Tutorial Lecture:
Non-Locality, Contextuality, and Sheaves

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Non-Locality and Contextuality

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Several approaches:

- Graph-theoretic: Cabello-Severini-Winter.
- Operational: Spekkens.
- "Contextuality by Default": Dzhafarov.
- Sheaf-theoretic: Abramsky-Brandenburger.

Outline

Three parts:

1. Illustrate that non-locality, and contextuality in general, consist in “local consistency + global inconsistency”.
2. Formulate this idea using topological and sheaf-theoretic terminology.
3. Demonstrate what this formulation can do.

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The high-level formalism will show

- Ubiquity of contextuality:
  Phenomena formally isomorphic to contextuality can be found in various other fields.
- Many mathematical faces of contextuality:
  It admits applications of logic, algebraic topology, combinatorics, etc.

Part I. Bell Non-Locality

- Review the concepts of Bell non-locality and no-signalling,
- Massage their definitions and Fine's theorem, and
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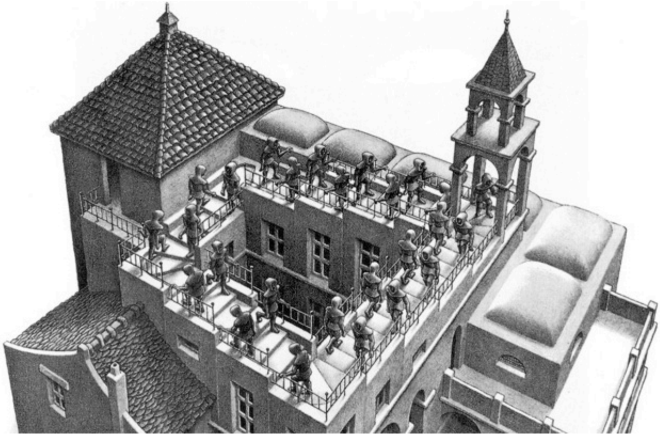
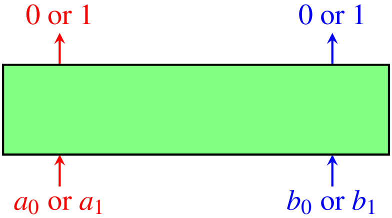


Bell Non-Locality

Bell-type setup. Input-output box for $(2, 2, 2)$ scenario:



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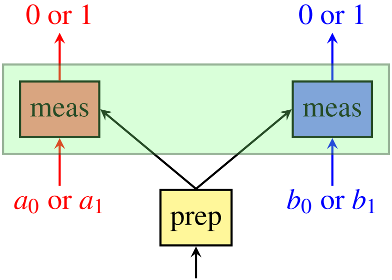

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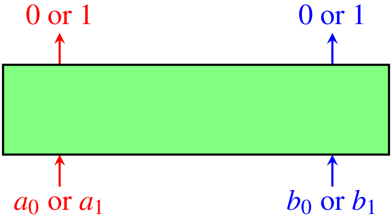

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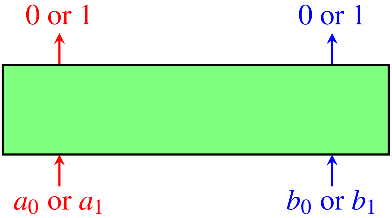

Distribution $p(o_A, o_B \mid a_i, b_j)$ for each **context** $\{a_i, b_j\}$.

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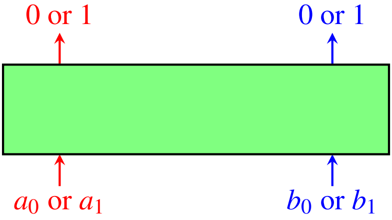

Distribution $p(o_A, o_B \mid a_i, b_j)$ for each **context** $\{a_i, b_j\}$.

So a probability table:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$1/2$	0	0	$1/2$
(a_0, b_1)	$3/8$	$1/8$	$1/8$	$3/8$
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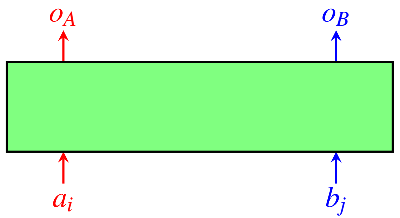
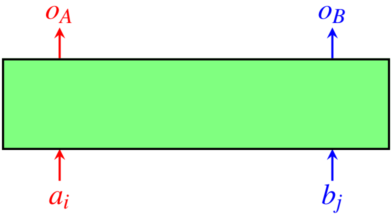
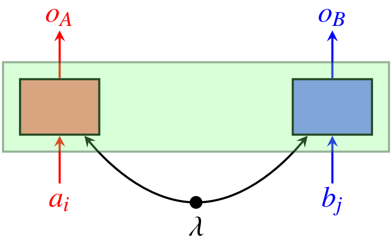

o_A
o_B
a_i
b_j



Locality (local causality)



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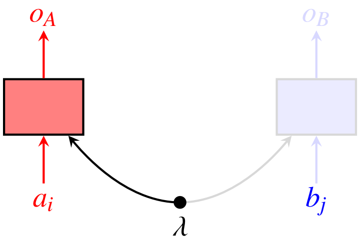
There is a model with a "local hidden variable" λ s.th.



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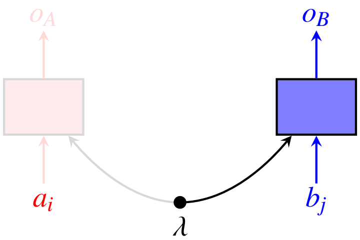

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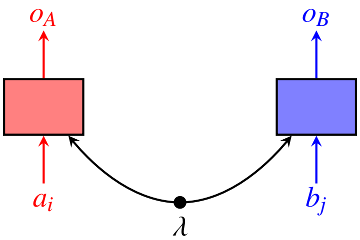

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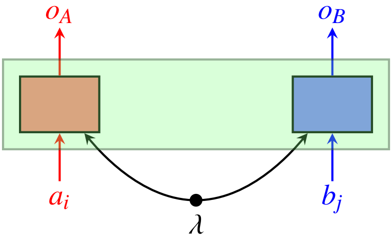
$$p(o_A \,|\, a_i, \lambda)\, p(o_B \,|\, b_j, \lambda)$$



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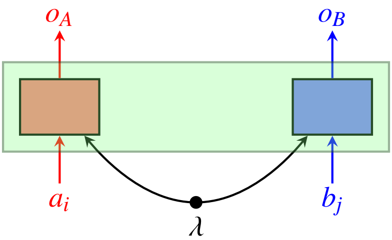
$$p(o_A, o_B \mid a_i, b_j) = \sum_\lambda p(o_A \mid a_i, \lambda)\, p(o_B \mid b_j, \lambda)\, p(\lambda).$$



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Violated by the Bell table:

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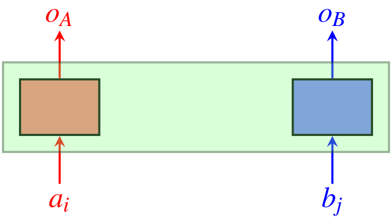
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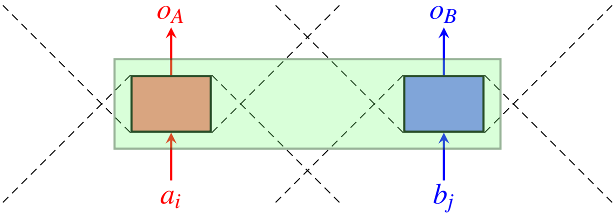
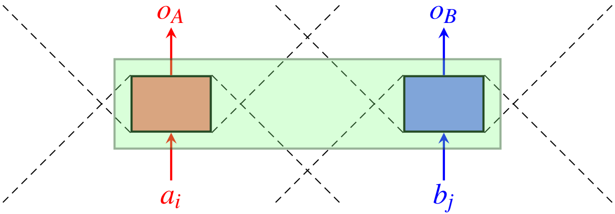


No-signalling

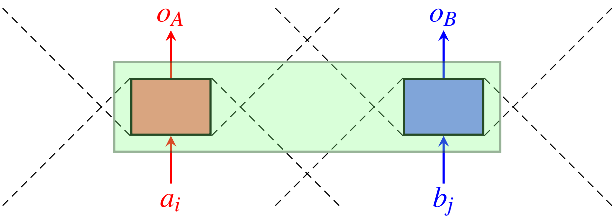
$p(o_A \mid a_i)$ is independent of b_j:



No-signalling

$p(o_A\,|\,a_i)$ is independent of b_j:

$$\begin{aligned} & p(o_A\,|\,a_i, b_0) \\ &\quad = p(o_A, 0\,|\,a_i, b_0) \\ &\quad\;\; + p(o_A, 1\,|\,a_i, b_0) \end{aligned}$$



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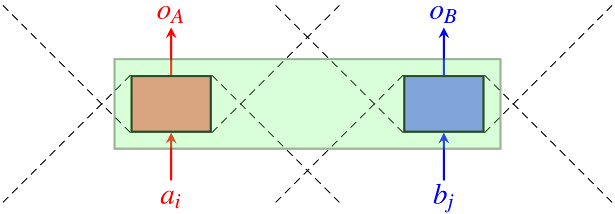

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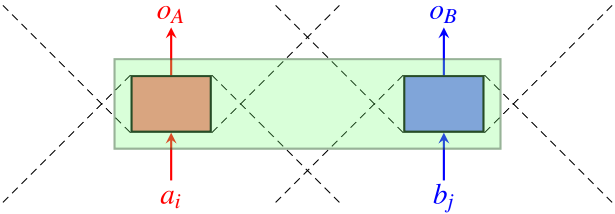

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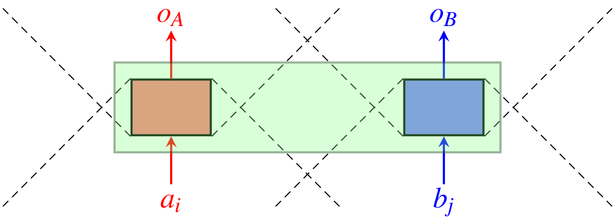

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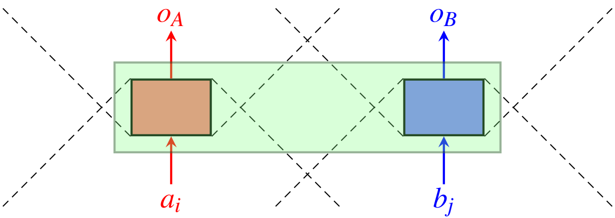

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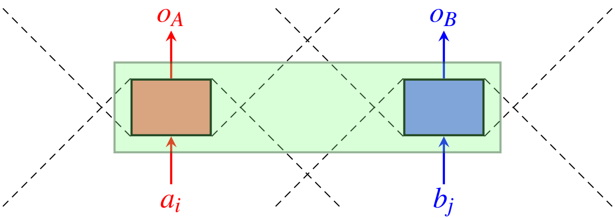

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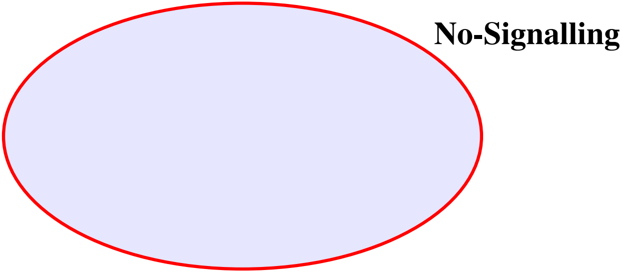

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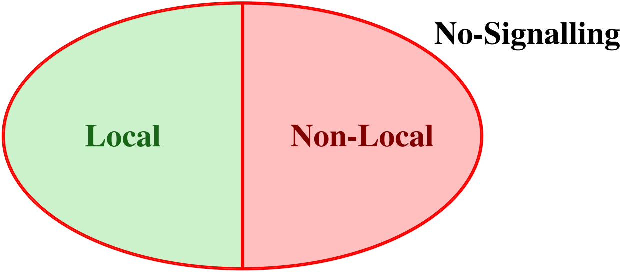

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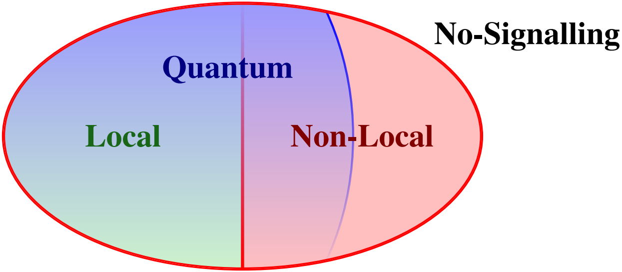

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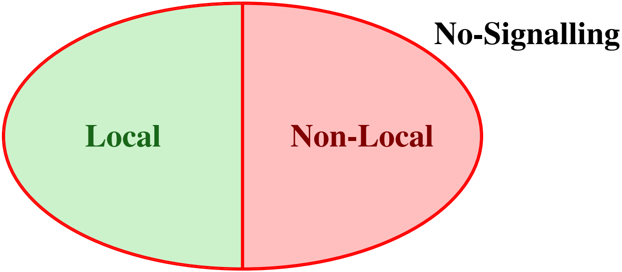

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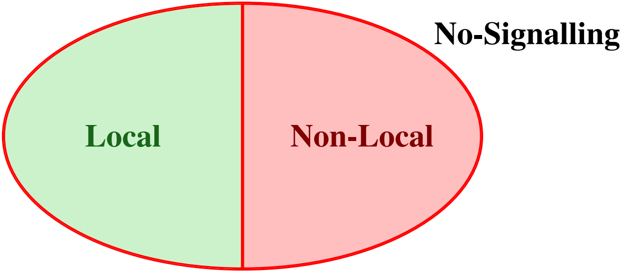

… beyond “2 parties, 2 inputs, 2 outputs”.

Convex combination mixes probability tables:

$$p_1(o_A, o_B \,|\, a_i, b_j) \qquad \cdots \qquad p_n(o_A, o_B \,|\, a_i, b_j)$$

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$$\alpha_1\, p_1(o_A, o_B \,|\, a_i, b_j) + \cdots + \alpha_n\, p_n(o_A, o_B \,|\, a_i, b_j)$$

for $\alpha_1, \ldots, \alpha_n \geqslant 0$ and $\alpha_1 + \cdots + \alpha_n = 1$.

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for $\alpha_1, \ldots, \alpha_n \geqslant 0$ and $\alpha_1 + \cdots + \alpha_n = 1$.

$$\alpha \left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & \cdot & \cdot & \cdot & \cdot \\ a_0b_1 & \cdot & x & \cdot & \cdot \\ a_1b_0 & \cdot & \cdot & \cdot & \cdot \\ a_1b_1 & \cdot & \cdot & \cdot & \cdot \end{array}\right) + (1-\alpha) \left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & \cdot & \cdot & \cdot & \cdot \\ a_0b_1 & \cdot & y & \cdot & \cdot \\ a_1b_0 & \cdot & \cdot & \cdot & \cdot \\ a_1b_1 & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

$$= \left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & \cdot & \cdot & \cdot & \cdot \\ a_0b_1 & \cdot & \alpha x + (1-\alpha) y & \cdot & \cdot \\ a_1b_0 & \cdot & \cdot & \cdot & \cdot \\ a_1b_1 & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

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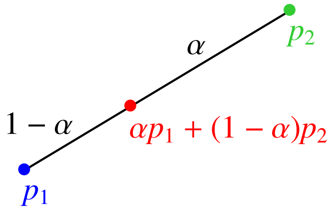

Deterministic tables

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	1	0	0	0
(a_0, b_1)	0	1	0	0
(a_1, b_0)	1	0	0	0
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This describes the assignment

$$(a_0,a_1,b_0,b_1) \mapsto (0,0,0,1).$$

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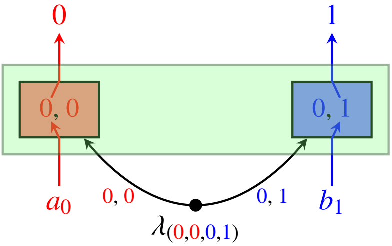
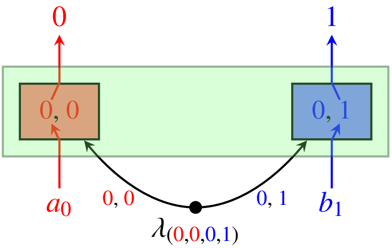


This describes the assignment

$$(a_0, a_1, b_0, b_1) \mapsto (0, 0, 0, 1).$$

Deterministic tables

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	1	0	0	0
(a_0, b_1)	0	1	0	0
(a_1, b_0)	1	0	0	0
(a_1, b_1)	0	1	0	0



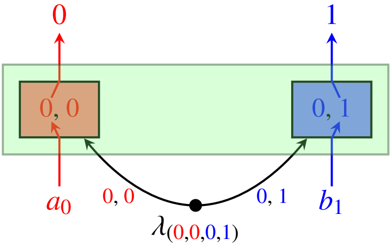
This describes the assignment

$$(a_0, a_1, b_0, b_1) \mapsto (0, 0, 0, 1).$$

- Instruction set + choosing and reading bit registers.

Deterministic tables

	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(a_0, b_0)	1	0	0	0
(a_0, b_1)	0	1	0	0
(a_1, b_0)	1	0	0	0
(a_1, b_1)	0	1	0	0



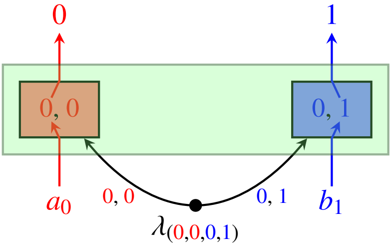
This describes the assignment

$$(a_0, a_1, b_0, b_1) \mapsto (0, 0, 0, 1).$$

- Instruction set + choosing and reading bit registers.
- State of a classical system.

Deterministic tables

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	1	0	0	0
(a_0, b_1)	0	1	0	0
(a_1, b_0)	1	0	0	0
(a_1, b_1)	0	1	0	0



This describes the assignment

$$(a_0, a_1, b_0, b_1) \mapsto (0, 0, 0, 1).$$

- Instruction set + choosing and reading bit registers.
- State of a classical system.

A deterministic table cannot be a mixture of other tables.

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$1/2$	0	0	$1/2$
(a_0, b_1)	$1/2$	0	0	$1/2$
(a_1, b_0)	$1/2$	0	0	$1/2$
(a_1, b_1)	$1/2$	0	0	$1/2$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$1/2$	0	0	$1/2$
(a_0, b_1)	$1/2$	0	0	$1/2$
(a_1, b_0)	$1/2$	0	0	$1/2$
(a_1, b_1)	$1/2$	0	0	$1/2$

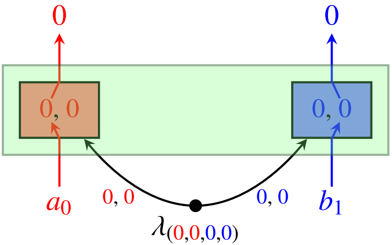
$$= \tfrac{1}{2}\left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & 1 & 0 & 0 & 0 \\ a_0b_1 & 1 & 0 & 0 & 0 \\ a_1b_0 & 1 & 0 & 0 & 0 \\ a_1b_1 & 1 & 0 & 0 & 0 \end{array}\right) + \tfrac{1}{2}\left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & 0 & 0 & 0 & 1 \\ a_0b_1 & 0 & 0 & 0 & 1 \\ a_1b_0 & 0 & 0 & 0 & 1 \\ a_1b_1 & 0 & 0 & 0 & 1 \end{array}\right)$$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$^1/_2$	0	0	$^1/_2$
(a_0, b_1)	$^1/_2$	0	0	$^1/_2$
(a_1, b_0)	$^1/_2$	0	0	$^1/_2$
(a_1, b_1)	$^1/_2$	0	0	$^1/_2$

$$= \tfrac{1}{2}\left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & 1 & 0 & 0 & 0 \\ a_0b_1 & 1 & 0 & 0 & 0 \\ a_1b_0 & 1 & 0 & 0 & 0 \\ a_1b_1 & 1 & 0 & 0 & 0 \end{array}\right) + \tfrac{1}{2}\left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & 0 & 0 & 0 & 1 \\ a_0b_1 & 0 & 0 & 0 & 1 \\ a_1b_0 & 0 & 0 & 0 & 1 \\ a_1b_1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$(0, 0, 0, 0)$ w/ prob. $^1/_2$ + $(1, 1, 1, 1)$ w/ prob. $^1/_2$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$1/2$	0	0	$1/2$
(a_0, b_1)	$1/2$	0	0	$1/2$
(a_1, b_0)	$1/2$	0	0	$1/2$
(a_1, b_1)	$1/2$	0	0	$1/2$



$= 1/2$

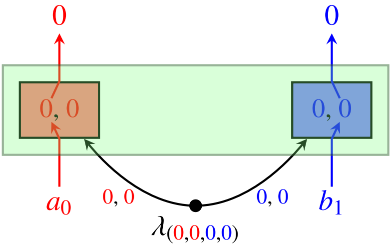
	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

$+ 1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1

$(0, 0, 0, 0)$ w/ prob. $1/2$ + $(1, 1, 1, 1)$ w/ prob. $1/2$

	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(a_0, b_0)	1/2	0	0	1/2
(a_0, b_1)	1/2	0	0	1/2
(a_1, b_0)	1/2	0	0	1/2
(a_1, b_1)	1/2	0	0	1/2

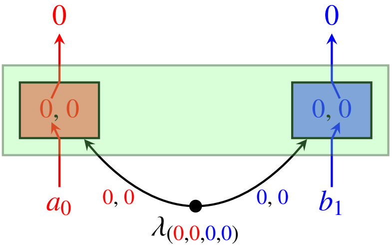


$$= \tfrac{1}{2}\left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & 1 & 0 & 0 & 0 \\ a_0b_1 & 1 & 0 & 0 & 0 \\ a_1b_0 & 1 & 0 & 0 & 0 \\ a_1b_1 & 1 & 0 & 0 & 0 \end{array}\right) + \tfrac{1}{2}\left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline a_0b_0 & 0 & 0 & 0 & 1 \\ a_0b_1 & 0 & 0 & 0 & 1 \\ a_1b_0 & 0 & 0 & 0 & 1 \\ a_1b_1 & 0 & 0 & 0 & 1 \end{array}\right)$$

(0, 0, 0, 0) w/ prob. 1/2 + (1, 1, 1, 1) w/ prob. 1/2

	(0, 0, 0, 0)	(0, 0, 0, 1)	⋯	(1, 1, 1, 0)	(1, 1, 1, 1)
(a_0, a_1, b_0, b_1)	1/2	0	⋯	0	1/2

	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(a_0, b_0)	1/2	0	0	1/2
(a_0, b_1)	1/2	0	0	1/2
(a_1, b_0)	1/2	0	0	1/2
(a_1, b_1)	1/2	0	0	1/2



marginals

= 1/2

	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

+ 1/2

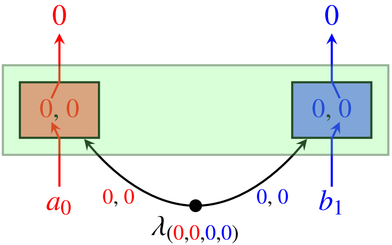
	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1

(0, 0, 0, 0) w/ prob. 1/2 + (1, 1, 1, 1) w/ prob. 1/2

	(0, 0, 0, 0)	(0, 0, 0, 1)	⋯	(1, 1, 1, 0)	(1, 1, 1, 1)
(a_0, a_1, b_0, b_1)	1/2	0	⋯	0	1/2

"Deterministic hidden variable" models

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_0, b_0)	$1/2$	0	0	$1/2$
(a_0, b_1)	$1/2$	0	0	$1/2$
(a_1, b_0)	$1/2$	0	0	$1/2$
(a_1, b_1)	$1/2$	0	0	$1/2$



↑ marginals

$= 1/2$

	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

$+ 1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1

$(0, 0, 0, 0)$ w/ prob. $1/2$ $+$ $(1, 1, 1, 1)$ w/ prob. $1/2$

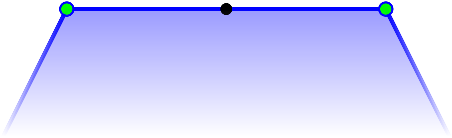
	$(0,0,0,0)$	$(0,0,0,1)$	$\cdots$	$(1,1,1,0)$	$(1,1,1,1)$
(a_0, a_1, b_0, b_1)	$1/2$	0	$\cdots$	0	$1/2$

Tables that admit deterministic hidden variable models form a **polytope** (within $\mathbb{R}^{16}$ for $(2, 2, 2)$), whose edges are exactly the deterministic tables:

	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$1/2$	0	0	$1/2$
a_1b_0	$1/2$	0	0	$1/2$
a_1b_1	$1/2$	0	0	$1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1



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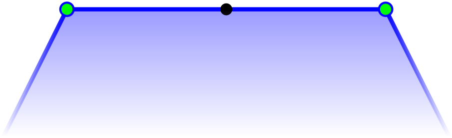
	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$1/2$	0	0	$1/2$
a_1b_0	$1/2$	0	0	$1/2$
a_1b_1	$1/2$	0	0	$1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1



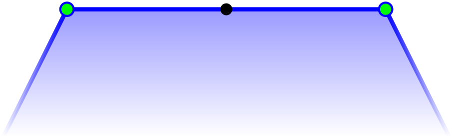
Admits a deterministic hidden variable model

Tables that admit deterministic hidden variable models form a **polytope** (within $\mathbb{R}^{16}$ for $(2,2,2)$), whose edges are exactly the deterministic tables:

	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$1/2$	0	0	$1/2$
a_1b_0	$1/2$	0	0	$1/2$
a_1b_1	$1/2$	0	0	$1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1



Admits a deterministic hidden variable model

$\Downarrow$

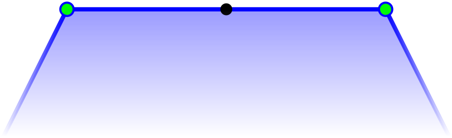
Local $=_{\text{def}}$ Admits a hidden variable model

Tables that admit deterministic hidden variable models form a **polytope** (within $\mathbb{R}^{16}$ for $(2,2,2)$), whose edges are exactly the deterministic tables:

	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$1/2$	0	0	$1/2$
a_1b_0	$1/2$	0	0	$1/2$
a_1b_1	$1/2$	0	0	$1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1



Admits a deterministic hidden variable model

$\Downarrow \Uparrow$

Local $=_{\text{def}}$ Admits a hidden variable model

Theorem (Fine 1982 for $(2, 2, 2)$;
Abramsky-Brandenburger 2011 beyond $(2, 2, 2)$).

A probability table $p(\cdot \mid a_i, b_j)_{i,j \in \{0,1\}}$ is local iff

- it is a convex combination of deterministic tables,

Theorem (Fine 1982 for $(2,2,2)$;
Abramsky-Brandenburger 2011 beyond $(2,2,2)$).

A probability table $p(\cdot \mid a_i, b_j)_{i,j\in\{0,1\}}$ is local iff

- it is a convex combination of deterministic tables, i.e.,
- there is a distribution $p(\cdot \mid a_0, a_1, b_0, b_1)$ over assignments

$$(a_0, a_1, b_0, b_1) \mapsto (h, i, j, k)$$

that gives each $p(\cdot \mid a_i, b_j)$ as a marginal.

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$$(a_0, a_1, b_0, b_1) \mapsto (h, i, j, k)$$

that gives each $p(\cdot \mid a_i, b_j)$ as a marginal.

	00	01	10	11
a_0b_0	1/2	0	0	1/2
a_0b_1	1/2	0	0	1/2
a_1b_0	1/2	0	0	1/2
a_1b_1	1/2	0	0	1/2

$= 1/2$

	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

$(0, 0, 0, 0)$

$+ 1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1

$(1, 1, 1, 1)$

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that gives each $p(\cdot \mid a_i, b_j)$ as a marginal.

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$1/2$	0	0	$1/2$
a_1b_0	$1/2$	0	0	$1/2$
a_1b_1	$1/2$	0	0	$1/2$

$= 1/2$

	00	01	10	11
a_0b_0	1	0	0	0
a_0b_1	1	0	0	0
a_1b_0	1	0	0	0
a_1b_1	1	0	0	0

$(0,0,0,0)$

$+ 1/2$

	00	01	10	11
a_0b_0	0	0	0	1
a_0b_1	0	0	0	1
a_1b_0	0	0	0	1
a_1b_1	0	0	0	1

$(1,1,1,1)$

	$(0,0,0,0)$	$(0,0,0,1)$	$\cdots$	$(1,1,1,0)$	$(1,1,1,1)$
(a_0, a_1, b_0, b_1)	$1/2$	0	$\cdots$	0	$1/2$

Fine's theorem:

A family $p(\cdot\,|\,a_i, b_j)_{i,j\in\{0,1\}}$ is local iff
it is given by some single $p(\cdot\,|\,a_0, a_1, b_0, b_1)$.

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Conceptual upshot. A set of empirical data may be

- no-signalling:

  able to assign probabilities consistently
  to the family of contexts $\{a_i, b_j\}$;

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  able to assign probabilities consistently
  to the family of contexts $\{a_i, b_j\}$;

- non-local:

  not able to extend the same
  to the entire set $\{a_0, a_1, b_0, b_1\}$ of all measurements.

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Conceptual upshot. A set of empirical data may be

- no-signalling: "locally consistent",
  able to assign probabilities consistently
  to the family of contexts $\{a_i, b_j\}$;
- non-local: "globally inconsistent",
  not able to extend the same
  to the entire set $\{a_0, a_1, b_0, b_1\}$ of all measurements.

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A family $p(\cdot \mid a_i, b_j)_{i,j \in \{0,1\}}$ is local iff
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- no-signalling: "locally consistent",
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  to the entire set $\{a_0, a_1, b_0, b_1\}$ of all measurements.

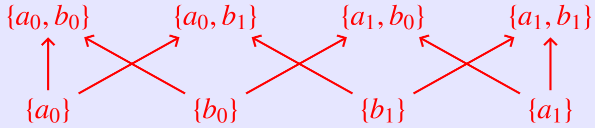
This is a sort of thing **sheaf theory** is good at dealing with.

Part II. Topological Model for Contextuality

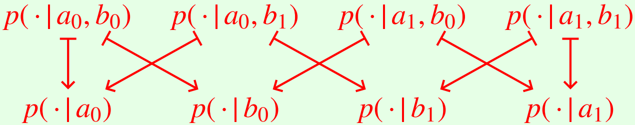
- The idea of “local consistency + global inconsistency” is a topological one;
- Formalize it in topological, sheaf-theoretic terms;
- And in a form highly independent of the QM formalism, applicable also to, e.g. relational database;
- It shows non-locality as a special case of contextuality;
- And naturally characterizes degrees of contextuality in ways capturing the structure of the probability polytope.

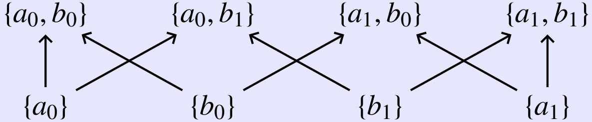
A nice diagram

A nice diagram



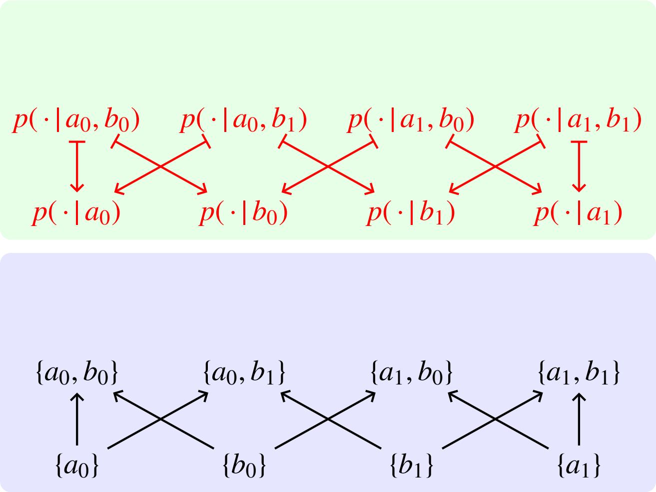
A nice diagram





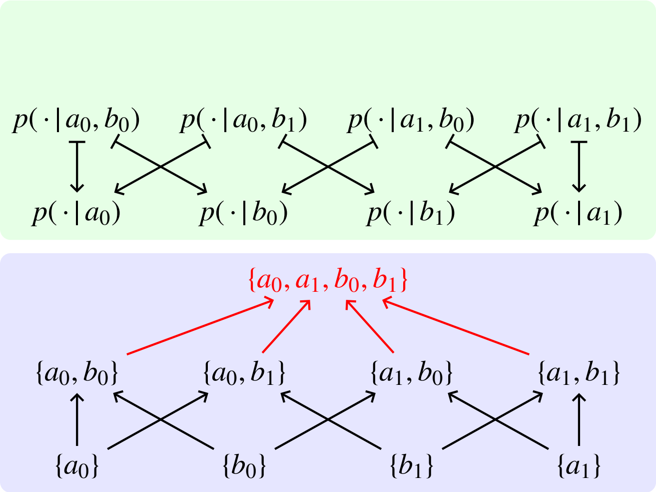
A nice diagram

- No-signalling: "$\mapsto$" marginals, any colliding pair agrees.



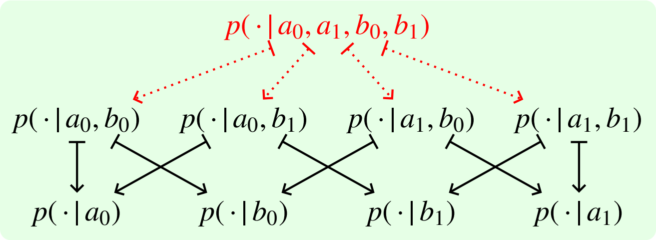
A nice diagram

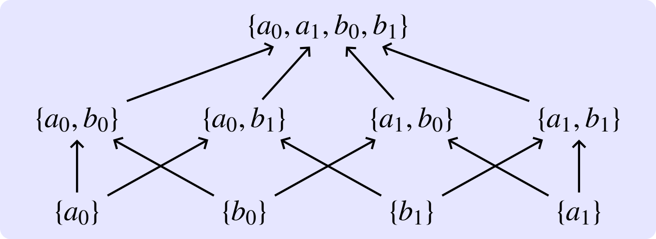
- No-signalling: "↦" marginals, any colliding pair agrees.



A nice diagram

- No-signalling: "↦" marginals, any colliding pair agrees.
- Locality: An extension over $\{a_0, a_1, b_0, b_1\}$ exists.





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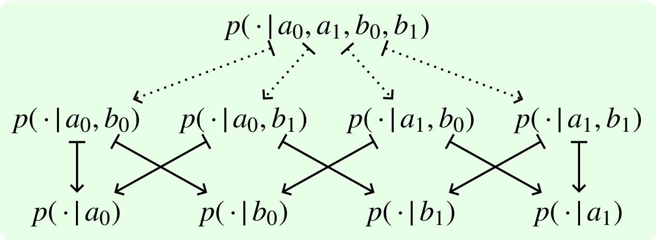

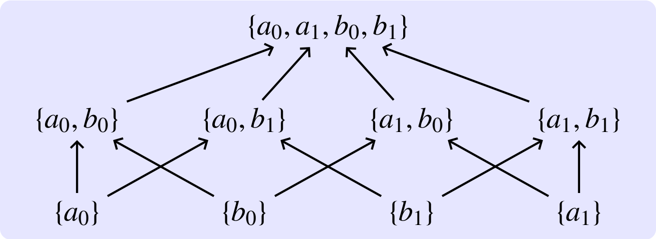

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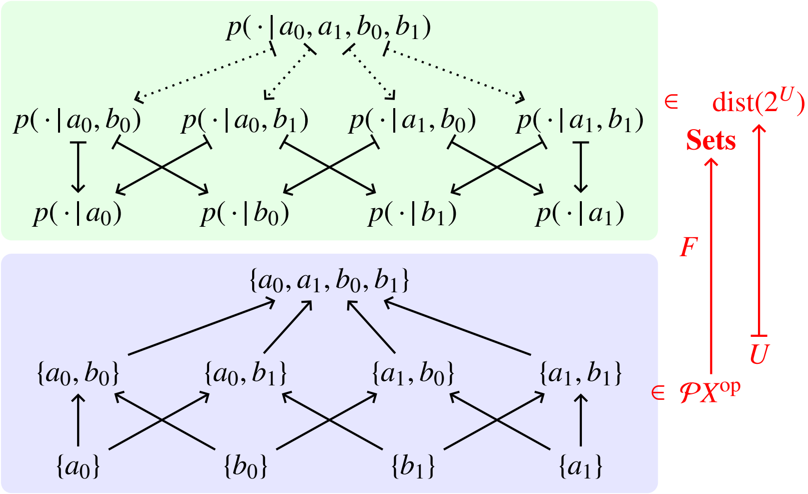

A nice diagram … Look, there's a functor! It's a presheaf!

- No-signalling: $p(\cdot \mid a_i, b_j)_{i,j}$ is a matching family in F.
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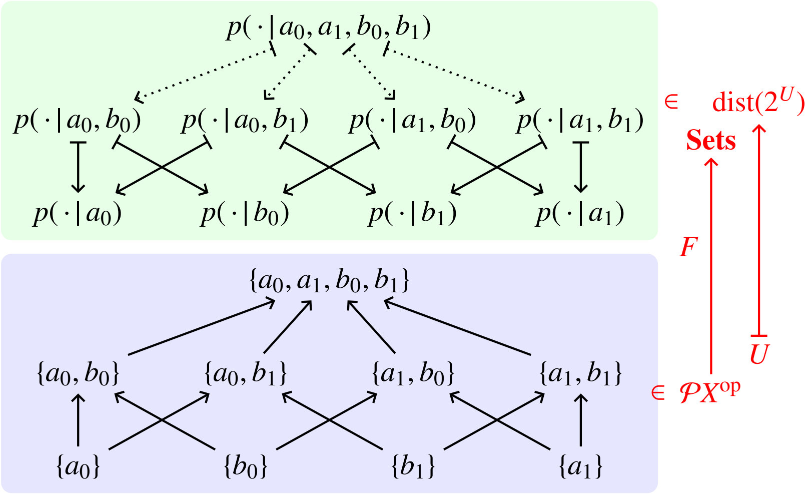

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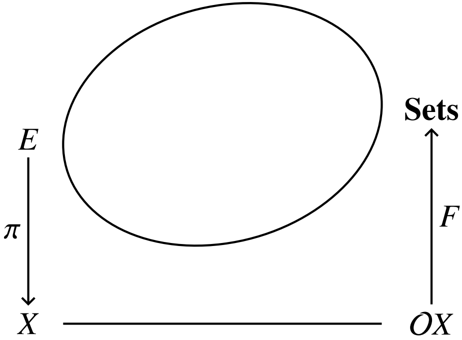

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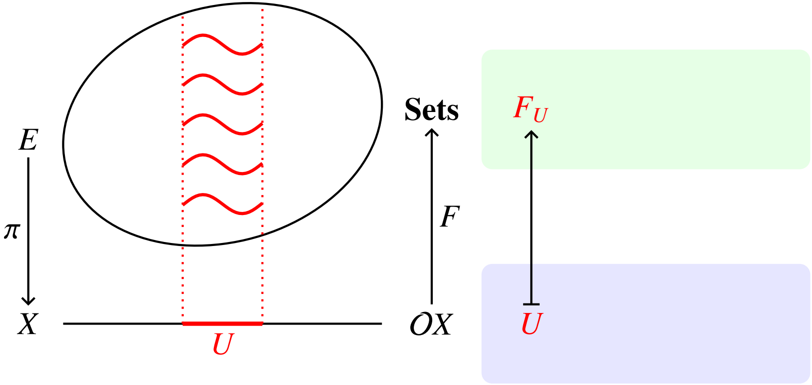

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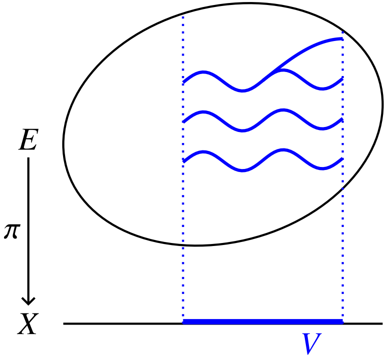

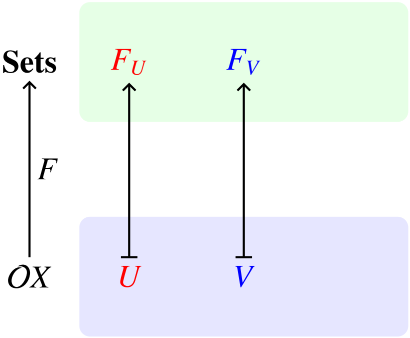

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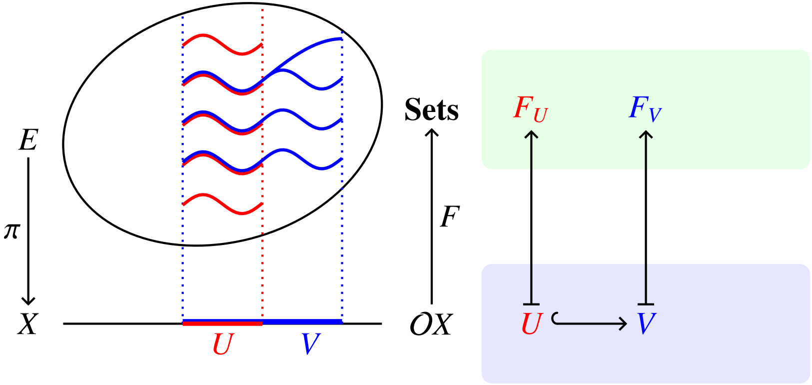

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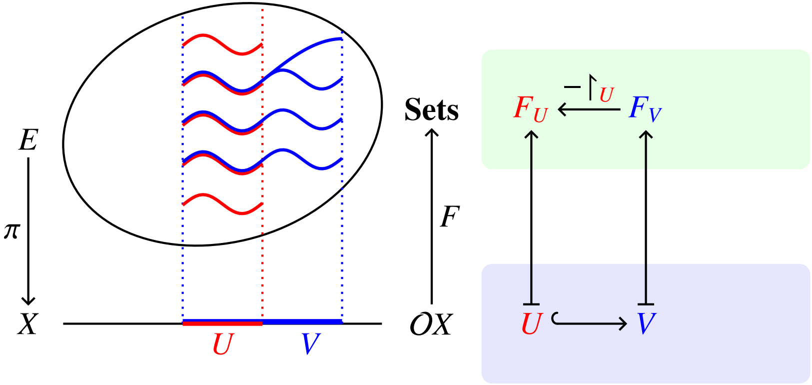

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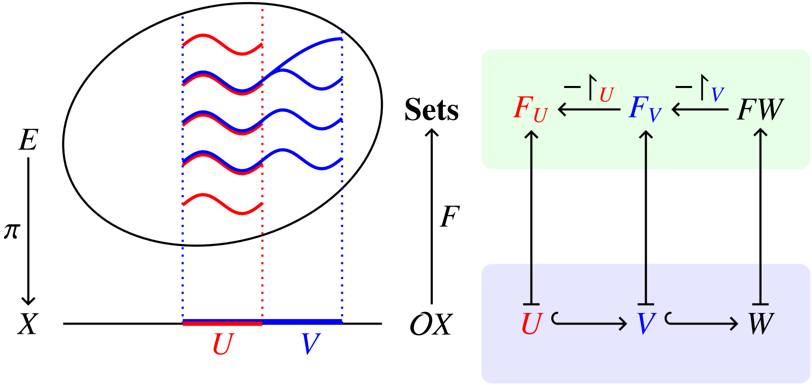

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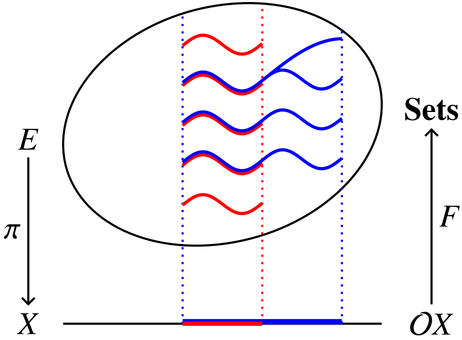

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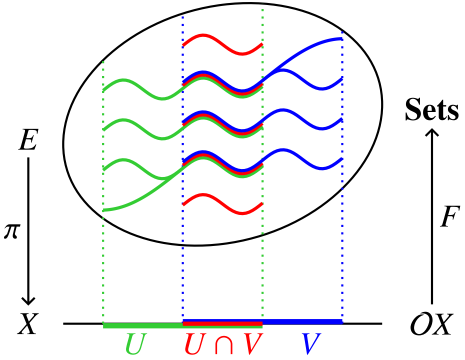

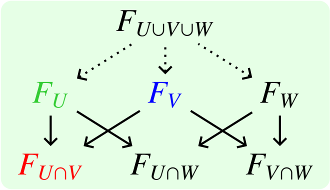

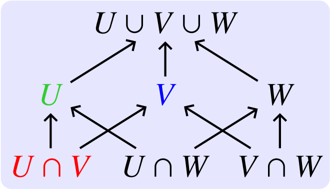

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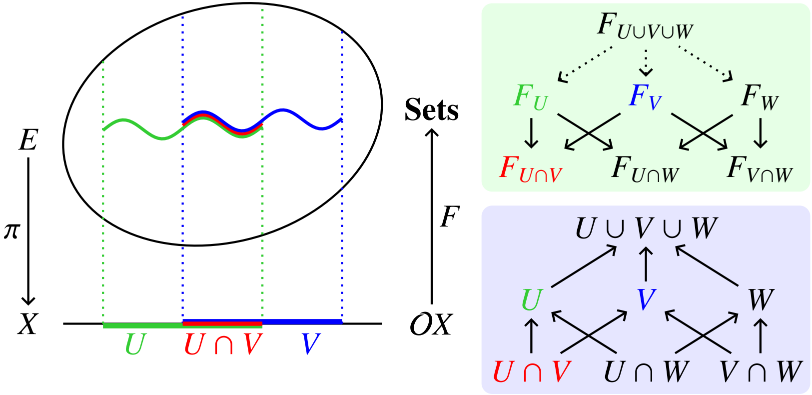

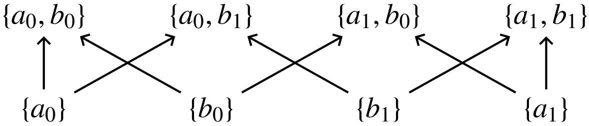
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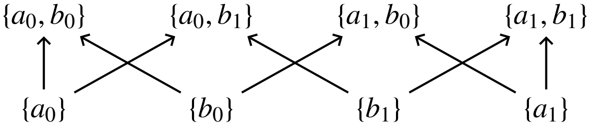
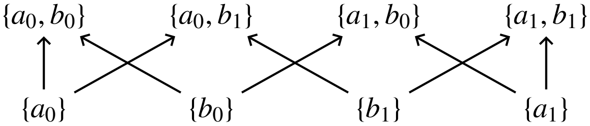


Def. An (abstract) simplicial complex C (on a set X) is a $\subseteq$-downward closed family of finite subsets of X.



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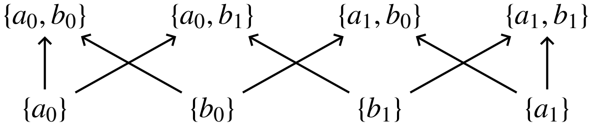

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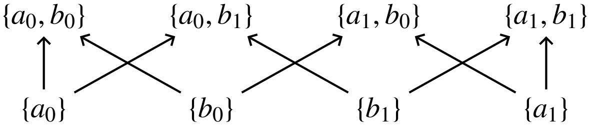

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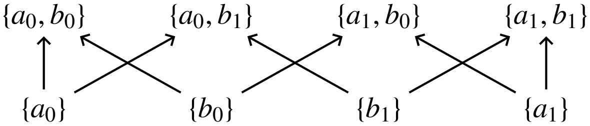

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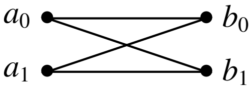

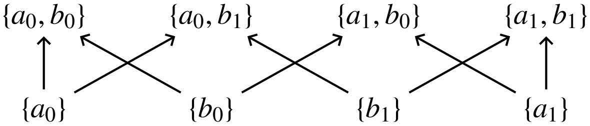

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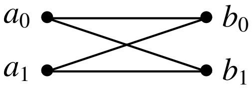

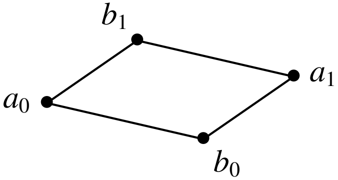

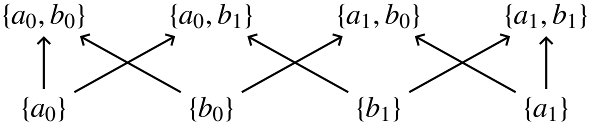

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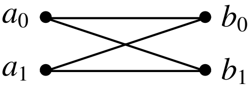

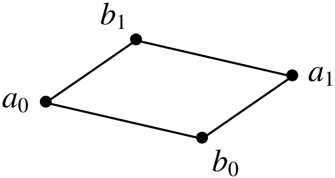

"Simplices" $U \in C$ are local regions, whereas X is global.

Topological Models for Contextuality

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Spaces of variables and of their values:

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- measurements + outcomes
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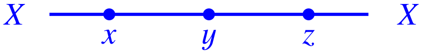

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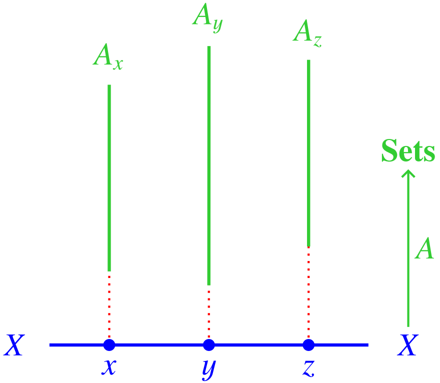

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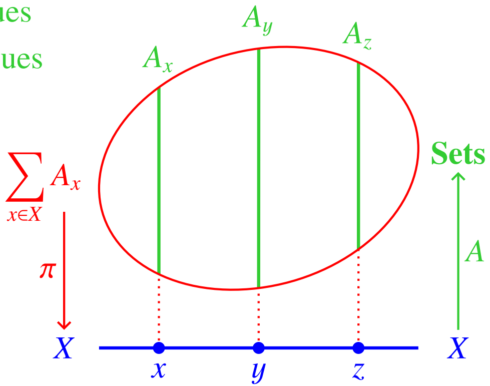

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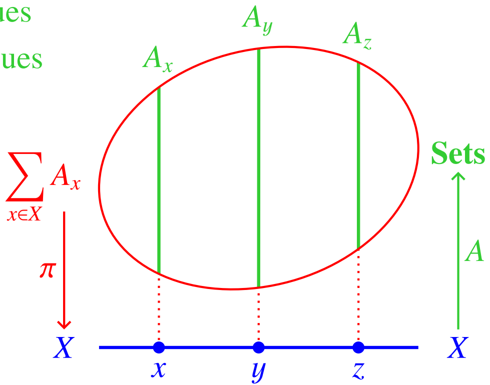

$\mathbf{Sets}/X \simeq \mathbf{Sets}^X.$

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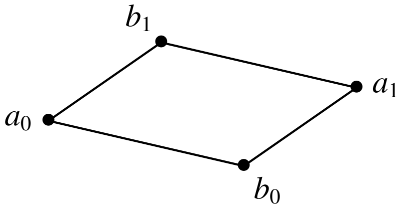

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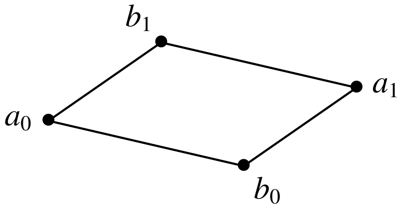

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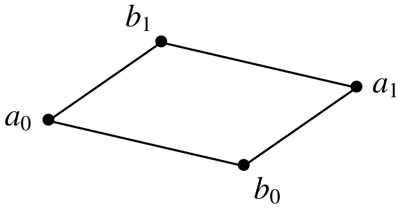

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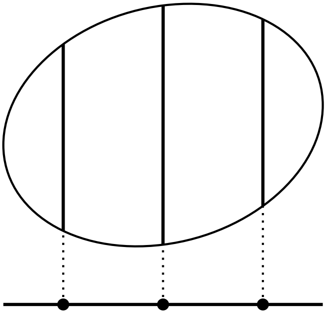
Write C for the simplicial complex of contexts $U \subseteq X$.

When we ask several questions, answers may obey constraints:



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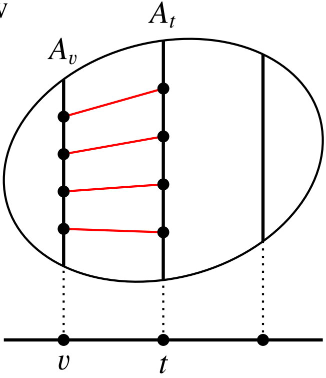

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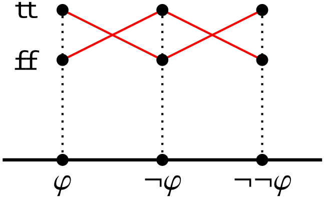

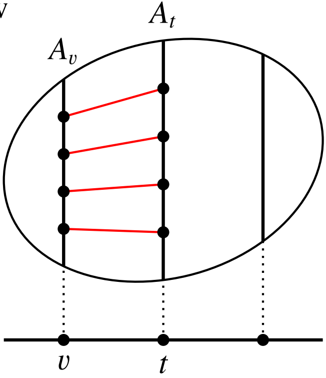

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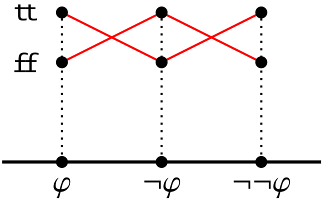

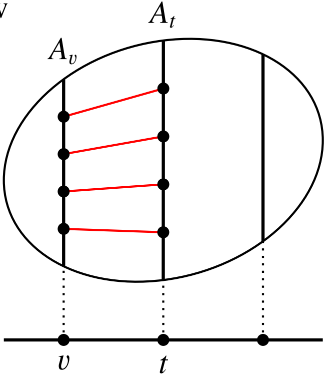

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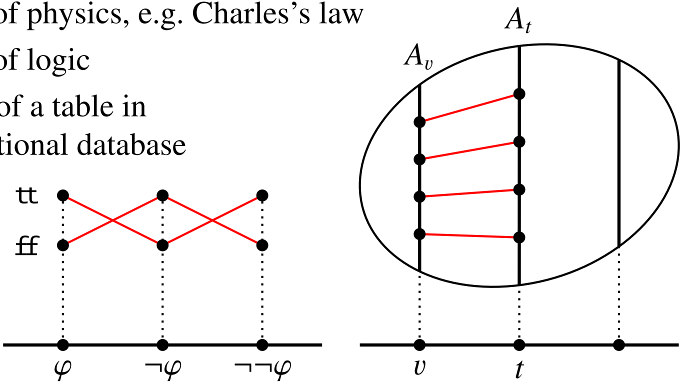

Distinguish good and bad ways of connecting dots in bundles
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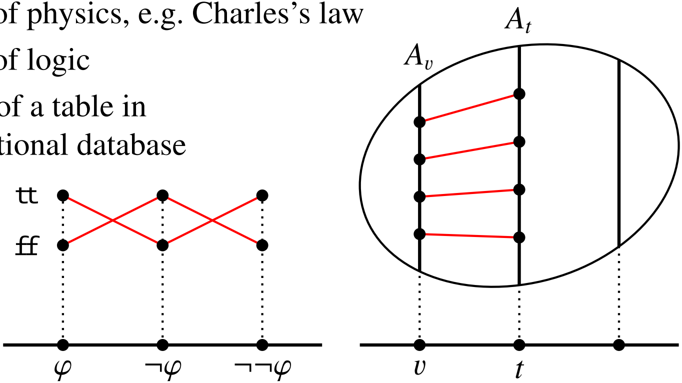

Distinguish good and bad ways of connecting dots in bundles
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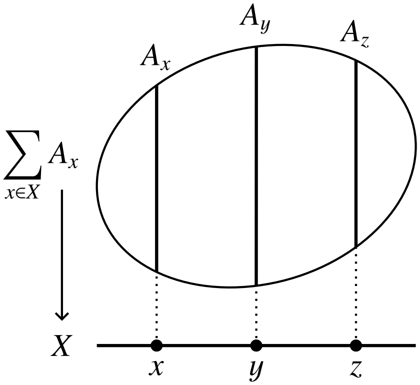
Write $\mathscr{A}$ for the set of good combinations of answers;
this makes $\sum_{x \in X} A_x$ another simplicial complex.


A_y
A_z
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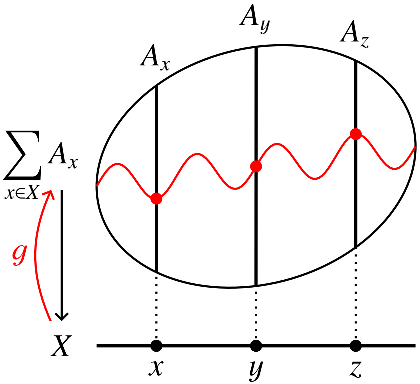
X
x
y
z

Def. A global section is a function $g : X \to \Sigma$, i.e.

- an assignment of answers to all the questions,



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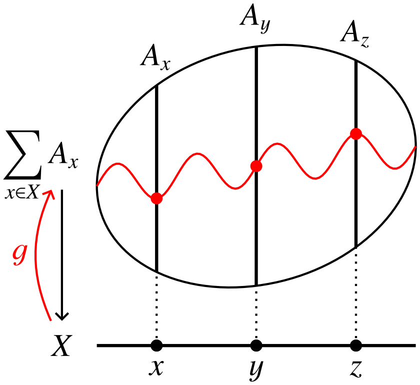

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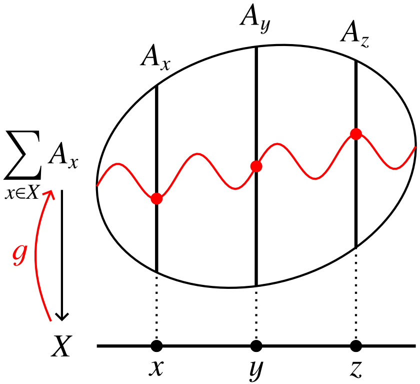
E.g.

- Models of classical logic.



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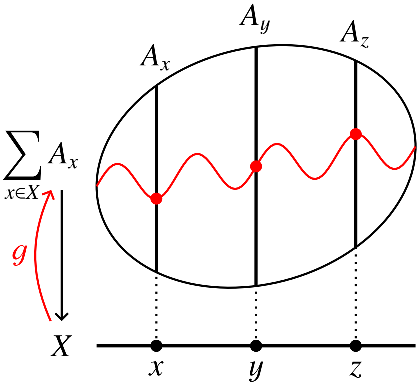
x is consistent $\iff$ $(x \mapsto \mathtt{tt})$ extends to a global section.



Def. A global section is a function $g : X \to \Sigma$, i.e.

- an assignment of answers to all the questions,
- that satisfies all the constraints, i.e.

$$g[U] \in \mathcal{A} \text{ for all } U \in \mathcal{C}.$$

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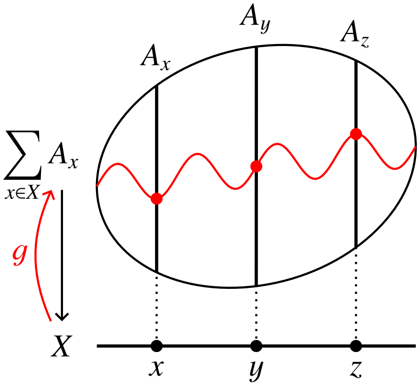

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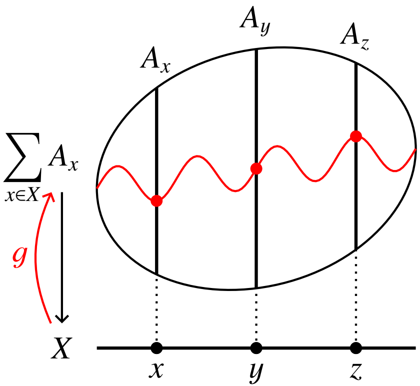
  x is consistent $\iff$ $(x \mapsto \mathtt{tt})$ extends to a global section.

- States of a physical system?

  . . . Classically yes, but no in QM!

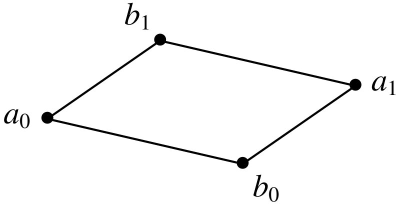


Hardy 1993:

	00	01	10	11
a_0b_0	✓	✓	✓	✓
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0

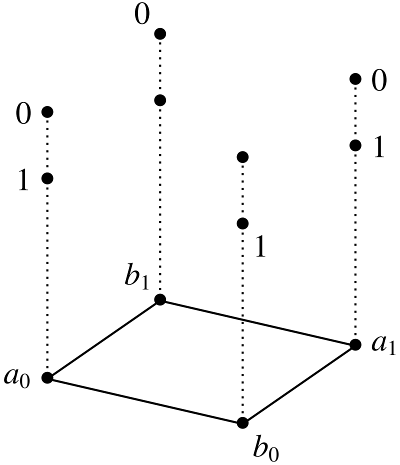
Hardy 1993:

	00	01	10	11
a_0b_0	✓	✓	✓	✓
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0



Hardy 1993:

	00	01	10	11
$a_0 b_0$	✓	✓	✓	✓
$a_0 b_1$	0	✓	✓	✓
$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0



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	00	01	10	11
a_0b_0	✓	✓	✓	✓
a_0b_1	0	✓	✓	✓
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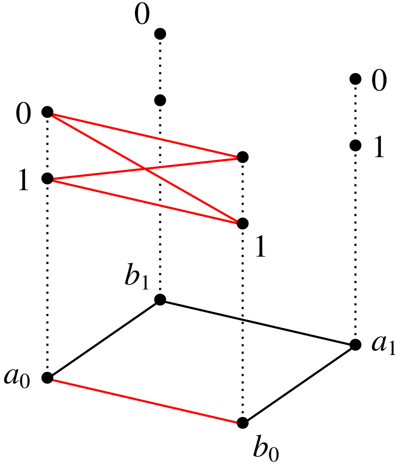
a_1b_1	✓	✓	✓	0



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a_0b_1	0	✓	✓	✓
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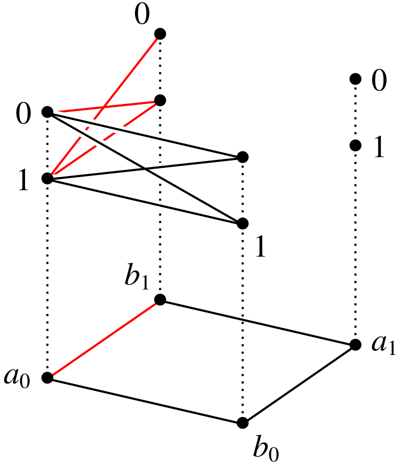
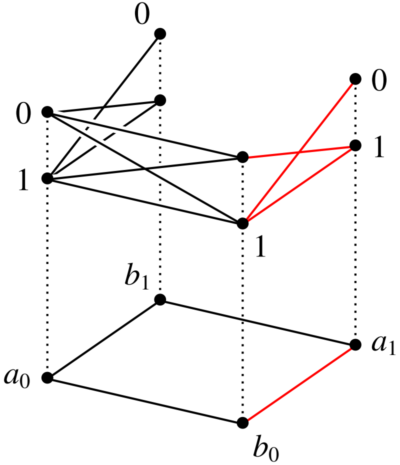
a_1b_1	✓	✓	✓	0



Hardy 1993:

	00	01	10	11
a_0b_0	✓	✓	✓	✓
a_0b_1	0	✓	✓	✓
a_1b_0	0	✓	✓	✓
a_1b_1	✓	✓	✓	0



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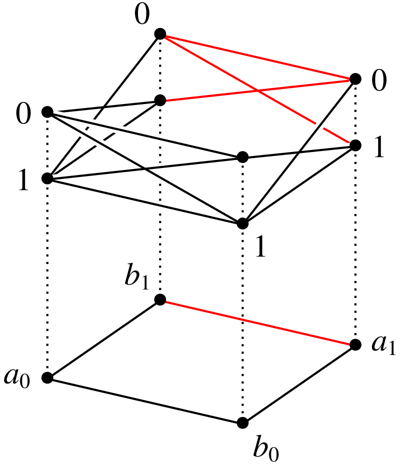

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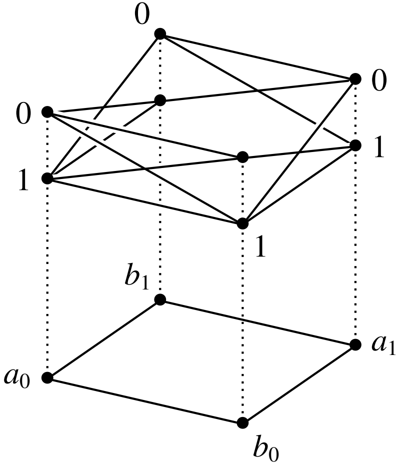

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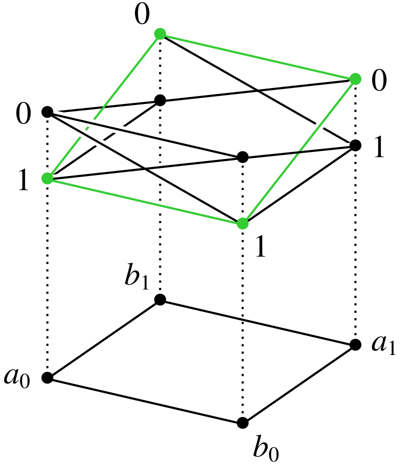
a_1b_1	✓	✓	✓	0



Some global sections, e.g.

$$(a_0, a_1, b_0, b_1) \mapsto (1, 0, 1, 0);$$

Hardy 1993:

	00	01	10	11
a_0b_0	✓	✓	✓	✓
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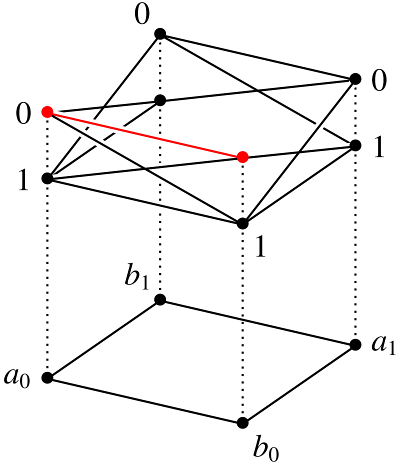
a_1b_1	✓	✓	✓	0



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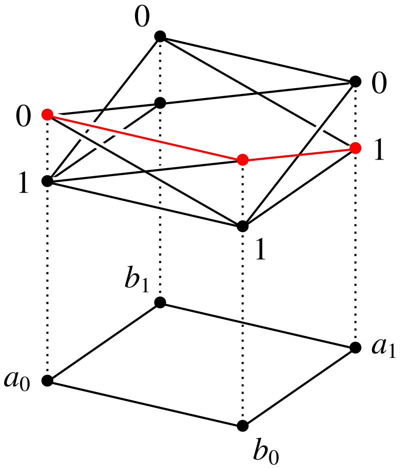

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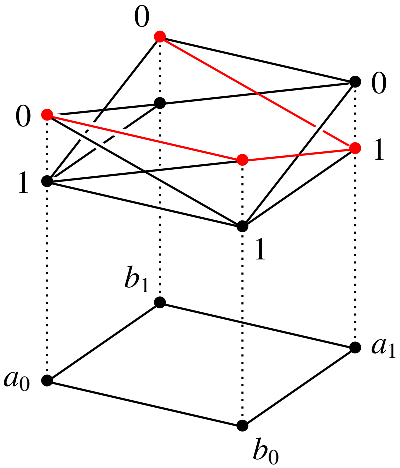
$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0



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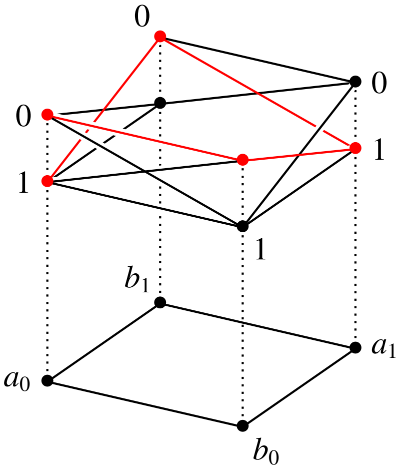

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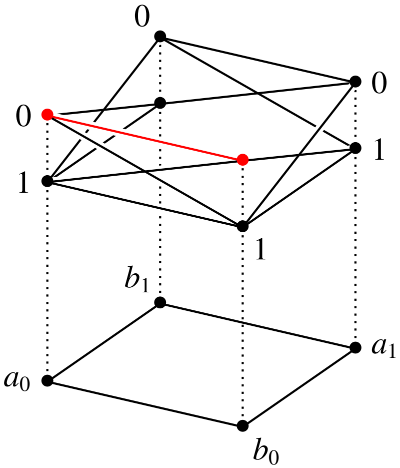
$a_1 b_1$	✓	✓	✓	0



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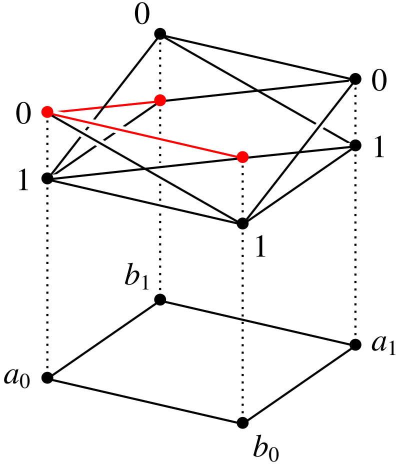

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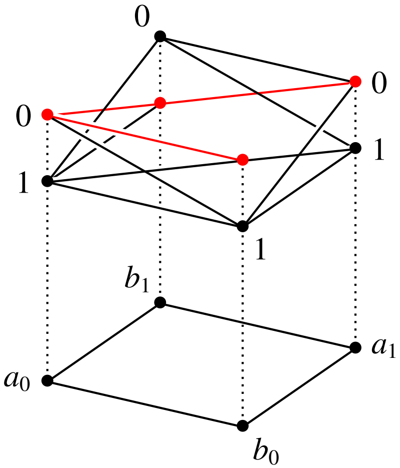

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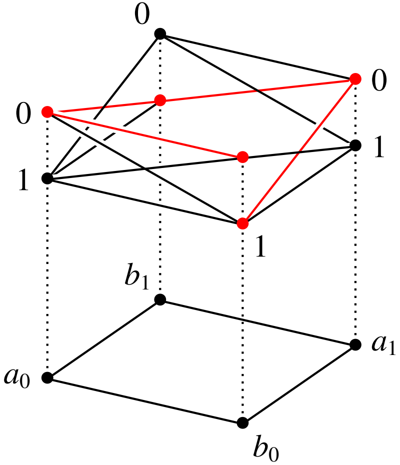

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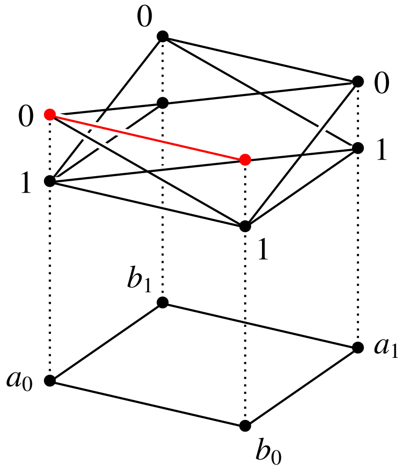

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Logical contextuality: Not all sections extend to global ones.

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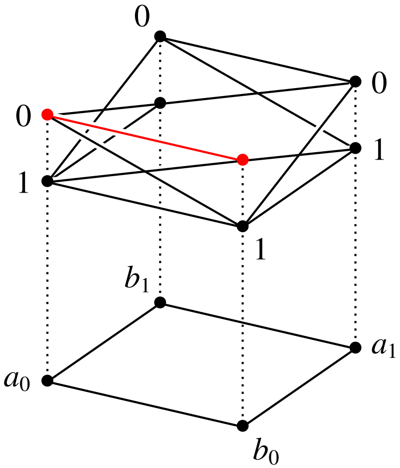
$a_1 b_0$	0	✓	✓	✓
$a_1 b_1$	✓	✓	✓	0



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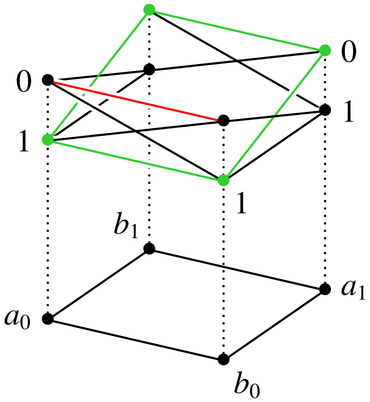
Contextuality = local consistency + global inconsistency

Hardy:



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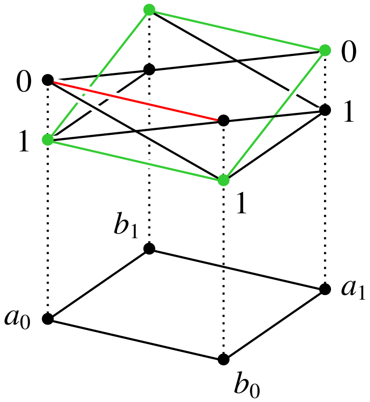
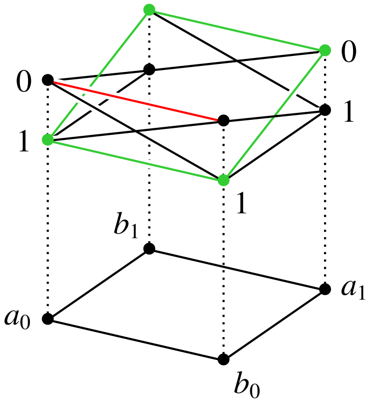


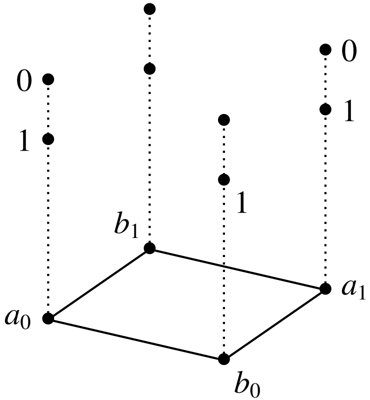
PR box:

Logical contextuality: Not all sections extend to global ones.

Hardy:



PR box:



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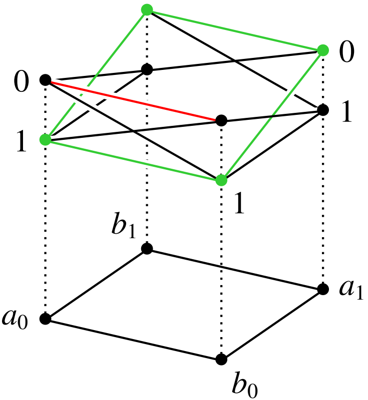

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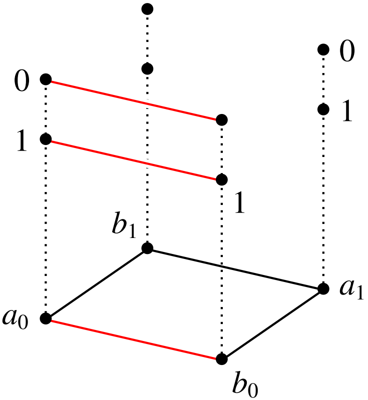

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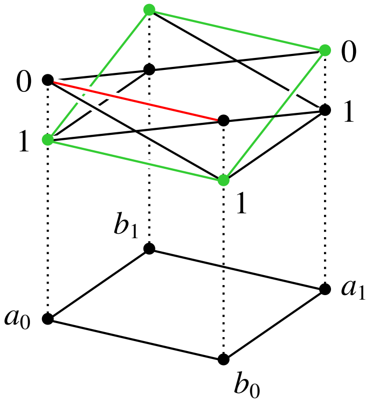

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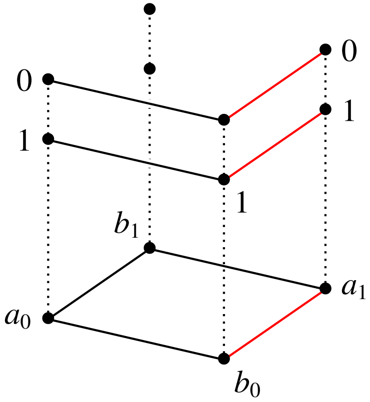

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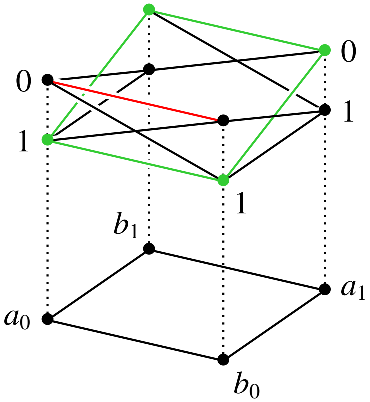

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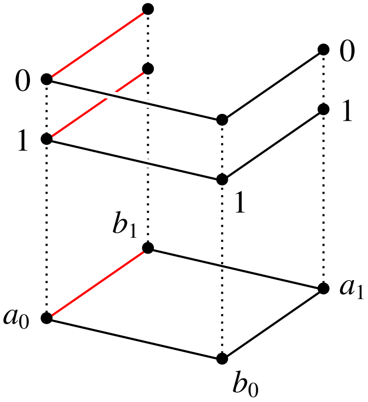

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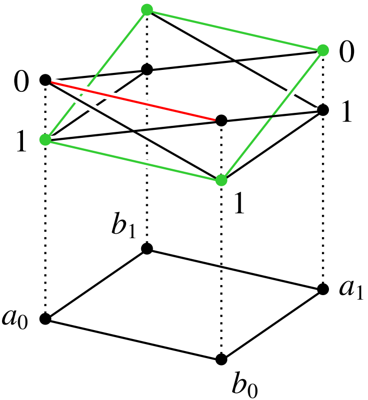

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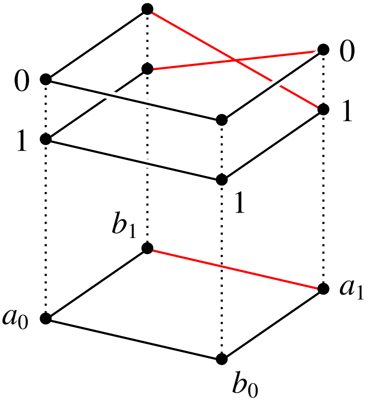

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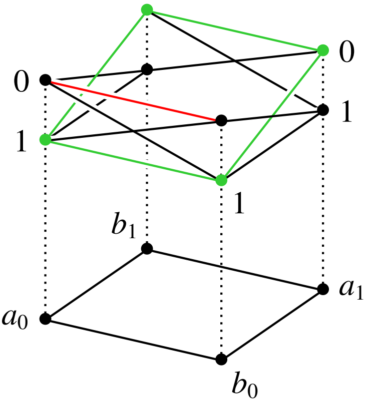

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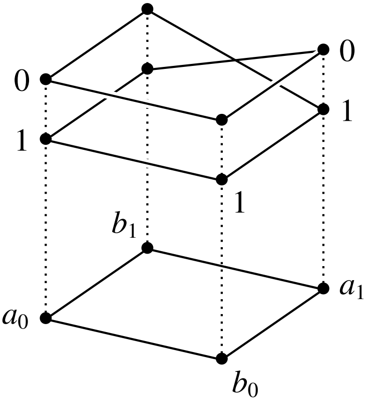

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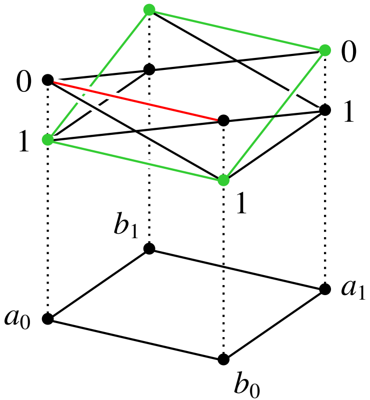

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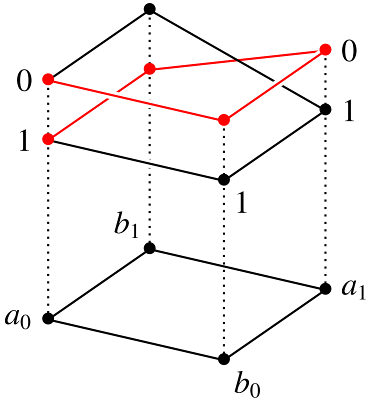

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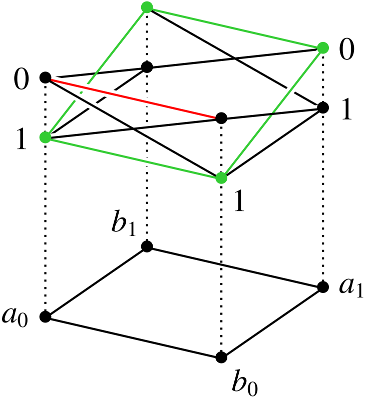

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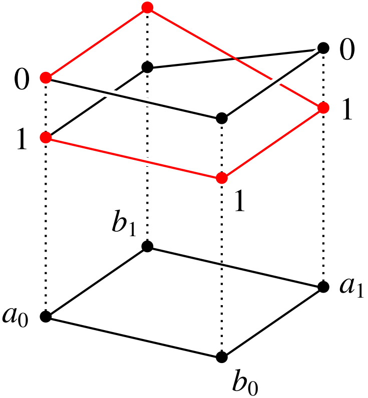

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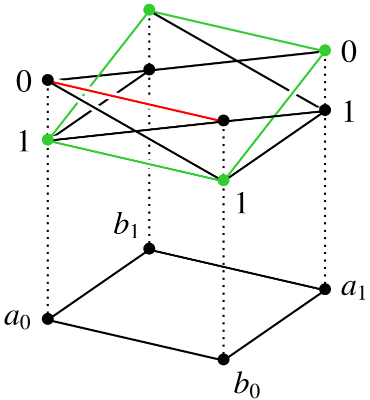

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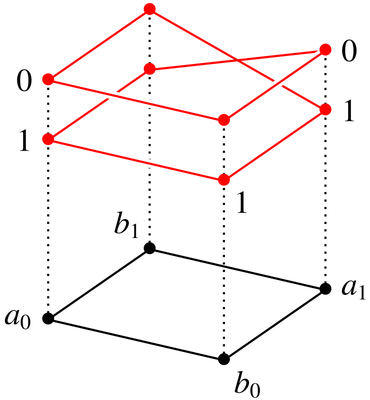

Logical contextuality: Not all sections extend to global ones.

Strong contextuality: No global section at all.

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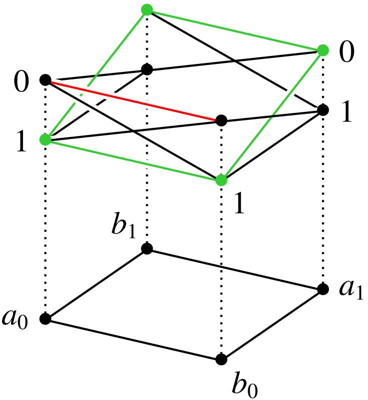

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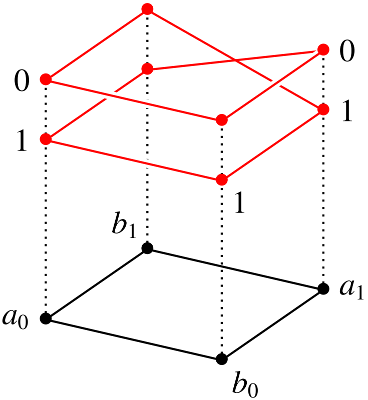

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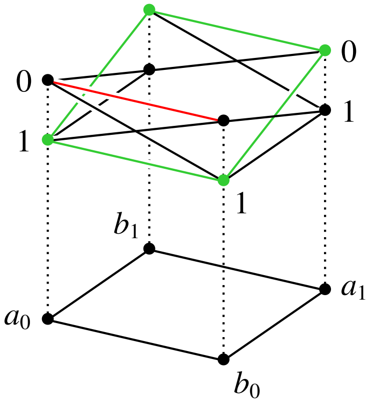
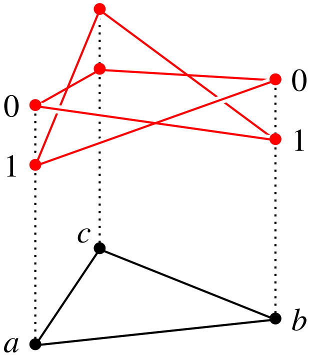
$$\text{Strongly contextual} \implies \text{Logically contextual.}$$

Hardy:



Specker triangle:



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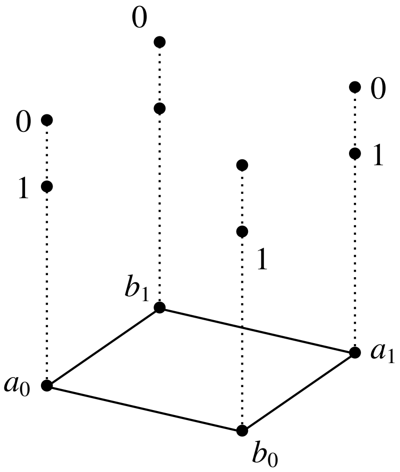
Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_1	$1/8$	$3/8$	$3/8$	$1/8$



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a_0b_0	$^1/_2$	0	0	$^1/_2$
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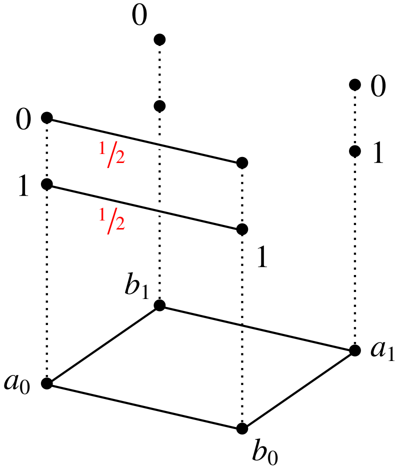
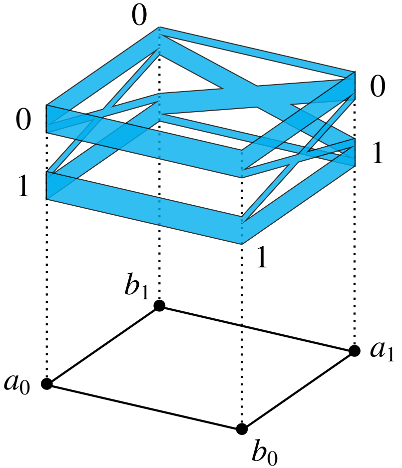
a_1b_0	$^3/_8$	$^1/_8$	$^1/_8$	$^3/_8$
a_1b_1	$^1/_8$	$^3/_8$	$^3/_8$	$^1/_8$



Bell vs. Hardy

	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_1	$1/8$	$3/8$	$3/8$	$1/8$



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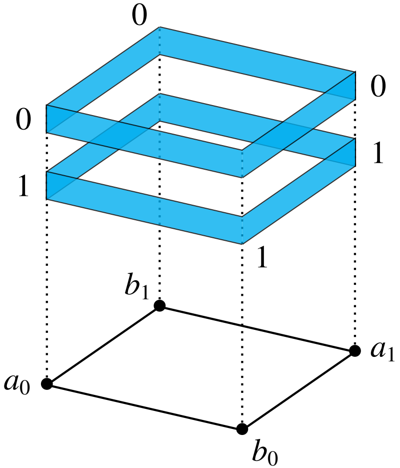
a_1b_1	$1/2$	0	0	$1/2$



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	00	01	10	11
a_0b_0	$1/2$	0	0	$1/2$
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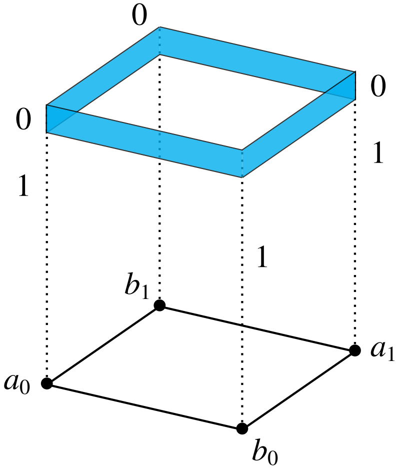
a_1b_0	$1/2$	0	0	$1/2$
a_1b_1	$1/2$	0	0	$1/2$



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	00	01	10	11
$a_0 b_0$	$1/2$	0	0	$1/2$
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$a_1 b_0$	$1/2$	0	0	$1/2$
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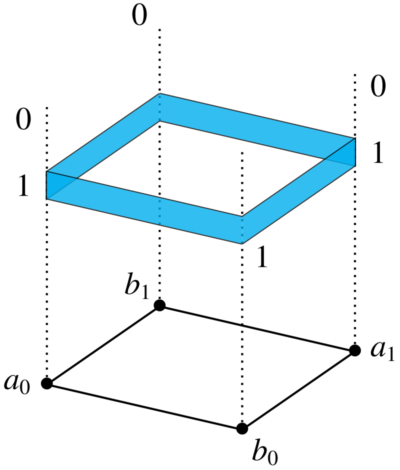

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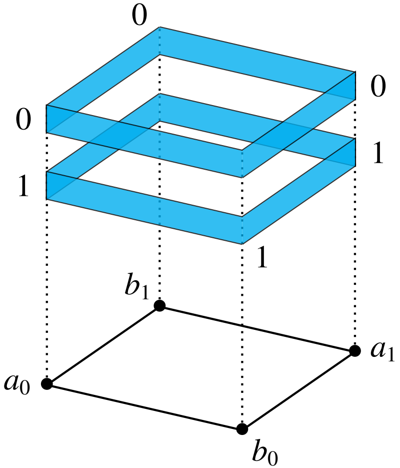

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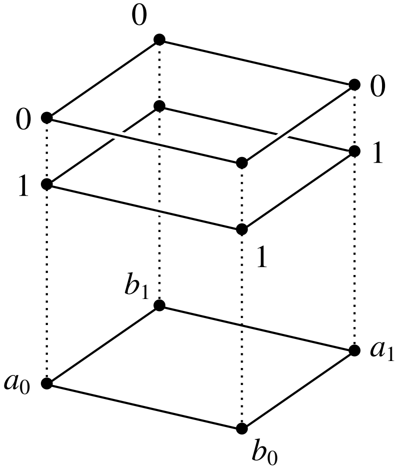

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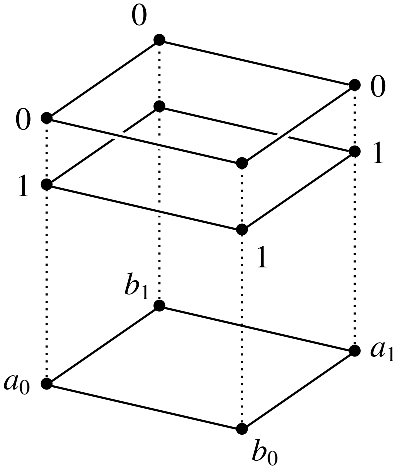

$$\text{Bell local} \implies \text{Logically non-contextual,}$$

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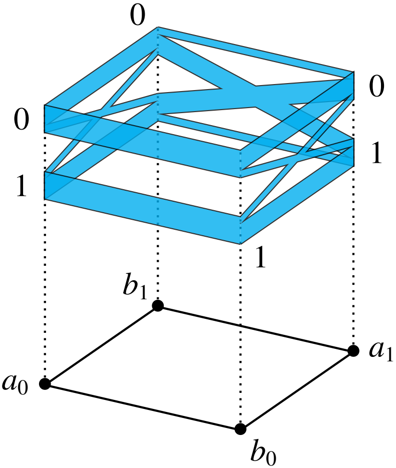
Bell vs. Hardy

	00	01	10	11
$a_0 b_0$	$1/2$	0	0	$1/2$
$a_0 b_1$	$3/8$	$1/8$	$1/8$	$3/8$
$a_1 b_0$	$3/8$	$1/8$	$1/8$	$3/8$
$a_1 b_1$	$1/8$	$3/8$	$3/8$	$1/8$



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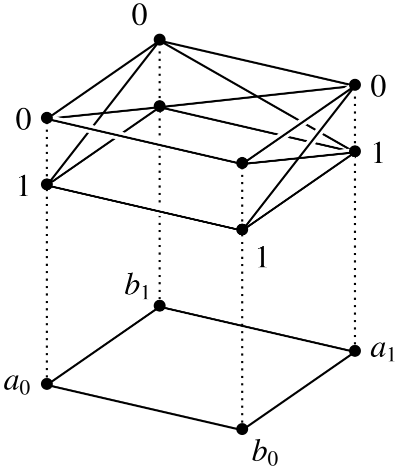

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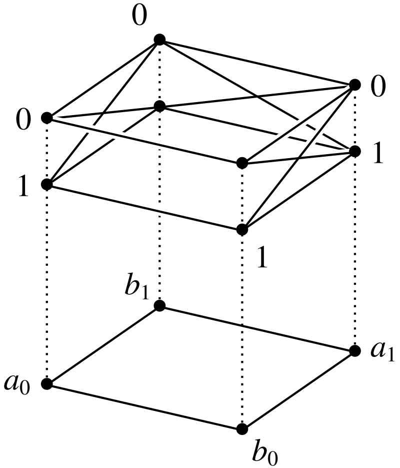

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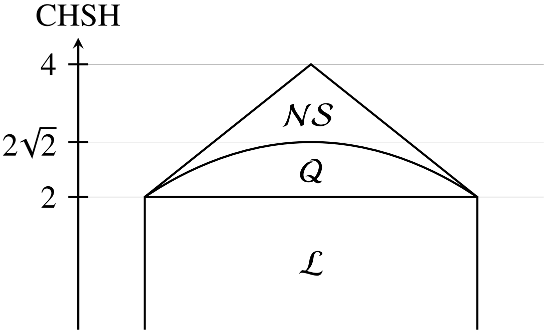
Polytope of $(2, 2, 2)$ no-signalling tables:



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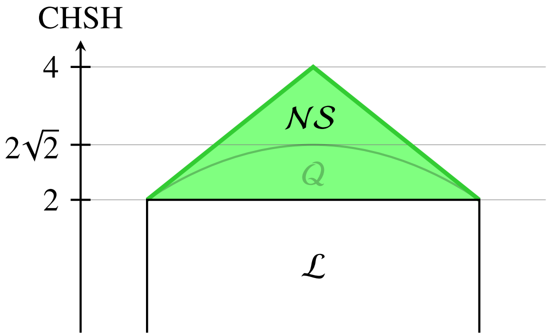

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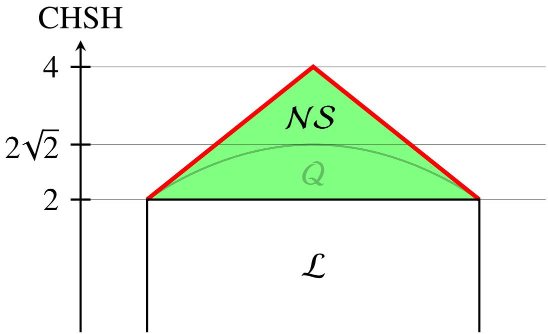

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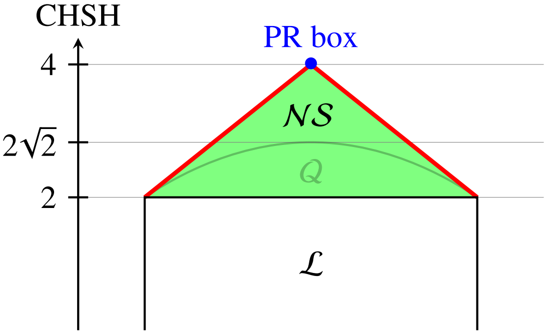
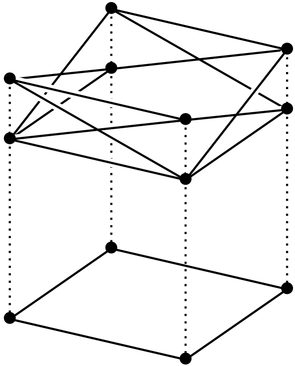
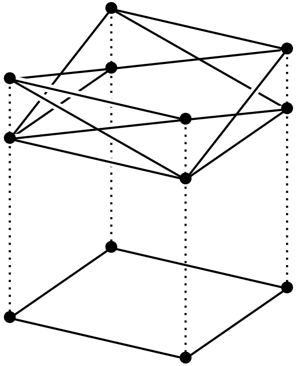




To **define models formally**,



To **define models formally**,
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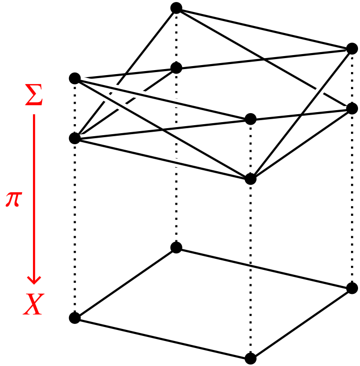
❶

❷



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1. Map / bundle of simplicial complexes

$$\pi : \sum_{x \in X} A_x \to X;$$

$s \in \mathcal{A}$ implies $\pi[s] \in C$.

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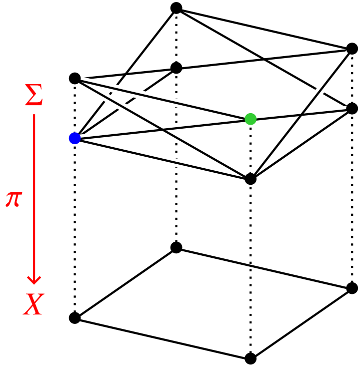

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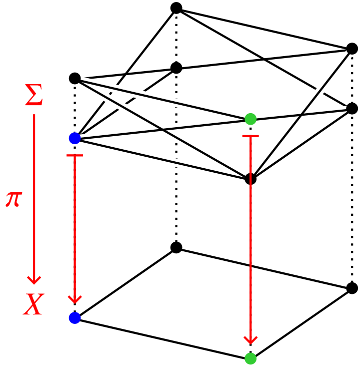

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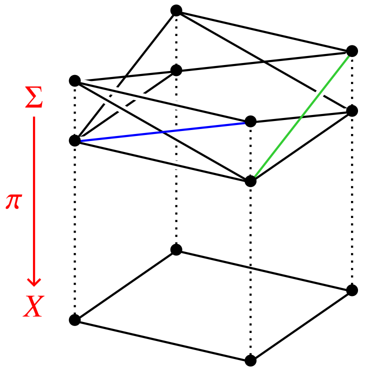

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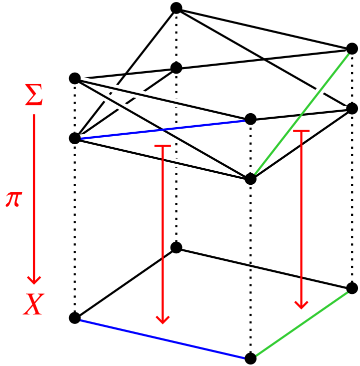

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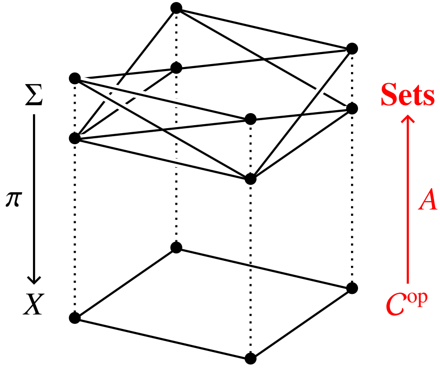

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$$A : C^{op} \to \textbf{Sets},$$

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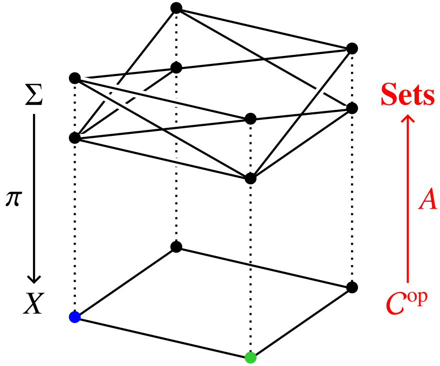

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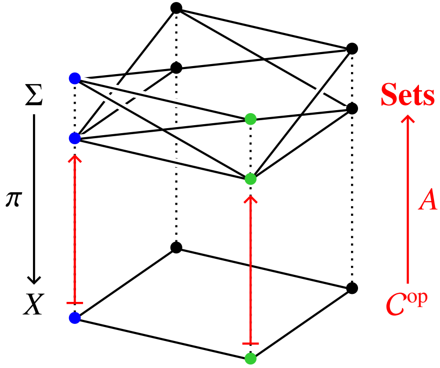

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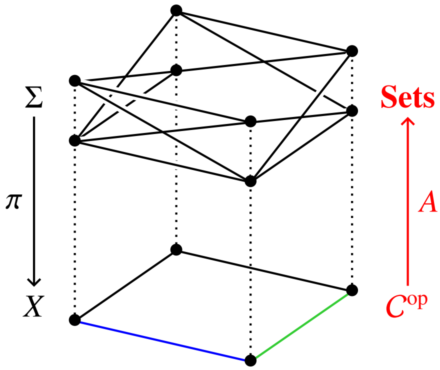

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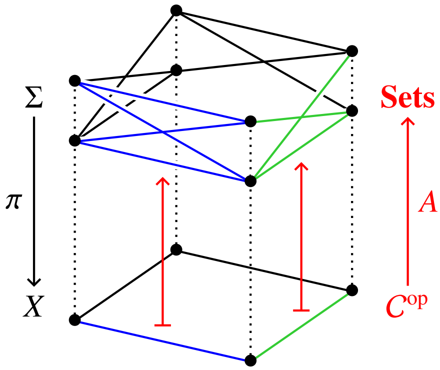

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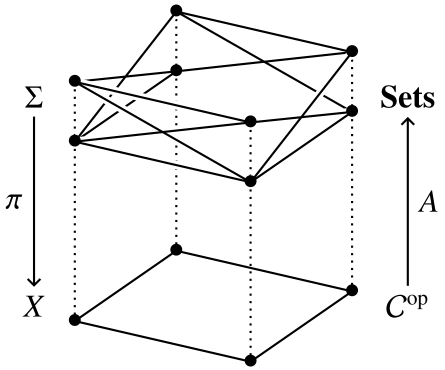

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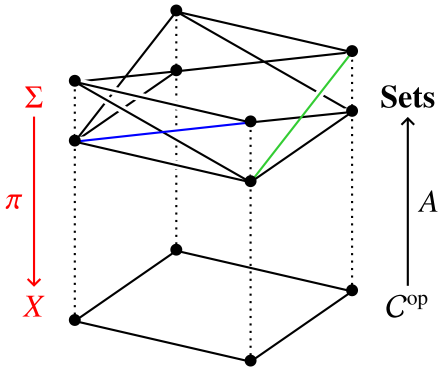

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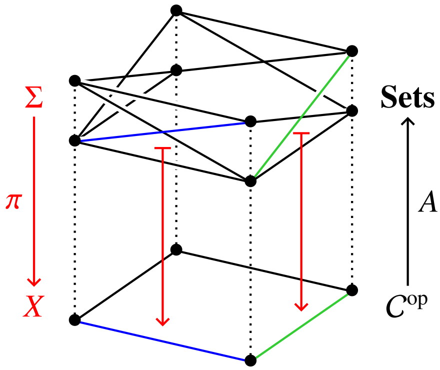

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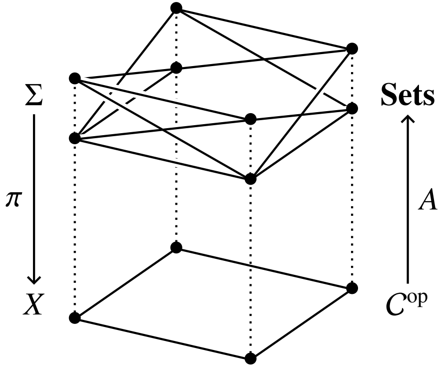

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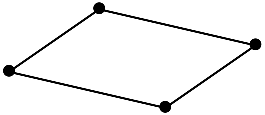

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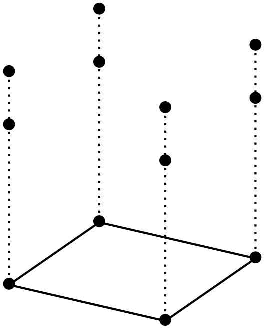

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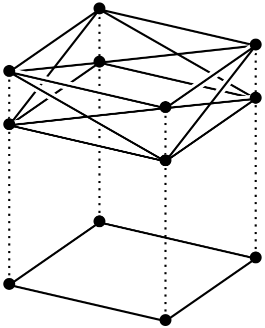

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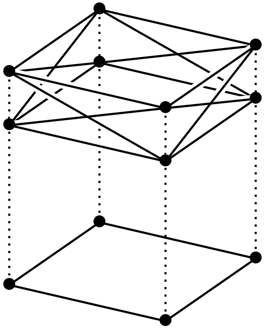

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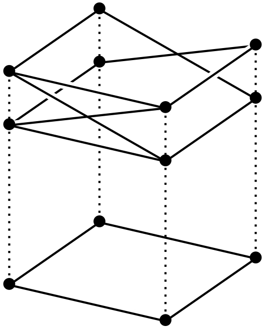

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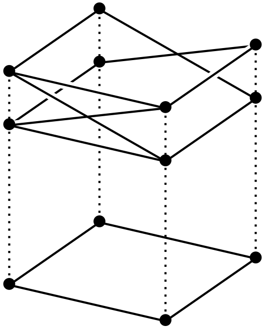

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So, a separated presheaf on $\mathcal{C}$ assigns a relation A_U on $(A_x)_{x \in U}$ to each $U \in \mathcal{C}$.

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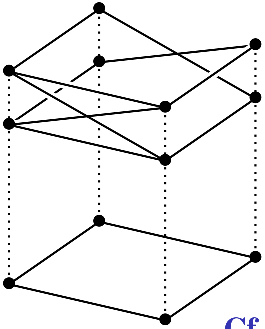

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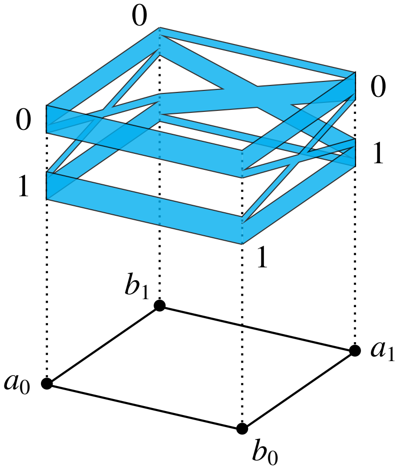
i.e. iff $A_U \subseteq \prod_{x \in U} A_x$ for all $U \in \mathcal{C}$.

So, a separated presheaf on $\mathcal{C}$
assigns a relation A_U on $(A_x)_{x \in U}$
to each $U \in \mathcal{C}$.

Cf. Given relations $(A_U)_{U \in \mathcal{C}}$, their natural join,
$\bowtie A = \{ g : \prod_{x \in X} A_x \mid g{\restriction}_U \in A_U \text{ for all } U \in \mathcal{C} \}$,
is the set of global sections.

Def. A separated presheaf A is no-signalling if



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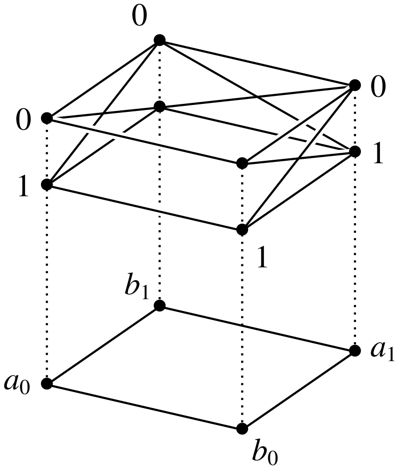

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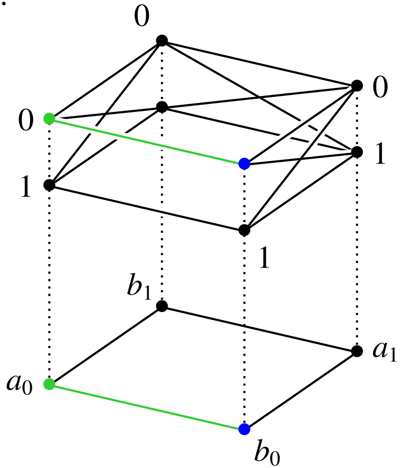

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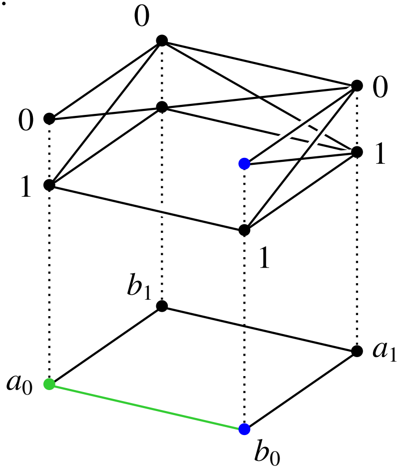

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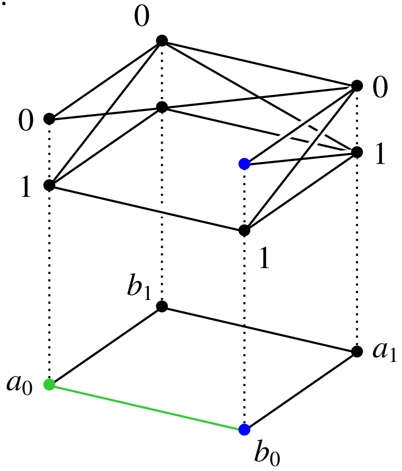

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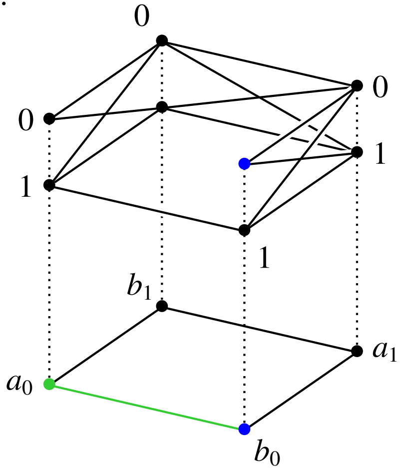

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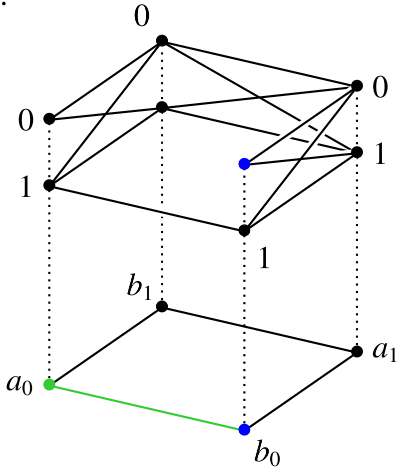

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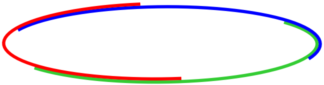

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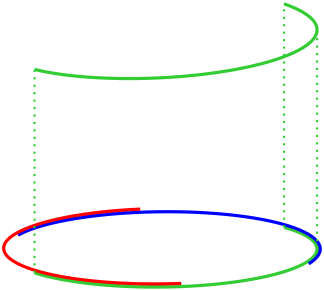

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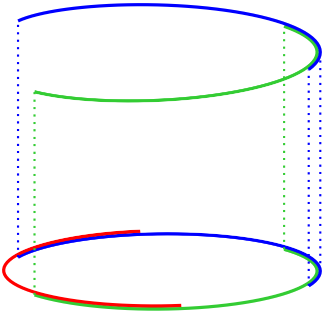

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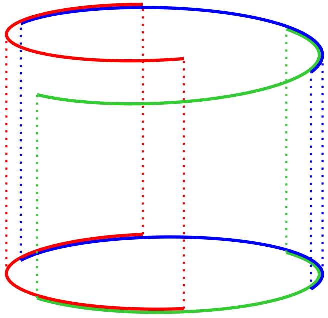

Part III. Contextuality Arguments

After seeing a new formalism, a natural question is "So what can we do using it?"

E.g. two families of contextuality argument:

- Logical methods (incl. equational, algebraic ones);
- Algebraic-topological method using cohomology;
- And some structural connection between the two families.

Logical Argument for Contextuality

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Using presheaves as semantic models, e.g.:

$(0,0) \vDash x \oplus y = 0$
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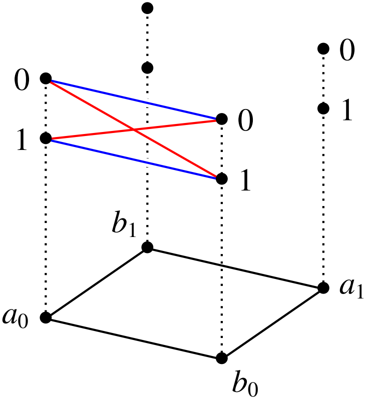

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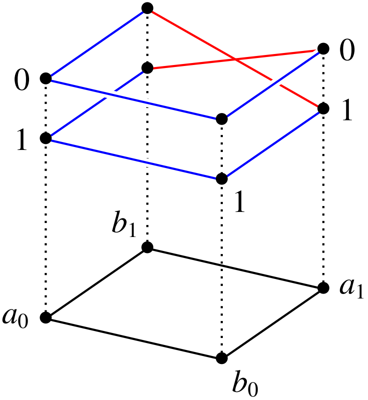

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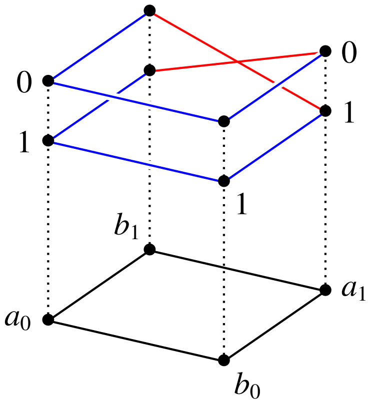

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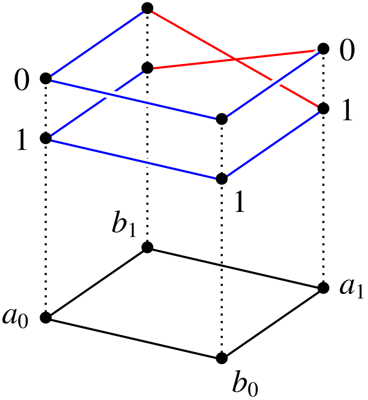

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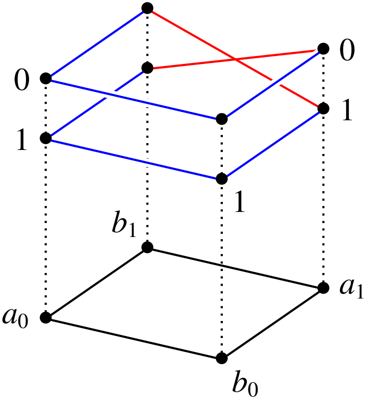

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The equations are inconsistent,
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- $k_0x_0 + \cdots + k_mx_m = p$ for $k_0, \ldots, k_m, p \in R$.
- Equations are inconsistent if a subset of them is s.th.
  - coefficients k of each variable x add up to 0,
  - parities p do not.

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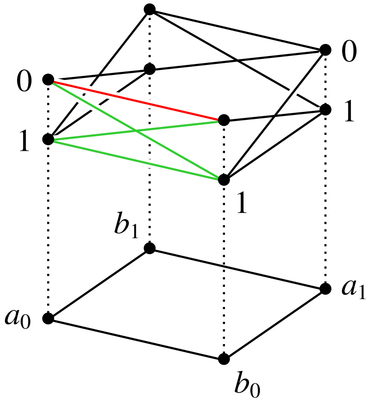

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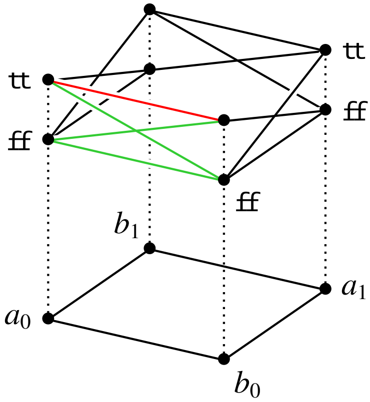

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$$\neg(a_0 \wedge b_1)$$
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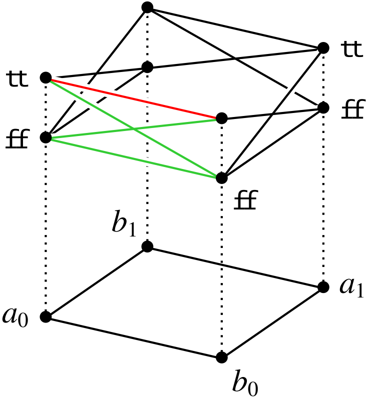

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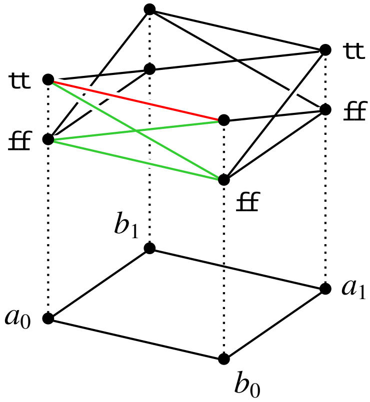

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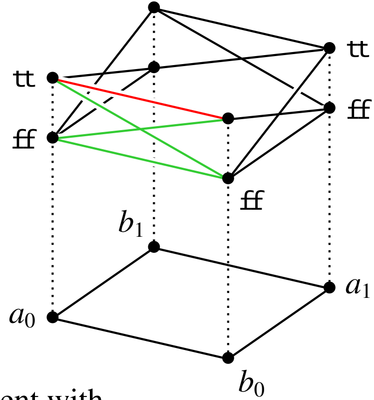

No global assignment (consistent with the other constrants) satisfies $a_0 \wedge b_0$, i.e. logically contextual!

No-signalling as semantic coherence

A formula φ can be "in context U" if free-var(φ) $\subseteq U$, and a model A satisfies φ by $A_U \subseteq [\![\varphi]\!]$.

- A satisfies $\neg a_0 \vee \neg b_0$
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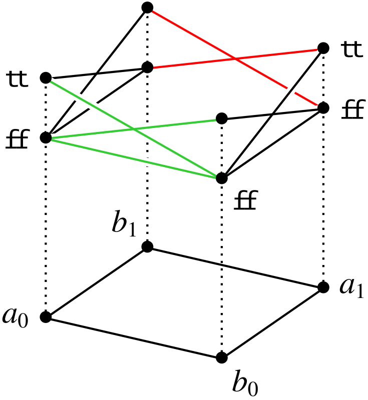

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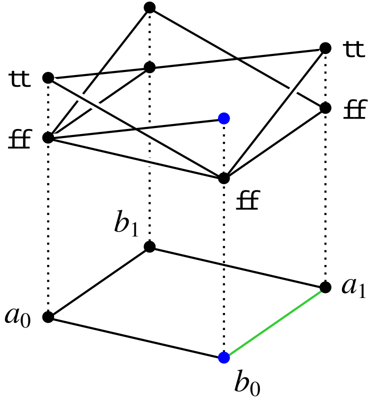

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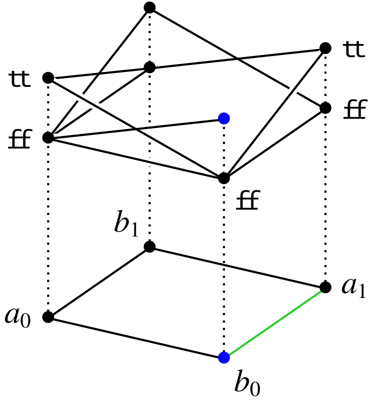

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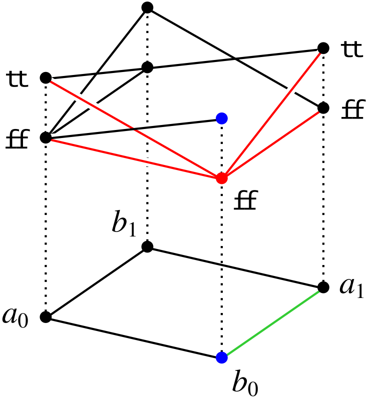

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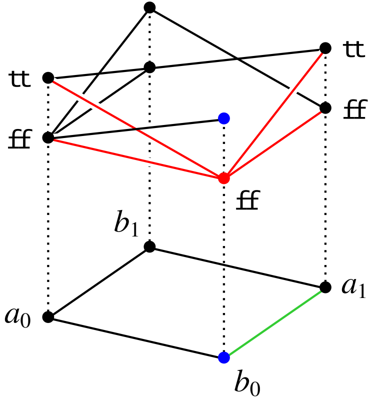

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A formula φ can be "in context U" if free-var(φ) $\subseteq U$, and a model A satisfies φ by $A_U \subseteq [\![\varphi]\!]$.

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When A fails no-signalling,

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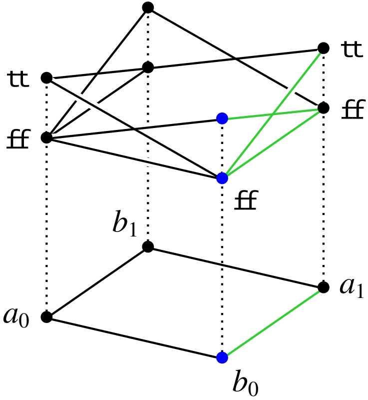

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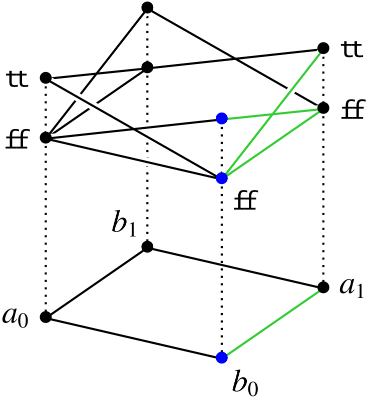

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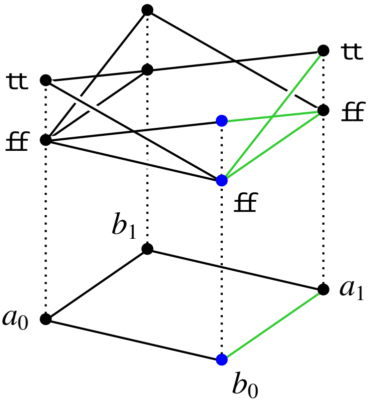

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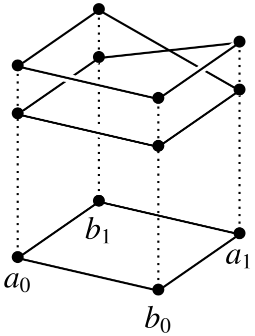

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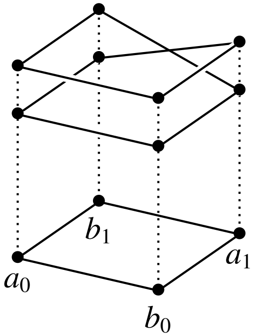

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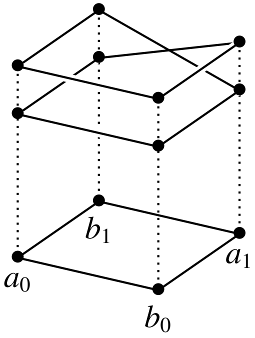

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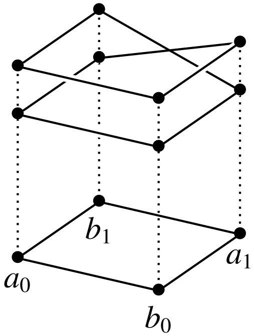

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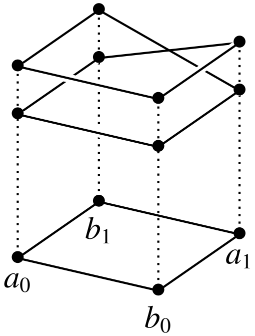

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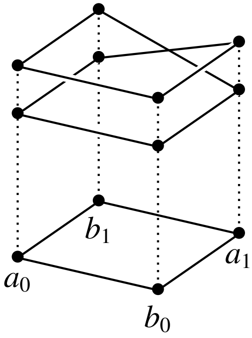

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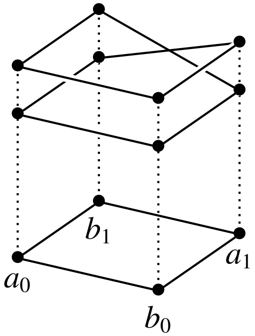

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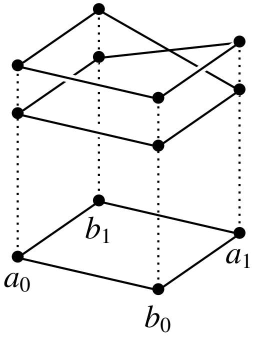

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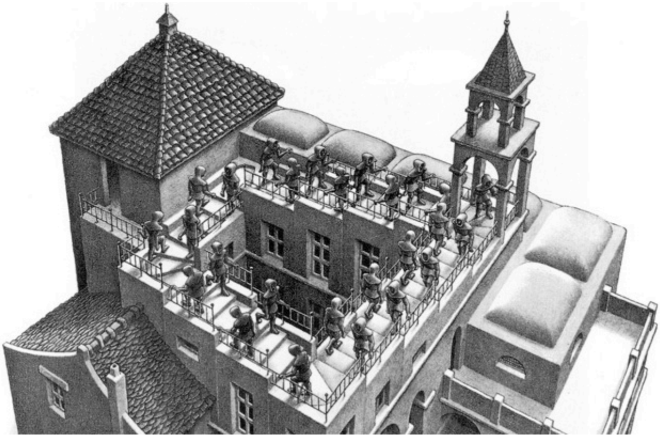
—Logic of contextual models? See arXiv:1605.08949.

Cohomological Argument for Contextuality



This sort of situation can be analyzed with cohomology.

Cf. Penrose 1991, “On the Cohomology of Impossible Figures”.

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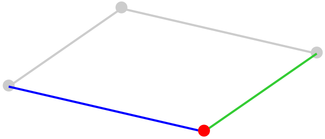
2. List "$N\mathcal{C}^1$" of intersecting pairs of contexts:

$$U, V \in \mathcal{C} \text{ s.th. } U \cap V \neq \varnothing.$$



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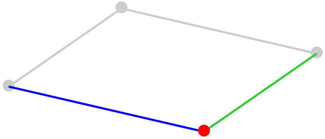

(We are now treating $U \in \mathcal{C}$ like vertices,
$(U, V) \in N\mathcal{C}^1$ like edges
in a new simplicial complex. . . .)

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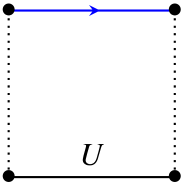

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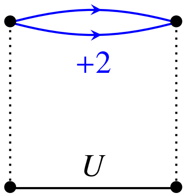

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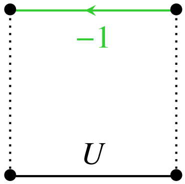

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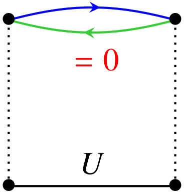

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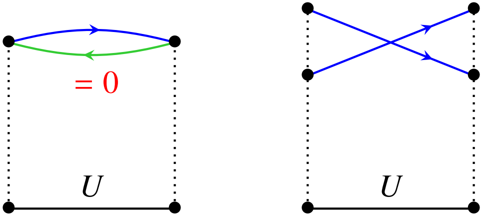

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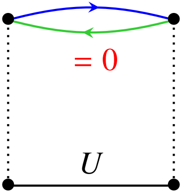

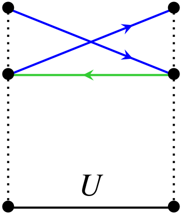

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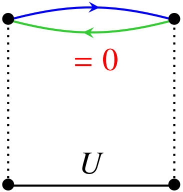
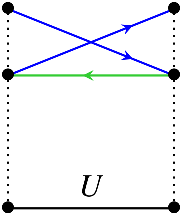
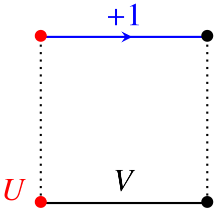




F is a presheaf w/ restriction $\rho_U^V : F(V) \to F(U)$,



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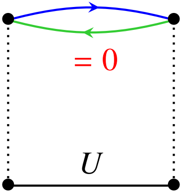

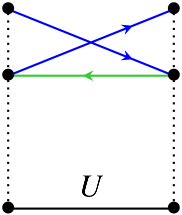

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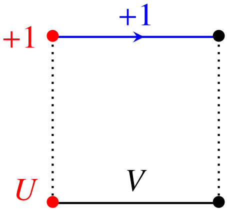

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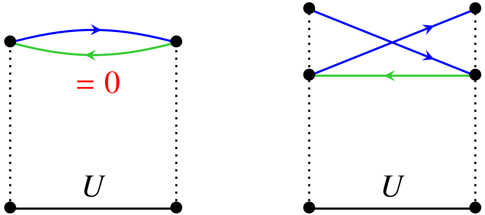

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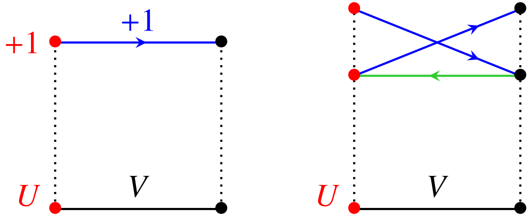

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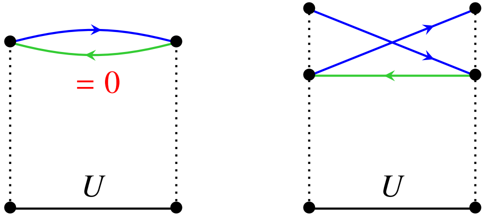

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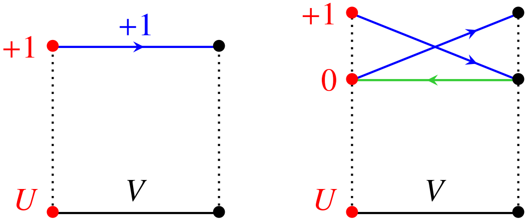

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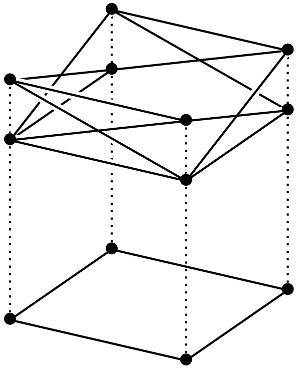

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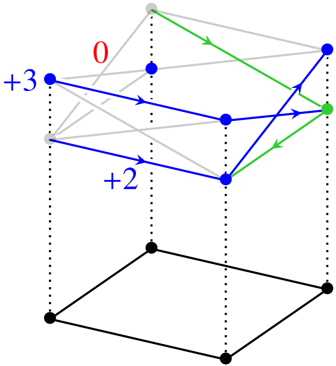

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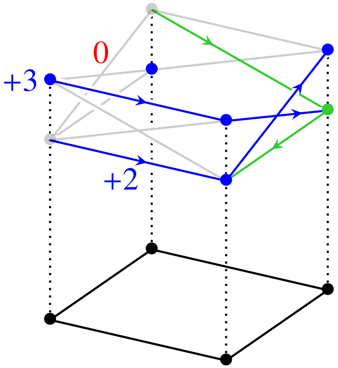
Such a family

$$\omega \in \prod_{U \in \mathcal{C}, U \neq \emptyset} F(U)$$

is called a "0-cochain".



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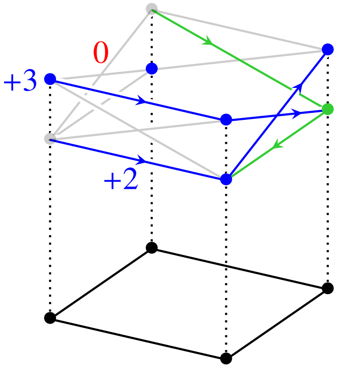

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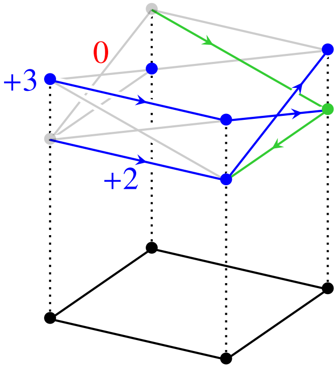

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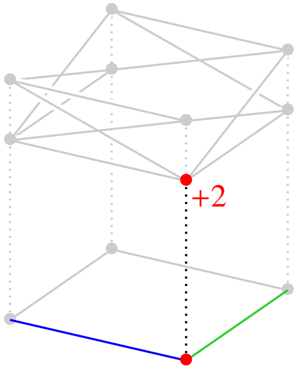

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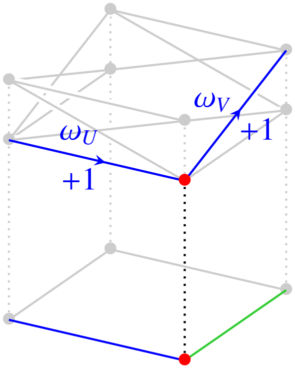

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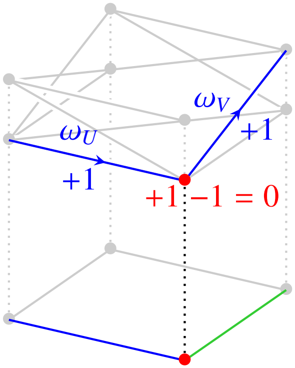

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ω s.th. $\delta^0(\omega) = 0$ is

- called a "0-cocycle",
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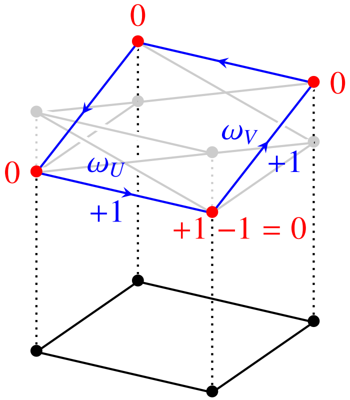

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Caveat:

global section
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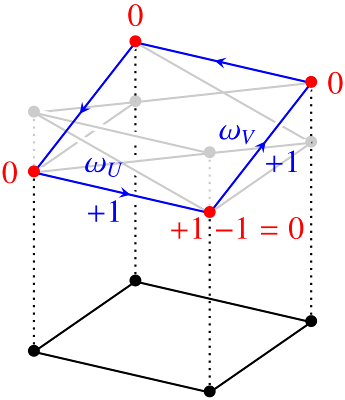

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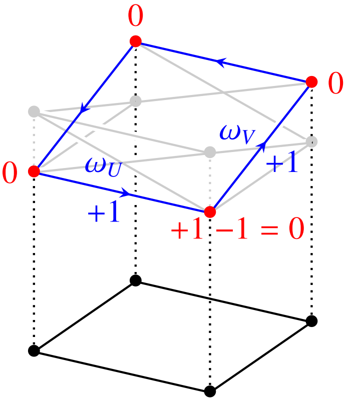

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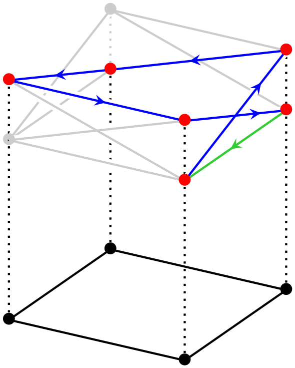

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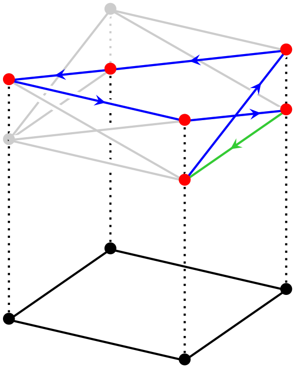

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The group of 0-cocycles is written $\check{H}^0(\mathcal{C}, F)$.

$$
\begin{array}{ccccccccc}
 & & \check{H}^0(C, F_{\bar{U}}) & \longrightarrow & \check{H}^0(C, F) & \longrightarrow & \check{H}^0(C, F|_U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{0} & \longrightarrow & C^0(C, F_{\bar{U}}) & \longrightarrow & C^0(C, F) & \longrightarrow & C^0(C, F|_U) & \longrightarrow & \mathbf{0} \\
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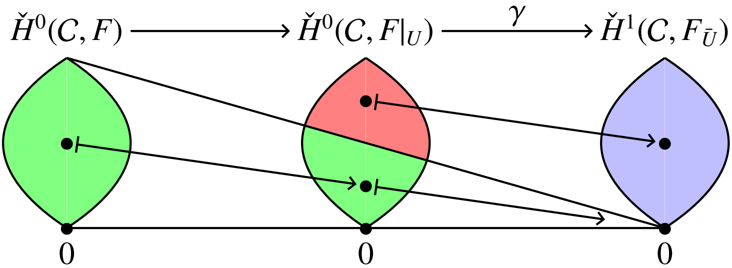

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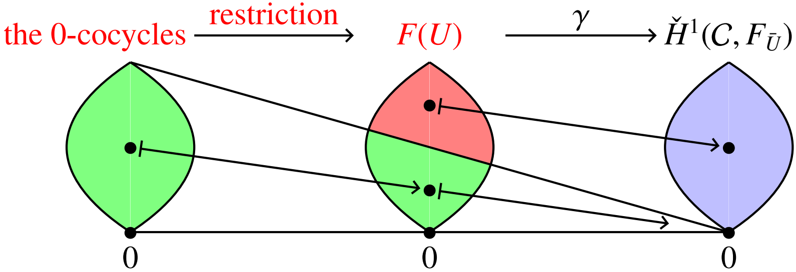
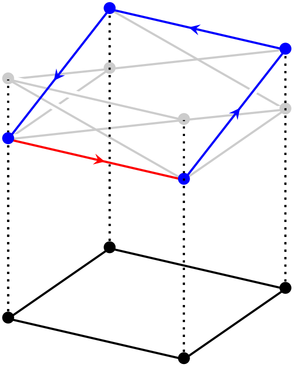


Cohomological test for contextuality:

Each section $s \in A_U \subseteq F(U)$ has the "obstruction" $\gamma(s)$:

s extends to a cocycle $\iff \gamma(s) = 0$.



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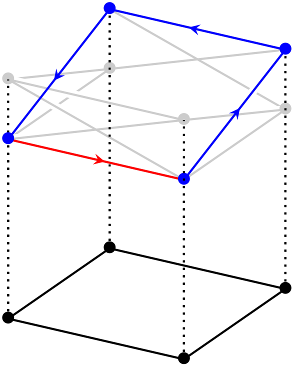
$\Uparrow$

s extends to global



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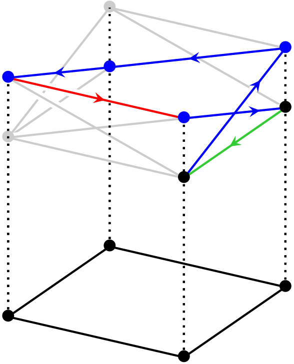

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- False positives,
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- Works for many cases;
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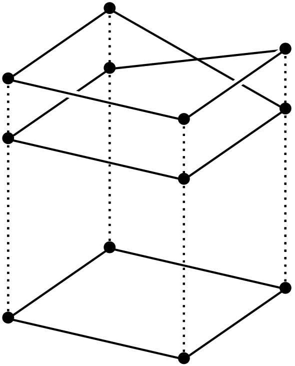

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"Strongly contextual by AvN argument"
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Theorem (Abramsky et al. 2015)**.**

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Hieararchy of strong contextuality:

$$\text{AvN} \quad \subsetneq \quad \text{gen. AvN} \quad \subsetneq \quad \text{cohom. SC} \quad \subsetneq \quad \text{SC}$$

Conclusion

- Contextuality—local consistency, global inconsistency—is topological in nature, expressed nicely with bundles.
- Applying cohomology shows that contextuality is a topological invariant of these bundles.
- Our topological models also serve as semantic models underlying the all-vs-Nothing argument in QM or even more general ones.
- On the other hand, our general formalism makes it clear that contextuality is a ubiquitous phenomenon.
- We expect contextuality to form a new juncture to which, from which, and through which techniques and insights from various fields are transported.

References

[1] Abramsky, Barbosa, Kishida, Lal, and Mansfield (2015), “Contextuality, cohomology and paradox”, *CSL2015*

[2] Abramsky and Brandenburger (2011), “The sheaf-theoretic structure of non-locality and contextuality”, *NJP*

[3] Fine (1982), “Hidden variables, joint probability, and the Bell inequalities”, *PRL*

[4] Hardy (1993), “Nonlocality for two particles without inequalities for almost all entangled states”, *PRL*

[5] Howard, Wallman, Veitch, and Emerson (2014), “Contextuality supplies the ‘magic’ for quantum computation”, *Nature*

[6] Mermin. Extreme quantum entanglement in a superposition of macroscopically distinct states, *PRL*

[7] Raussendorf (2013), “Contextuality in measurement-based quantum computation”, *PRA*

[8] Pironio, Jean-Daniel Bancal, and Valerio Scarani, Extremal correlations of the tripartite no-signaling polytope, *JPhysA*

[9] Penrose (1991), “On the cohomology of impossible figures”, *Structural Topology*

Contextuality in Logical Paradoxes

Read bundles $\pi : \sum_{x \in X} A(x) \to X$ in logic terms:

$x \in X$ are sentences,

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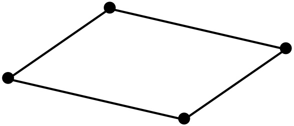

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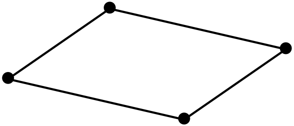

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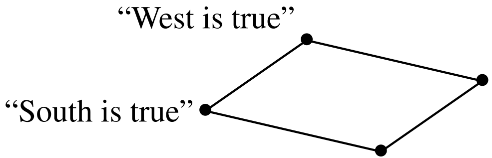

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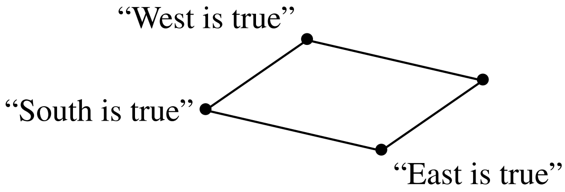

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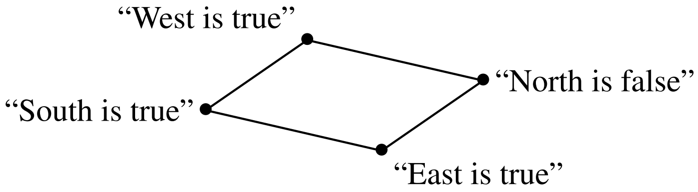

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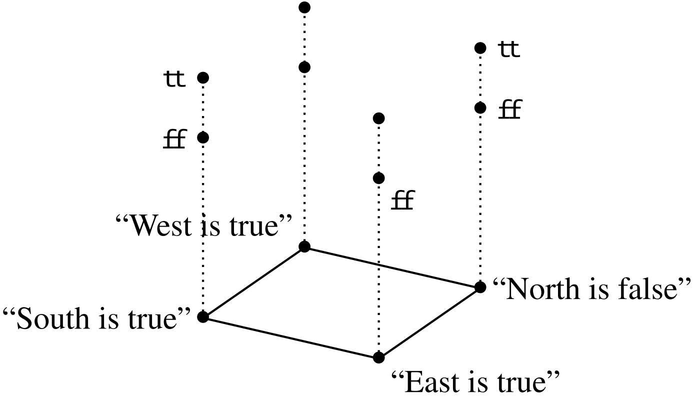

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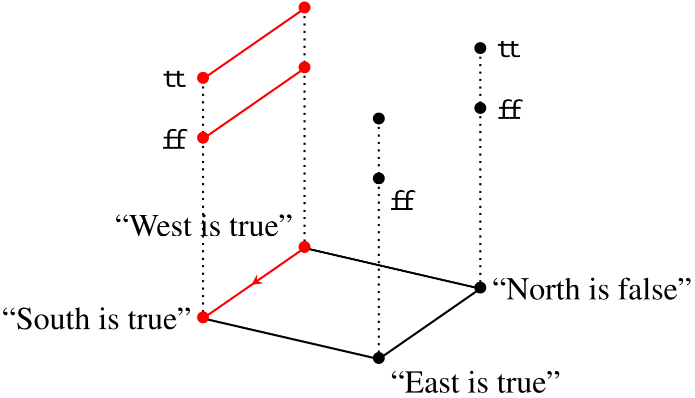

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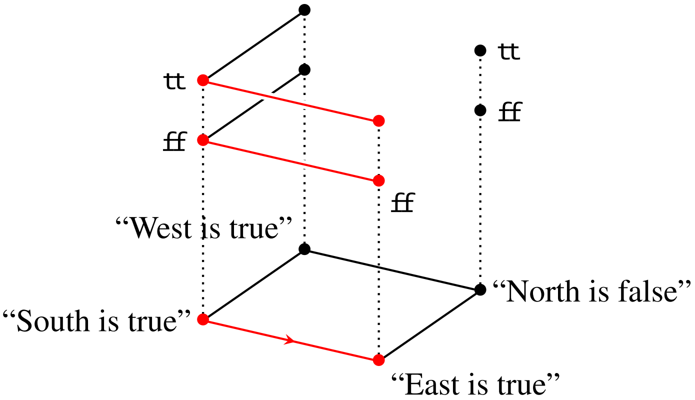

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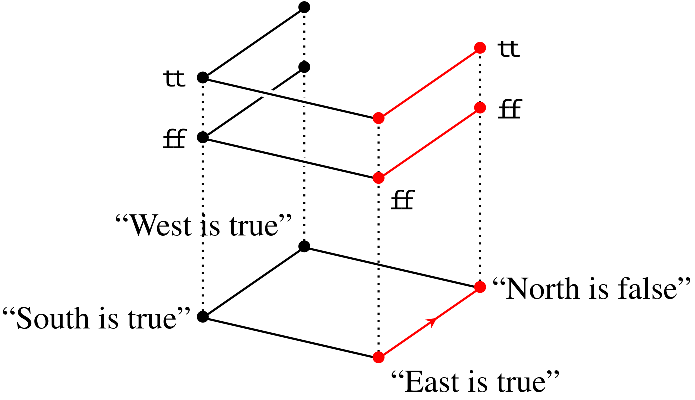

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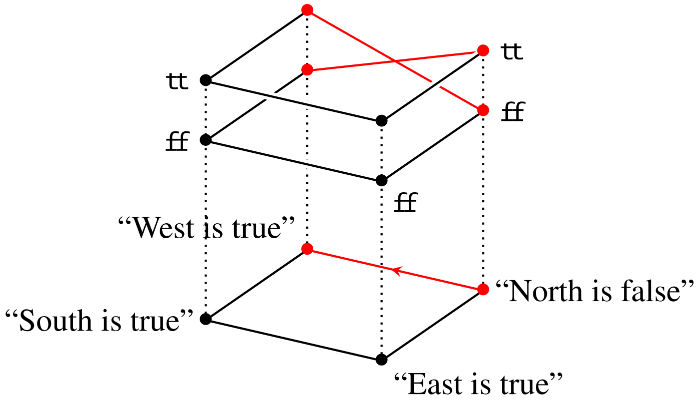

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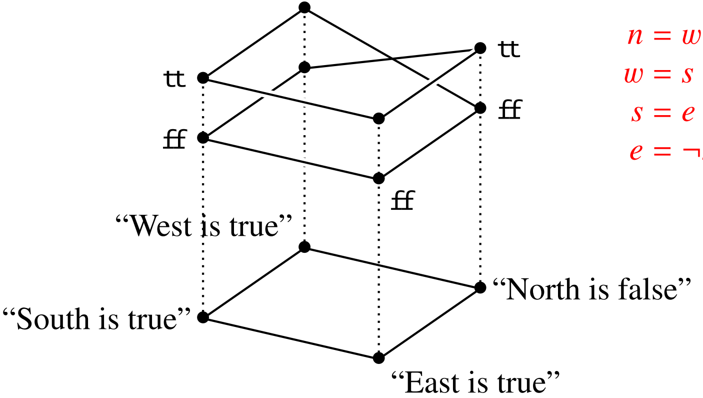

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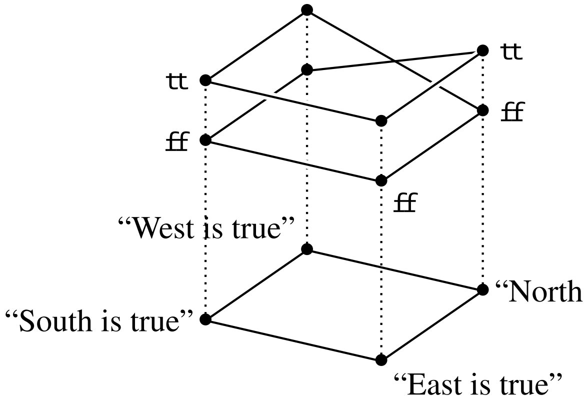

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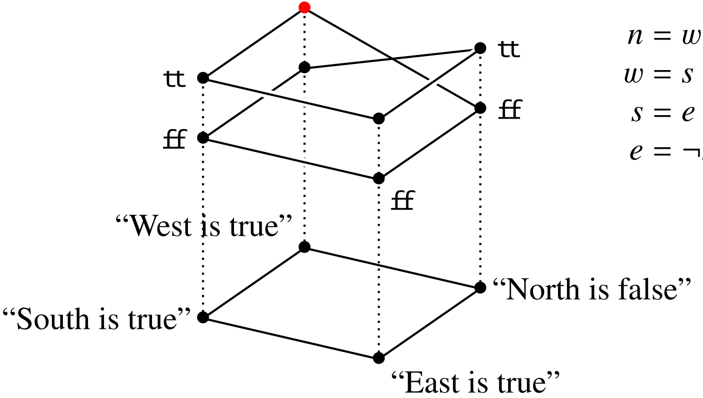

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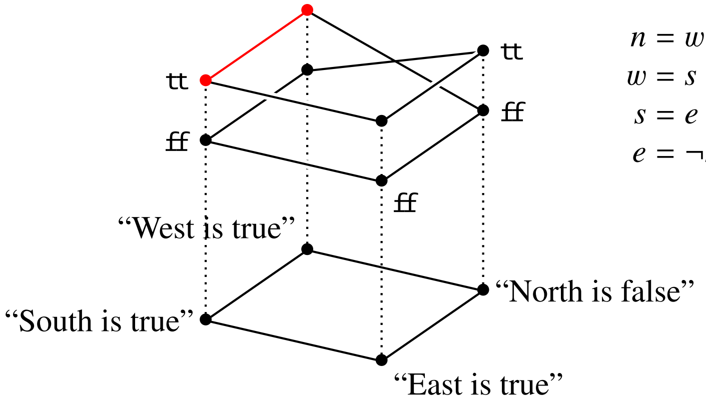

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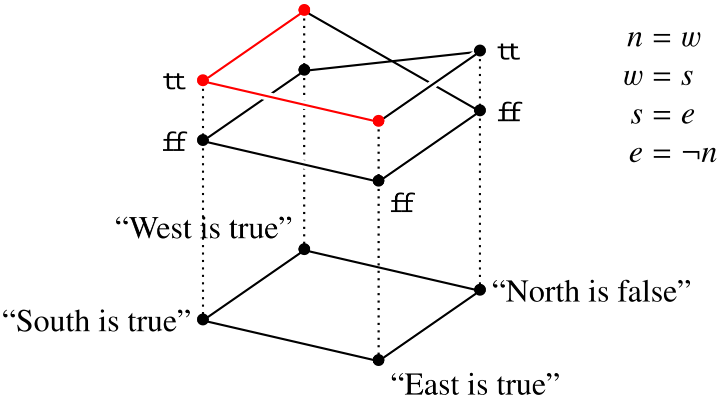

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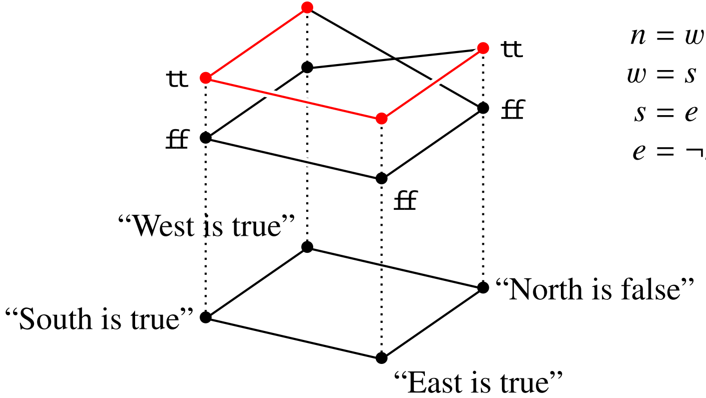

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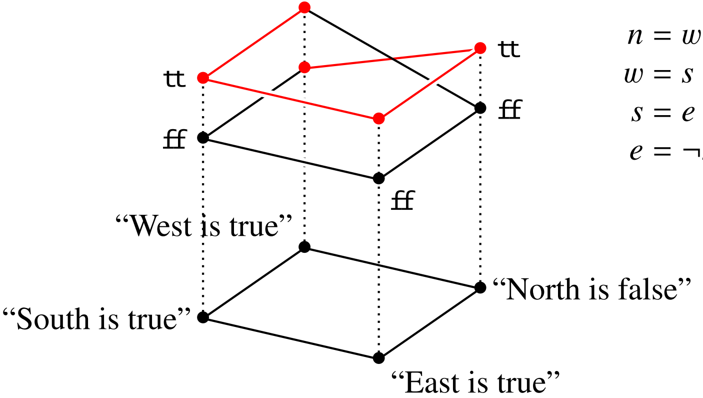

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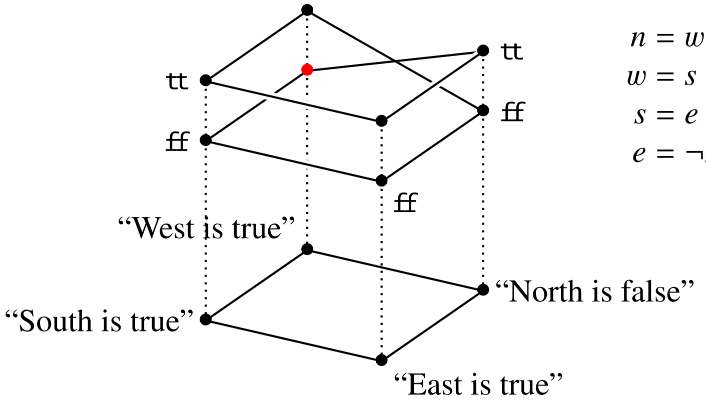

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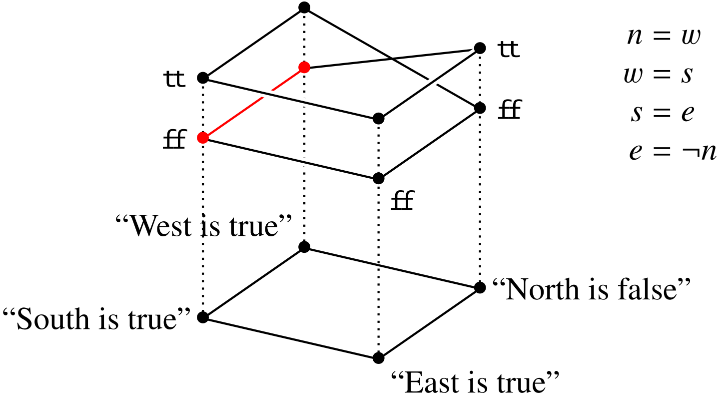

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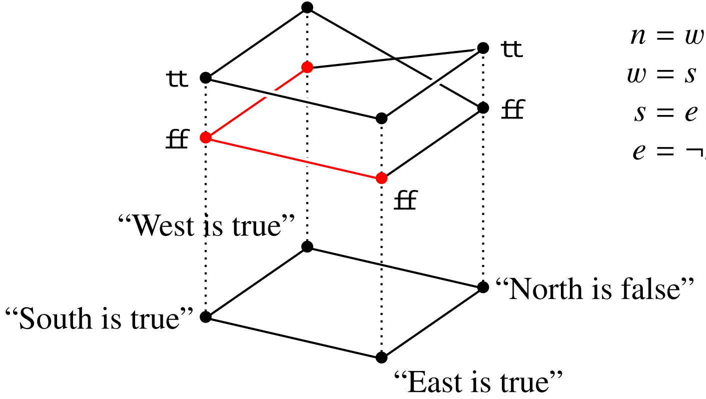

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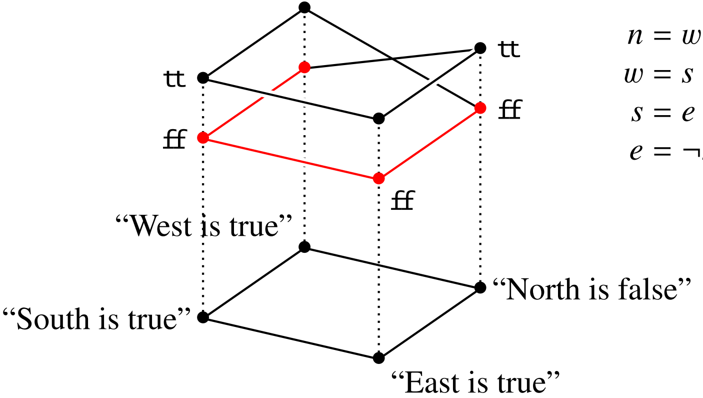

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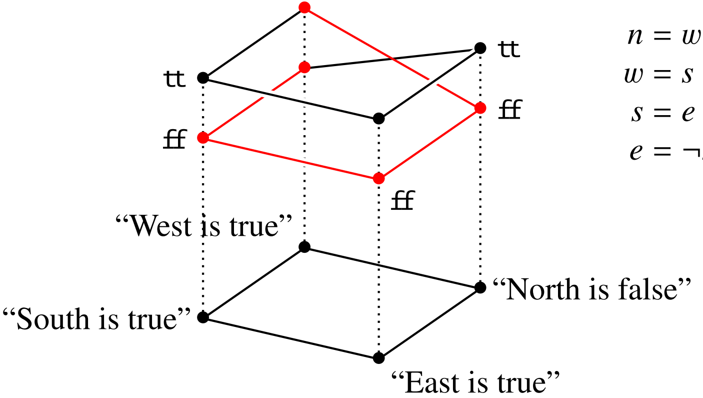

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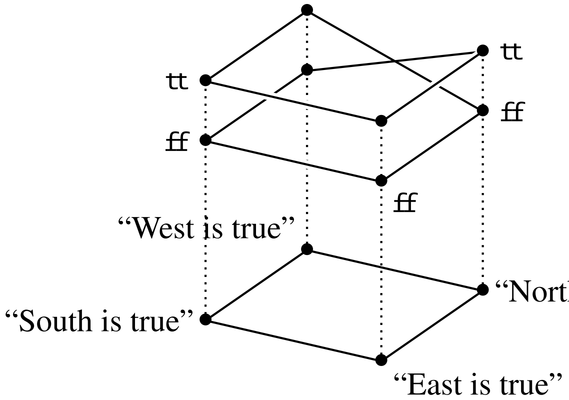

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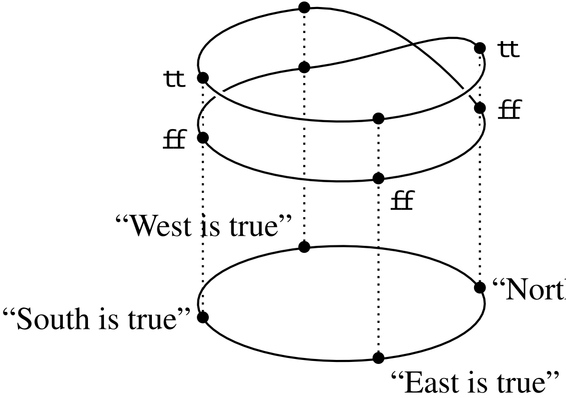

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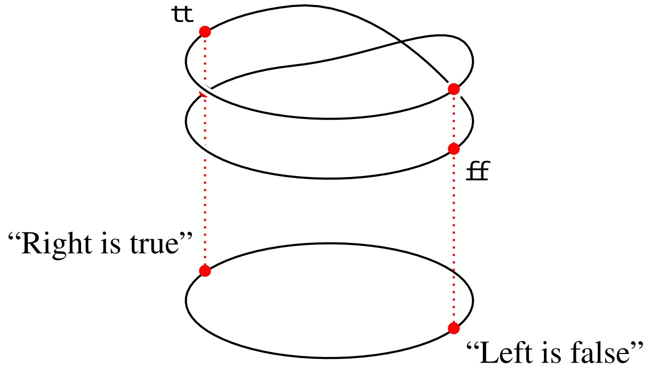

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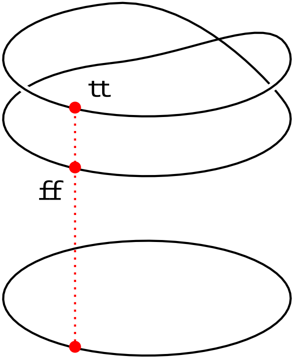
Contextuality in Logical Paradoxes



"This sentence is false"

- AvN argument read as Boolean equations.
- Paths capture our inference deriving contradiction.
- Logical paradoxes have the same topology as "paradoxes" of (strong) contextuality.