Quantifying Contextuality
via linear programming



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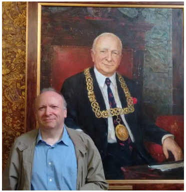

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- Contextuality as a resource
- There may be more than one useful measure

Overview

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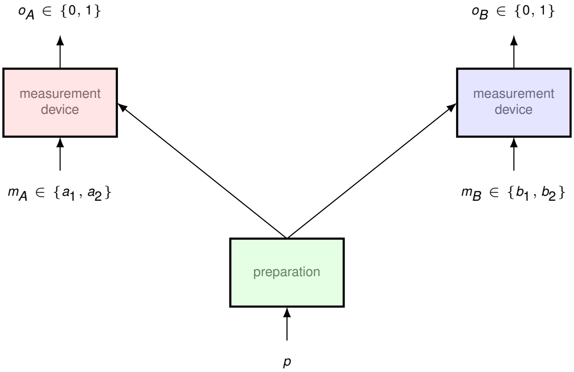
- Computable, using linear programming
- Precise relationship to **violations of Bell inequalities**

Contextuality

Empirical data

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$



Abramsky–Brandenburger framework

Measurement scenario $\langle X, \mathcal{M}, O \rangle$:

- X is a finite set of measurements or variables
- O is a finite set of outcomes or values
- $\mathcal{M}$ is a cover of X, indicating **joint measurability** (contexts)

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Example: (2,2,2) Bell scenario

- The set of variables is $X = \{a_1, a_2, b_1, b_2\}$.
- The outcomes are $O = \{0, 1\}$.
- The measurement contexts are:

$$\{ \{a_1, b_1\}, \quad \{a_1, b_2\}, \quad \{a_2, b_1\}, \quad \{a_2, b_2\} \}$$

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A joint outcome or **event** in a context C is $s \in O^C$, e.g.

$$s = [a_1 \mapsto 0, b_1 \mapsto 1]\ .$$

(These correspond to the cells of our probability tables.)

Another example: 18-vector Kochen–Specker

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- A set of 18 variables, $X = \{A, \ldots, O\}$
- A set of outcomes $O = \{0, 1\}$
- A measurement cover $\mathcal{M} = \{C_1, \ldots, C_9\}$, whose contexts C_i correspond to the columns in the following table:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O

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Compatibility condition: these distributions "agree on overlaps", i.e.

$$\forall_{C,C' \in \mathcal{M}}.\ e_C|_{C \cap C'} \ = \ e_{C'}|_{C \cap C'} \ .$$

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For multipartite scenarios, compatibility $=$ the **no-signalling** principle.

Contextuality

A (compatible) empirical model is **non-contextual** if there exists a **global distribution** $d \in \mathsf{Prob}(O^X)$ (on the joint assignments of outcomes to all measurements) that marginalises to all the e_C:

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The import of results such as Bell's and Bell–Kochen–Specker's theorems is that there are empirical models arising from quantum mechanics that are contextual.

Strong contextuality

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E.g. K–S models, GHZ, the PR box:

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a_1	b_1	✓	×	×	✓
a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
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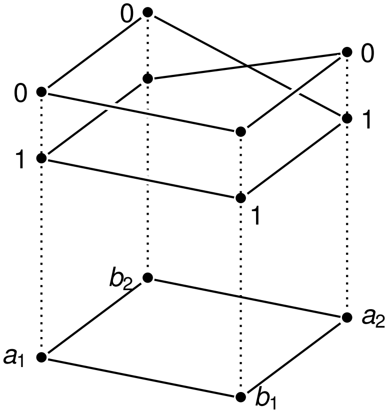

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Equivalently, maximum weight λ over all convex decompositions

$$e = \lambda e^{NC} + (1 - \lambda) e'$$

where e^{NC} is a non-contextual model.

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$$\text{NCF}(e) = \lambda \qquad \text{CF}(e) = 1 - \lambda$$

Computing the contextual fraction

Contextuality as a linear system

For a measurement scenario $\langle X, \mathcal{M}, O \rangle$, the **incidence matrix M** has

- m rows indexed by $\langle C, s \rangle$, $C \in \mathcal{M}$, $s \in O^C$
- n columns indexed by global assignments $g \in O^X$

$$\mathbf{M}[\langle C, s \rangle, g] := \begin{cases} 1 & \text{if } g|_C = s \\ 0 & \text{otherwise} \end{cases} .$$

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A model e is non-contextual if and only if there is $\mathbf{d} \in \mathbb{R}^n$ solving:

$$\mathbf{M}\,\mathbf{d} = \mathbf{v}^e \qquad \text{with} \qquad \mathbf{d} \geq \mathbf{0} .$$

(Non-)contextual fraction via linear programming

Checking contextuality of e corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\,\mathbf{d} = \mathbf{v}^e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array} \quad .$$

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Computing the non-contextual fraction corresponds to solving the following linear program:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}^e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \end{array} .$$

Violations of Bell inequalities

Generalised Bell inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

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For a model e, the inequality reads as

$$\mathcal{B}_\alpha(e) \; \leq \; R \,,$$

where

$$\mathcal{B}_\alpha(e) \; := \sum_{C \in \mathcal{M}, s \in \mathcal{E}(C)} \alpha(C, s) e_C(s) \,.$$

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It is called a **Bell inequality** if it is satisfied by every NC model. If it is saturated by some NC model, the Bell inequality is said to be **tight**.

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A Bell inequality establishes a bound for the value of $\mathcal{B}_\alpha(e)$ amongst NC models.

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The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model e is the value

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} .$$

Bell inequality violation and the contextual fraction

Proposition

Let e be an empirical model.

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Proposition

Let e be an empirical model.

- The normalised violation by e of any Bell inequality is at most $\mathsf{CF}(e)$.
- This is attained: there exists a Bell inequality whose normalised violation by e is exactly $\mathsf{CF}(e)$.
- Moreover, this Bell inequality is tight at “the” non-contextual model e^{NC}.

$$e = \mathsf{NCF}(e) e^{NC} + \mathsf{CF}(e) e^{SC}$$

Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}^e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \quad . \end{array}$$

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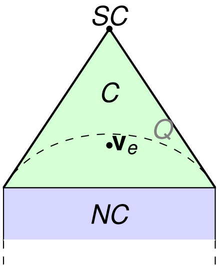

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`and`	$\mathbf{c} \geq \mathbf{0}$.

Dual LP:

`Find`	$\mathbf{y} \in \mathbb{R}^m$
`minimising`	$\mathbf{y} \cdot \mathbf{v}^e$
`subject to`	$\mathbf{M}^T\,\mathbf{y} \geq \mathbf{1}$
`and`	$\mathbf{y} \geq \mathbf{0}$.

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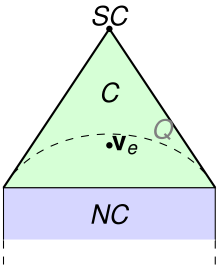

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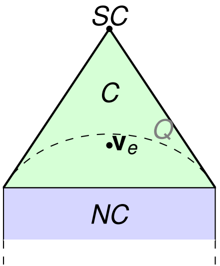

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$$\begin{array}{ll} \texttt{Find} & \mathbf{y} \in \mathbb{R}^m \\ \texttt{minimising} & \mathbf{y} \cdot \mathbf{v}^e \\ \texttt{subject to} & \mathbf{M}^T\,\mathbf{y} \geq \mathbf{1} \\ \texttt{and} & \mathbf{y} \geq \mathbf{0} \quad . \end{array}$$

$$\boxed{\mathbf{a} := \mathbf{1} - |\mathcal{M}|\mathbf{y}}$$

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Quantifying Contextuality LP:

Find	$\mathbf{c} \in \mathbb{R}^n$
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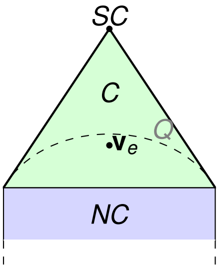

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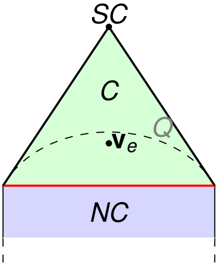

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computes tight Bell inequality (separating hyperplane)

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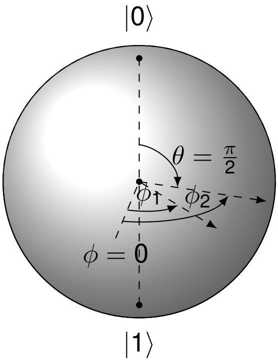

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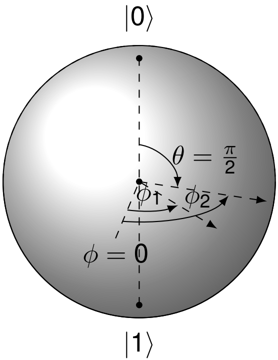
- e.g. $(\phi_1, \phi_2) = (0, \pi/3)$ gives Bell–CHSH model

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$



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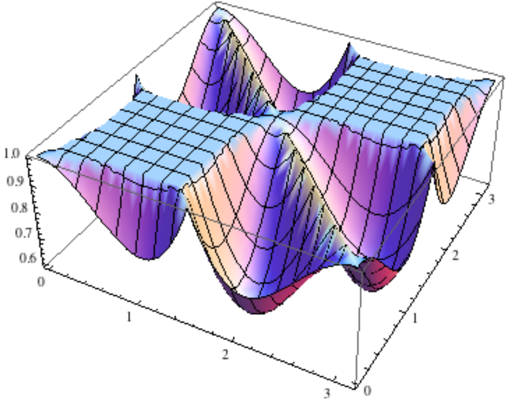

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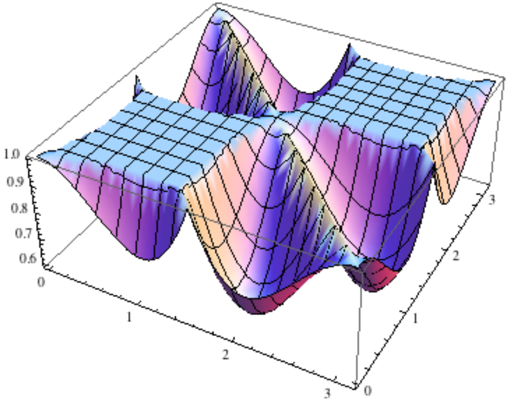

The minima of the plot (maximum contextuality) occur when

$$\{\phi_1, \phi_2\} \in \left\{ \left\{ \frac{\pi}{8}, \frac{5\pi}{8} \right\}, \left\{ \frac{7\pi}{8}, \frac{3\pi}{8} \right\} \right\} .$$

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Note that these achieve Tsirelson violation of the CHSH inequality.

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- n-partite GHZ states, given for $n > 2$ by:

$$|\psi_{\mathrm{GHZ}(n)}\rangle = \frac{|\uparrow\rangle^{\otimes n} + |\downarrow\rangle^{\otimes n}}{\sqrt{2}}$$

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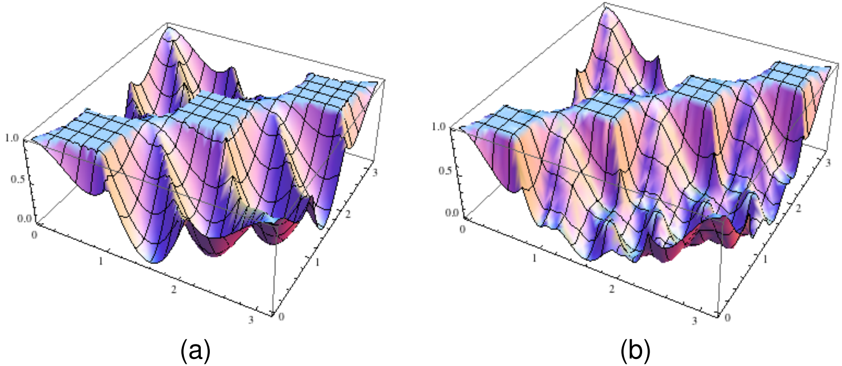

Figure: Non-contextual fraction of empirical models obtained with equatorial measurements at ϕ_1 and ϕ_2 on each qubit of $|\psi_{\mathrm{GHZ}(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$.

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- General n: equatorial measurements at

$$(\phi_1, \phi_2) \in \left\{ \left(\frac{(n+k)\pi}{2n}, \frac{k\pi}{2n} \right) \mid 0 \leq k < n \right\}$$

  on each qubit of the n-partite GHZ state give rise to the strongly contextual GHZ(n) model.

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  - Connections with Contextuality-by-Default (Dzhafarov et al.)

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  - Towards a resource theory as for entanglement (e.g. LOCC), non-locality, . . .

Questions...

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