}\to\mathrm{B}$ such that $\mathcal{U}^{-1}\mathcal{U}=\mathcal{I}_{\mathrm{A}}$, *and* $\mathcal{U}\mathcal{U}^{-1}=\mathcal{I}_{\mathrm{B}}$.

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In general we can't follow the approaches relying on the invariant state.

Purity via RaRe channels [Chiribella & Scandolo '15b]

Free operations (RaRe channels) of the form [Uhlmann]

$$\mathcal{R} = \sum_i p_i \mathcal{U}_i ,$$

with $\mathcal{U}_i$'s *reversible* channels and $\{p_i\}$ *probability distribution*.

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- ρ is purer than σ if $\mathcal{R}\rho = \sigma$, for some RaRe channel $\mathcal{R}$
- No reference to the invariant state!

Section 2

Other definitions of purity

Working plan

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We need to introduce some axioms...
with some thermodynamic flavour!

Purity Preservation

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The sequential and parallel composition of pure transformations is a pure transformation.

- The product of two pure states is pure.
- Without Purity Preservation, information could be lost simply by composing devices!

Causality

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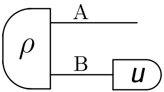
- We can use u to define the marginals of bipartite states:

$$\rho_{\mathrm{A}} := \mathrm{Tr}_{\mathrm{B}} \rho_{\mathrm{AB}} =$$



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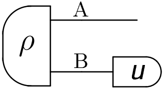
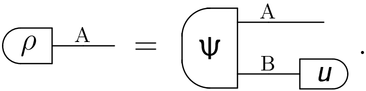


Important in thermodynamics: we need to consider subsystems of larger systems!

Purification [Chiribella et al. '10]

1. Every state ρ_{A} can be purified: there exists a pure state Ψ_{AB} such that



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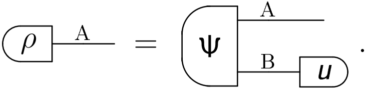
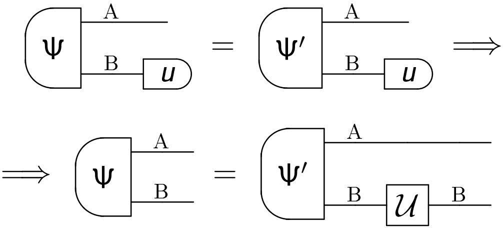


2. Different purifications of the same state differ by a reversible transformation $\mathcal{U}$ on the purifying system:



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It also guarantees the existence of a unique invariant state χ for every system!

Pure Sharpness

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- We can think of a as part of a yes/no test to check an *elementary property* of the system.
- Pure Sharpness guarantees that every system has an elementary property.

Permutability and Strong Symmetry

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Under the previous axioms, Permutability implies Strong Symmetry.

Real, complex, and fermionic QM (+ possible higher-order interference variants) satisfy all these axioms.

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1. For every non-trivial system A, an integer $d_{\mathrm{A}} \geq 2$ (dimension) gives the size of all *maximal* sets of perfectly distinguishable pure states.

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We strongly think that Permutability/Strong Symmetry isn't necessary in fact (work in progress. . .).

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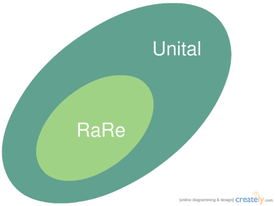
RaRe channels are unital, but not all unital channels are RaRe [Landau & Streater]!



Noisy operations

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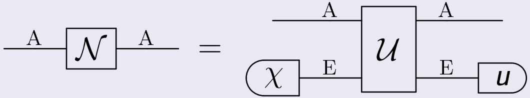
Channels $\mathcal{N}$ of the form



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and their topological closure.

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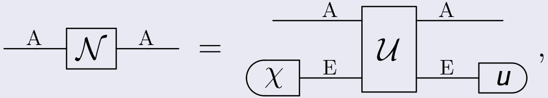

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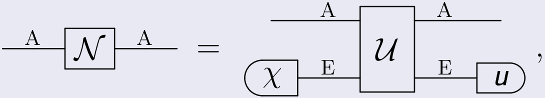

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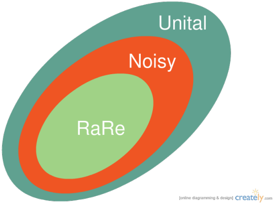
- Noisy operations are unital, but not all unital channels are noisy operations [Haagerup & Musat]!
- RaRe channels are noisy operations (via a tool in Lee & Selby's talk), but not all noisy operations are RaRe [Shor]!

Conclusions

- We've analysed some different ways to define a resource theory of purity: RaRe channels, noisy operations, and unital channels.



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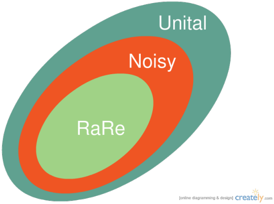

- Despite the *strict* inclusions, they're equivalent: the induced preorder is the same (majorisation).

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