Geometric Quantization and Epistemically Restricted Theories

Ivan Contreras [1] Ali Nabi Duman [2]

[1]University of Illinois at Urbana-Champain

[2]King Fahd University of Petroleum and Minerals

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Introduction

Spekkens' Toy Theory:

- An evidence of an epistemic view of quantum states

It contains many quantum features:

- complementarity
- no-cloning
- no-broadcasting
- teleportation
- entanglement
- Choi-Jamiolkowski isomorphism

Introduction

- Classical ontological theory → Statistical Theory → Epistemically restricted theories

Classical Complementarity

The valid epistemic states are those wherein an agent knows the values of a set of variables that commute relative to the Poisson bracket and is maximally ignorant otherwise.

- Mechanics → Liouville mechanics → Gaussian epistricted mechanics
- trits → Statistical theory of trits → Stabilizer subtheory for qutrits
- Optics → Statistical optics → Subtheory of quantum optics

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To investigate how the epistricted theories fit into mathematical methods of geometric quantization.

Geometric Quantization

Given a symplectic manifold (M, Ω) modelling a classical mechanic system and its geometric properties, construct a Hilbert space $\mathcal{H}$ and a set of operators on $\mathcal{H}$ which give the quantum analogue of the classical system.

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Quadrature Epistricted Theories

- The phase space: $\Omega = \mathbb{R}^{2n}$

The epistemic restrictions:

- A set of variables are *jointly knowable* if and only if it is commuting with respect to the Poisson bracket.
- An agent can know only the variables which are linear combination of the position and momentum variables.

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Quadrature Epistricted Theories

Correspondence between geometric quantization and epistricted theories

Object	Semi-classical version in quantization	Epistricted theories
phase space	$(\mathbb{R}^{2n}, \omega)$	$(\mathbb{R}^{2n}, \omega)$
state	lagrangian submanifold of $\mathbb{R}^{2n}$ with half-density $a : \mathbb{R}^{2n} \to \mathbb{R}$	lagrangian subspace with a valuation function $v : \mathbb{R}^{2n} \to \mathbb{R}$
transformations	hamiltonian H on $\mathbb{R}^{2n}$	affine symplectic transformation

Groupoid Quantization

Quantize epistricted theories by a twisted polarized convolution C^*-algebra of a symplectic groupoid in the sense of E. Hawkins.

Hawkins' Recipe

- Construct an symplectic groupoid Σ over Ω.
- Construct a prequantization (σ, L, ∇) of Σ.
- Choose a symplectic groupoid polarization P of Σ which satisfies both symplectic and groupoid polarization.
- Construct a "half form" bundle.
- Ω is quantized by twisted, polarized convolution algebra $C^*_P(\Sigma, \sigma)$.

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Groupoid Quantization (Hawkins '08, GBV '95)

- Start with the symplectic manifold $\Omega = \mathbb{R}^{2n}$ with symplectic form ω (epistricted theories)
- Construct the symplectic groupoid $\mathbb{R}^{2n} \oplus \bar{\mathbb{R}}^{2n}$ integrating the symplectic vector space $\mathbb{R}^{2n}$
- Identify $\mathbb{R}^{2n} \oplus \bar{\mathbb{R}}^{2n}$ with the cotangent bundle $T^*(\mathbb{R}^{2n})$ via a symplectomorphism $\Phi : \mathbb{R}^{2n} \oplus \bar{\mathbb{R}}^{2n} \to T^*(\mathbb{R}^{2n})$.
- Obtain the the Darboux coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ of $T^*(\mathbb{R}^{2n})$ from the symplectomorphism Φ.
- The projection $T^*(\mathbb{R}^{2n})$ to $\mathbb{R}^{2n*}$ gives the polarization

$$P = span\{\partial/\partial p_1, \ldots, \partial/\partial p_n\}$$

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- This polarization gives us the half-form pairing
- The kernels of operators on $L^2(\mathbb{R}^{2n}) \leftrightarrow$ Weyl symbols

Weyl transform

The kernel T of a functional f is given by (Weyl transform)

$$Tf(p,q) = C \int f(\frac{p+q}{2}, \zeta) e^{i\zeta(q-p)/\hbar} d\zeta.$$

- The quantization procedure gives the twisted group algebra $C^*(\mathbb{R}^{2n*}, \sigma)$ where $\sigma : \mathbb{R}^{2n*} \times \mathbb{R}^{2n*} \to \mathbb{T}$, $\sigma(x,y) = e^{\frac{-i}{\{q,p\}}}$.
- Moyal quantization of a Poisson vector space

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Quantum Subtheories (Spekkens '14, Gross '07)

- A *projector-valued measure* with outcome set K is a set of projectors $\{\Pi_k : k \in K\}$ such that $\Pi_k^2 = \Pi_k, \forall k \in K$ and $\sum_k \Pi_k = I$.
- Then the *quadrature observable* associated with f is

$$\mathcal{O}_f = \{\hat{\Pi}_f(\mathbf{f}) : \mathbf{f} \in \mathbb{R}\}$$

where

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Main Result

- The operational equivalence quantum subtheories and epistricted theories is proven using Wigner representation.
- The Wigner representation of an operator product is given by the Moyal product.

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Geometric quantization with an appropriate choice of polarization results in an algebraic structure which is operationally equivalent to epistricted theories. The group algebra $C^*(\mathbb{R}^{2n*}, \sigma)$ contains the algebraic structure of quadrature quantum subtheories.

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Functoriality

- In general, geometric quantization is not functorial.
- The geometric quantization picture for symplectic groupoids turns out to be functorial with respect to the choices, i.e. the polarizations (the groupoid one)
- objects $\leftrightarrow$ symplectic manifolds
- 1-morphism $\leftrightarrow$ Lagrangian polarizations
- 2-morphisms $\leftrightarrow$ affine transformations between Lagrangian polarizations
- The 2-morphims $\leftrightarrow$ C*-algebra automorphisms after quantization.

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Discrete Groupoid Quantization (odd-prime case)

- Start with the special dagger frobenius algebra of epstricted theories, **Spek**.
- Construct the groupoid $\mathcal{G}$ corresponding to **Spek** via the explicit connection in Heunen, Cattaneo, Contreras.
- Obtain the pair groupoid from $\mathcal{G}$ and introduce the symplectic structure on it which is compatible with the pair groupoid.
- Apply geometric quantization procedure on the pair groupoid by considering the complex valued function space on the groupoid and using discrete fourier transform (integral kernel) defined by Gross.
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