Infinite-dimensional Categorical Quantum Mechanics

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Motivation for this work

We want to do (diagrammatic) CQM in ∞-dimensions, but...

[1] Although there is a characterisation of orthonormal bases in terms of $H^\star$ algebras.

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(c) Some *genuinely new* finite vectors arise in non-standard Hilbert spaces:

$$\text{e.g. } \frac{1}{\sqrt{\nu}} \sum_{n=1}^{\nu} |e_n\rangle, \text{ where } \begin{cases} |e_n\rangle & \text{form an orthonormal basis} \\ \nu & \text{is an infinite natural} \end{cases}$$

The Category *Hilb - Objects

Objects are pairs $\mathcal{H} := (V, |e_n\rangle_{n=1}^{\kappa})$ specified by the following data:

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(iii) a non-standard natural number κ (the dimension of $\mathcal{H}$):

- V is finite-dim $\Rightarrow$ κ is the finite natural dim V
- V is ∞-dim $\Rightarrow$ κ is some infinite natural[2]

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The Category *Hilb - Morphisms

Morphisms $F : (V, |e_n\rangle_{n=1}^{\nu}) \to (W, |e'_m\rangle_{m=1}^{\mu})$ in *Hilb are uniquely determined by $\mu \times \nu$ matrices (like in finite dimensions):

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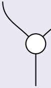
Equipped with matrix composition, tensor, adjoint and addition, *Hilb is an enriched $\dagger$-symmetric monoidal category, with ${}^\star\mathbb{C}$ as its field of scalars.

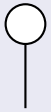
Classical Structures in *Hilb

Theorem (Classical structures)

The following co-multiplication and co-unit define a unital special commutative $\dagger$*-Frobenius algebra on any object* $(V, |e_n\rangle_{n=1}^{\kappa})$ *of* *Hilb*:*



$$:= \sum_{n=1}^{\kappa} |e_n\rangle \otimes |e_n\rangle \langle e_n|$$



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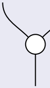
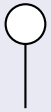
 $:= \sum_{n=1}^{\kappa} |e_n\rangle \otimes |e_n\rangle \, \langle e_n|$  $:= \sum_{n=1}^{\kappa} \langle e_n|$

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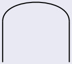
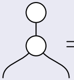
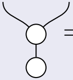
*Hilb *is †-compact, with the following cup and cap on object* $(V, |e_n\rangle_{n=1}^{\kappa})$*:*

 $:=$  $= \sum_{n=1}^{\kappa} \langle e_n| \otimes \langle e_n|$  $:=$  $= \sum_{n=1}^{\kappa} |e_n\rangle \otimes |e_n\rangle$

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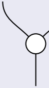
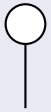
 $:= \sum_{n=1}^{\kappa} |e_n\rangle \otimes |e_n\rangle \langle e_n|$  $:= \sum_{n=1}^{\kappa} \langle e_n|$

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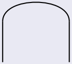
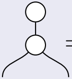
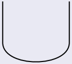
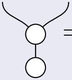
 $:=$  $= \sum_{n=1}^{\kappa} \langle e_n| \otimes \langle e_n|$  $:=$  $= \sum_{n=1}^{\kappa} |e_n\rangle \otimes |e_n\rangle$

The †-compact structure gives rise to a trace, and we have $\mathrm{Tr}[id] = \kappa$.

Wavefunctions with Periodic Boundary

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(i) standard separable Hilbert space $\mathrm{L}^2[\mathbb{R}/(L\mathbb{Z})]$

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$$\left(\frac{1}{\sqrt{L}}|\chi_{n\hbar}\rangle\right)_{n\in\mathbb{Z}}, \text{ where } \chi_{n\hbar} := x \mapsto e^{-i(2\pi/L)nx}$$

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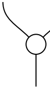
Classical structure corresponding to the momentum observable:


$$:= \sum_{n=-\omega}^{+\omega} |\chi_{n\hbar}\rangle \otimes |\chi_{n\hbar}\rangle \, \langle\chi_{n\hbar}|$$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Theorem (Position Eigenstates)

The following states act as position eigenstates, in the sense that for all standard smooth $f \in \mathrm{L}^2[\mathbb{R}/(L\mathbb{Z})]$ *we have* $\langle \delta_{x_0} | f \rangle \simeq f(x_0)$*:*

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In the categorical framework, the relationship between momenta and space translation symmetry is witnessed by the following group algebra:

$$\Big(\Big\{ \sqrt{L} |\delta_x\rangle \,\Big|\, x \in \mathbb{R}/(L\mathbb{Z}) \Big\} ,$$  $,$  $$\Big) \cong \Big(\mathbb{R}/(L\mathbb{Z}), +, 0 \Big)$$

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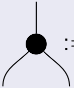
Theorem (Position Observable)

The following multiplication and unit define a unital quasi-special commutative †-Frobenius algebra, the group algebra for boosts:

 $$:= \frac{1}{\sqrt{L}^3} \sum_{n=-\omega}^{+\omega} \sum_{m=-\omega}^{+\omega} |\chi_{(n \oplus m)\hbar}\rangle \langle \chi_{n\hbar}| \otimes \langle \chi_{m\hbar}|$$  $$:= \frac{1}{\sqrt{L}} |\chi_0\rangle$$

This algebra is strongly complementary to the momentum observable, and copies the (re-scaled) position eigenstates $\sqrt{L}|\delta_{x_0}\rangle$.

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- Non-separable spaces and quantum field theory

Thanks for Your Attention!

Any Questions?

S Gogioso, F Genovese. *Infinite-dimensional Categorical Quantum Mechanics*. arXiv:1605.04305
S Abramsky, C Heunen. *H*-algebras and nonunital Frobenius algebras*. arXiv:1011.6123
A Robinson. *Non-standard analysis*. Princeton University Press, 1974